COMMON TRANSVERSALS

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(Received 5th July 1966)

1. A well-known theorem in group theory [(**8**), p. 11, Satz 3] asserts that, when H is a subgroup of finite index in a group G, there exists a system of common representatives of the right cosets and the left cosets of H in G. Various proofs and generalisations, mainly involving combinatorial rather than group-theoretical ideas, are known, and an excellent account of the subject is to be found in Chapter 5 of Ryser's book (**6**), where references to the literature are given. The purpose of the present paper is to use group-theoretical ideas to prove theorems of a similar nature. The motivation for this work comes from the theory of Hecke operators, and one of the main objects is to provide a simple proof of a result given by Petersson (**4**), which is needed in order to prove the normality of these operators.

If a set S is partitioned in r ways as a union of disjoint non-void subsets, a subset C of S is called a system of common representatives, or a *common transversal*, for the r partitions when C contains exactly one element in common with each of the subsets. Usually we shall take $r = 2$, but we also give some results for $r>2$ in § 3.

We shall find the following notation useful. Let S be a set on whose elements a binary operation (denoted by juxtaposition) is defined. Let A and B be subsets of S and put

$$C = \{c\colon\ c = ab,\ a \in A,\ b \in B\}.$$

Then we write, as usual,

$$C = AB.$$

If, however, each $c \in C$ can be expressed *uniquely* in the form $c = ab$, where $a \in A$ and $b \in B$, we write

$$C = A \,.\, B.$$

Thus, if H is a subgroup of a group G, we can write

$$G = H \,.\, R = L \,.\, H \tag{1}$$

to denote that R is a right transversal of H in G and L is a left transversal of H in G. The index of H in G is denoted by $[G\colon\ H]$ and the order of a finite set C by $|\,C\,|$. Hence, when the index is finite, we have, by (1),

$$[G\colon\ H] = |\,R\,| = |\,L\,|.$$

2. The following theorem, apart from the final remark, is essentially Theorem 1.7.1 of Marshall Hall (**1**); it is, however, the final remark which is of importance for our purpose.

Theorem 1. *Let H_1 and H_2 be subgroups of a group G and let g be any element of G. Write*

$$H_{12}(g) = g^{-1}H_1g \cap H_2,\ H_{21}(g^{-1}) = gH_2g^{-1} \cap H_1, \tag{2}$$

so that

$$H_{21}(g^{-1}) = gH_{12}(g)g^{-1}, \tag{3}$$

and suppose that the indices

$$n_1 = [H_2 : H_{12}(g)],\ n_2 = [H_1 : H_{21}(g^{-1})] \tag{4}$$

are finite. Then the double coset H_1gH_2 is a union of n_1 different right cosets of H_1 in G, and is also the union of n_2 different left cosets of H_2 in G. Further, each of the n_1 right cosets of H_1 has a non-void intersection with each of the n_2 left cosets of H_2.

For completeness we give a proof. Write

$$H_2 = H_{12}(g) \,.\, R, \quad H_1 = L \,.\, H_{21}(g^{-1}), \tag{5}$$

so that $| R | = n_1$ and $| L | = n_2$. Then

$$H_1gH_2 = H_1gH_{12}(g)R = (H_1g \cap H_1gH_2)R = H_1gR.$$

In fact

$$H_1gH_2 = H_1g \,.\, R. \tag{6}$$

For if $h_1gr_1 = h'_1gr_2$, where $h_1, h'_1 \in H_1$ and $r_1, r_2 \in R$, then $r_2r_1^{-1} \in g^{-1}H_1g$. Since $r_2r_1^{-1} \in H_2$, we have $r_2r_1^{-1} \in H_{12}(g)$ and hence, by (5), $r_2 = r_1$ and $h'_1 = h_1$. We can show similarly that

$$H_1gH_2 = L \,.\, gH_2. \tag{7}$$

Finally, if $h_1 \in H_1$ and $h_2 \in H_2$, the two cosets H_1gh_2 and h_1gH_2 have the common member h_1gh_2; note that this is the case even when n_1 or n_2 is infinite.

Theorem 2. *Suppose that the conditions of Theorem 1 hold and that $n_1 = n_2$. Then there exists a common transversal of those right cosets of H_1 and left cosets of H_2 that are contained in H_1gH_2. In particular, this is the case when H_1 and H_2 have equal finite index in G.*

This is an immediate consequence of the final sentence in the enunciation of Theorem 1; in fact a common transversal can be chosen in $n_1!$ different ways. To prove the last part of the theorem we observe that, if

$$[G : H_1] = [G : H_2] = m < \infty,$$

then $[G : H_{12}(g)] = mn_1$, $[G : H_{21}(g^{-1})] = mn_2$ and these are equal, by (3). If C is a common transversal, we have

$$H_1gH_2 = H_1 \,.\, C = C \,.\, H_2. \tag{8}$$

We note that Theorem 2 holds also when n_1 and n_2 are infinite, provided that the sets L and R in (5) have the same cardinal.

Theorem 3. *Suppose that H_1 and H_2 are subgroups of equal finite index in a group G. Then there exists a common transversal of the right cosets of H_1 and the left cosets of H_2 in G.*

For G is a disjoint union of double cosets $H_1gH_2(g \in G)$. When $H_1 = H_2$, Theorem 3 is the theorem proved by Zassenhaus [(8), p. 11]; for different subgroups H_1 and H_2 of the same index it is well known in the case when G is a finite group, but I cannot find a reference to it in the case when G is infinite.

There is a corresponding result when only right (or only left) cosets are considered.

Theorem 4. *Suppose that H_1 and H_2 are subgroups of equal finite index in a group G. Then there exists a common right (left) transversal of H_1 and H_2 in G.*

I cannot give a group-theoretical proof of this result. However, it can be deduced from the following combinatorial theorem, of which several proofs are known [(6), Theorem 2.2, p. 51].

Theorem 5. *If a set S of mn elements is partitioned in two ways as a union of m disjoint sets each containing n elements, then there exists a common transversal of both partitions.*

To deduce Theorem 4, put $[G\colon H_1] = [G\colon H_2] = m$ and

$$[G\colon H_1 \cap H_2] = mn,$$

so that $n \leqq m$, by Theorem 1.5.5 of (1). We take S in Theorem 5 to be the set of mn right cosets $(H_1 \cap H_2)g$ of $H_1 \cap H_2$ in G. Each right coset $H_i a$ is a union of n of these cosets ($i = 1, 2$), and the result follows. Theorem 5 can also be used to provide an alternative proof of Theorem 3.

The following theorem enables one to conclude that $n_1 = n_2$ in Theorem 1 in certain cases, so that Theorem 2 can be applied.

Theorem 6. *Let H be a subgroup of a group G and g an element of G. Suppose that $g^{-1} = agb$, where a and b belong to the normaliser of H in G, and that $[H\colon H \cap g^{-1}Hg]$ is finite. Then there exists a common transversal of those right and left cosets of H that are contained in the double coset HgH.*

Proof. Let $H_{12} = g^{-1}Hg \cap H$, $H_{21} = gHg^{-1} \cap H$, as in Theorem 1. Then, since $g^{-1} = agb$, $H_{21} = a^{-1}H_{12}a$. Let $H = H_{12} \,.\, R$ so that

$$H = a^{-1}Ha = H_{21} \,.\, a^{-1}Ra$$

and it follows that

$$[H\colon H_{12}] = |R| = |a^{-1}Ra| = [H\colon H_{21}],$$

so that Theorem 2 is applicable.

3. We consider here partitions of a group G into several families of cosets with respect to subgroups of equal index.

Theorem 7. *Suppose that $H_1, H_2, \ldots, H_k$ are subgroups of a group G of finite index m and that K and N are subgroups of G such that*

$$[K\colon N] = m, N = K \cap H_i, \quad (1 \leqq i \leqq k). \tag{9}$$

Then it is possible to find a common transversal for the $2k$ partitions of G into right and left cosets of the k subgroups $H_i (1 \leqq i \leqq k)$.

Proof. By Theorem 3 we can find a subset A of K such that

$$K = N\,.\,A = A\,.\,N. \tag{10}$$

Then

$$G = H_i\,.\,A = A\,.\,H_i \quad (1 \leqq i \leqq k). \tag{11}$$

For, if $a, a' \in A$, then $H_i a = H_i a'$ if and only if $a'a^{-1} \in H_i$; but $a'a^{-1} \in K$ and so $a'a^{-1} \in N$, which gives $a' = a$. Hence the m cosets $H_i a (a \in A)$ are distinct and, since $[G: H_i] = m$, we have $G = H_i.\,A$; similarly, $G = A\,.\,H_i$.

We illustrate this theorem by taking G to be the inhomogeneous modular group $\Gamma(1)$, and write

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where a, b, c and d are integers and $ad-bc = \det T = 1$. Two matrices T and $-T$ yield the same element of $\Gamma(1)$. For any positive integer Q we write, as usual,

$$\Gamma(Q) = \{T\colon\ T \equiv \pm I (\text{mod } Q)\},$$

$$\Gamma_0(Q) = \{T\colon\ c \equiv 0 (\text{mod } Q)\},$$

$$\Gamma^2 = \{T\colon\ ab+bc+cd \equiv 0 (\text{mod } 2)\},$$

and

$$\Gamma^3 = \{T\colon\ ab+cd \equiv 0 (\text{mod } 3)\}.$$

We also denote by Γ^4 any one of the four conjugate " cycloidal " subgroups of level 4 and index 4 in $\Gamma(1)$; see Petersson (5).

In the accompanying table n_Q denotes the number of different conjugates of the group $\Gamma_0(Q)$ in $\Gamma(1)$. The conditions of Theorem 7 are satisfied with K and N as in the table and $k = n_Q$. The subgroups $H_i (i \leqq k)$ are the groups conjugate to $\Gamma_0(Q)$.

Q	2	3	4
$[\Gamma(1)\colon \Gamma_0(Q)]$	3	4	6
n_Q	3	4	3
K	Γ^2	Γ^3	Γ^4
N	$\Gamma(2)$	$\Gamma(3)$	$\Gamma(4)$

We deduce that, in each of the cases $Q = 2, 3, 4$, there exists a common transversal of the $2n_Q$ partitions of $\Gamma(1)$ into right and left cosets of the n_Q subgroups conjugate to $\Gamma_0(Q)$. In all these cases it is easy to give the common transversal explicitly.

A similar result holds whenever Q is a prime $p \equiv 3 (\text{mod } 4)$; we take $p > 3$. There are then $p+1$ groups $H_i (1 \leqq i \leqq p+1)$ conjugate to $H_1 = \Gamma_0(p)$, and $[H_1\colon \Gamma(p)] = \frac{1}{2}p(p-1)$. Take K to be any one of the $\frac{1}{2}p(p-1)$ conjugate groups containing $\Gamma(p)$ as a subgroup of index $p+1$. Here $K/\Gamma(p)$ is a dihedral group of order $p+1$ [(7), § 114] and is the normaliser in $\Gamma(1)/\Gamma(p)$ of one of the $\frac{1}{2}p(p-1)$ conjugate cyclic subgroups of order $\frac{1}{2}(p+1)$. Since $p+1$ and $\frac{1}{2}p(p-1)$ are coprime, $K \cap H_i = N = \Gamma(p)$ $(1 \leqq i \leqq p+1)$, and Theorem 7 shows that there exists a common transversal of the $2(p+1)$ partitions of $\Gamma(1)$ into right and left cosets of the groups $H_i (1 \leqq i \leqq p+1)$.

Finally, we observe that, when a family of subgroups of $\Gamma(1)$ has a common right transversal, these subgroups possess a common fundamental region.

4. We now turn to the problem that initiated this investigation. Petersson [(**4**), Hilfssatz 2] shows that there exists a common transversal for the right and left classes of 2×2 integral matrices S of determinant p, such that

$$S \equiv \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix} \pmod{Q}$$

for each matrix S in the transversal. Here p is a prime that does not divide Q. The proof given by Petersson is hard to follow since part of it appears to have been omitted.

In this section we apply Theorem 2 to yield a proof of a generalisation of this result. Throughout n, N and Q denote fixed non-zero integers satisfying

$$n \geqq 1, \quad Q \geqq 1, \quad (N, Q) = 1. \tag{12}$$

We denote by Γ the group of all $n \times n$ matrices T with integral entries and with $\det T = 1$. The group of all non-singular $n \times n$ matrices with rational entries is denoted by G, so that G contains Γ as a subgroup of infinite index. Let also Ω be the semigroup of all non-singular matrices with integral entries. Then

$$\Gamma \subseteq \Omega \subseteq G.$$

Further, write

$$\Omega_N = \{T\colon T \in \Omega,\ \det T = N\} \tag{13}$$

for the set of all integral matrices of " order " N, and put

$$\Gamma_Q = \{T\colon T \in \Gamma,\ T \equiv I \pmod{Q}\}, \tag{14}$$

where I is the identity in Γ. Γ_Q is a normal subgroup in Γ of finite index (**3**)

$$Q^{n^2-1} \prod_{p \mid n} \left\{ \prod_{j=2}^{n} (1-p^{-j}) \right\}.$$

We now reduce each element T of Ω_N to Hermite's normal form [(**2**), Theorem 22.1]. It follows that

$$\Omega_N = \Gamma \,.\, P_N \tag{15}$$

where $T \in P_N$ if and only if T is of the form

$$T = \begin{Bmatrix} t_1 & Qt_{12} & Qt_{13} & \cdots & Qt_{1n} \\ 0 & t_2 & Qt_{23} & \cdots & Qt_{2n} \\ 0 & 0 & t_3 & \cdots & Qt_{3n} \\ \cdots & \cdots & & \cdots & \\ 0 & 0 & 0 & \cdots & Qt_{n-1,n} \\ 0 & 0 & 0 & \cdots & t_n \end{Bmatrix}, \tag{16}$$

https://doi.org/10.1017/S0013091500011482 Published online by Cambridge University Press

where

$$t_1 t_2 t_3 \ldots t_n = N,\ 0 \leqq t_{ij} < | t_j | \quad (1 \leqq i < j \leqq n)$$

and

$$t_i > 0 \quad (1 \leqq i < n),\ \mathrm{sgn}\, t_n = \mathrm{sgn}\, N.$$

(In Hermite's normal form as given in (**2**) the zeros are above the main diagonal, rather than below it, but the form (16) is more usual in the application to modular forms, where $n = 2$.)

Alternatively, each element T of Ω_N can be expressed in Smith's normal form [(**2**), Theorem 26.2], from which it follows that

$$\Omega_N = \Gamma \Sigma_N \Gamma, \tag{17}$$

where $H \in \Sigma_N$ if and only if

$$H = \mathrm{diag}\,(h_1, h_2, \ldots, h_n), \tag{18}$$

where $h_1 h_2 \ldots h_n = N$, $h_i > 0 (1 \leqq i < n)$, sgn $h_n =$ sgn N and $h_i \mid h_{i+1}$ $(1 \leqq i < n)$.

Both P_N and Σ_N are finite sets and we put

$$f_n(N) = | P_N |,\ g_n(N) = | \Sigma_N |,$$

and

$$N = \pm p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}, \tag{19}$$

where $\alpha_i > 0$ $(1 \leqq i \leqq k)$ and $p_1, p_2, \ldots, p_k$ are different primes. Then it is easily shown that

$$f_n(N) = \prod_{i=1}^{k} \prod_{j=1}^{n-1} \left(\frac{p_i^{\alpha_i + j} - 1}{p_i^j - 1} \right) \tag{20}$$

and

$$g_n(N) = \sum_{d^n \mid N} g_{n-1}(N/d^n) \leqq f_n(N). \tag{21}$$

From (21), $g_n(N)$ can be found in a finite number of steps, since $g_1(N) = 1$ and

$$g_2(N) = \prod_{i=1}^{k} (1 + [\alpha_i/2]). \tag{22}$$

Theorem 8. *Let n, N and Q be integers satisfying* (12) *and let J be any fixed member of Ω_N. Then there exists a subset C_N of Ω_N consisting of $f_n(N)$ elements such that*

$$\Omega_N = \Gamma \,.\, C_N = C_N \,.\, \Gamma, \tag{23}$$

and such that, for each $T \in C_N$,

$$T \equiv J (\mathrm{mod}\ Q). \tag{24}$$

Proof. By (17) we can find $S_1, S_2 \in \Gamma$ and $H \in \Sigma_N$ such that

$$J = S_1 H S_2^{-1}; \tag{25}$$

we suppose H given by (18). Write

$$\Omega_N^J = \{T : T \in \Omega_N, \quad T \equiv J (\mathrm{mod}\ Q)\}.$$

For each $T \in P_N$, as given by (16), choose $S_T \in \Gamma$ such that

$$S_T \equiv \mathrm{diag}\,(h_1 t_1^{-1}, h_2 t_2^{-1}, \ldots, \ h_n t_n^{-1}) \quad (\mathrm{mod}\ Q). \tag{26}$$

https://doi.org/10.1017/S0013091500011482 Published online by Cambridge University Press

Here t^{-1} means any integer x such that $xt \equiv 1 (\bmod\ Q)$. Since the product of the diagonal elements on the right of (26) is congruent to 1 modulo Q, such a matrix S_T exists. Now write

$$P_N^J = \{S_1 S_T T S_2^{-1}:\ T \in P_N\}, \tag{27}$$

so that

$$\Omega_N = S_1 \Omega_N S_2^{-1} = \Gamma . P_N^J. \tag{28}$$

Note that, if $T_1 \in P_N^J$, then

$$T_1 \equiv S_1 S_T T S_2^{-1} \equiv S_1 H S_2^{-1} \equiv J \ (\bmod\ Q).$$

It follows from this and (28) that

$$\Omega_N^J = \Gamma_Q . P_N^J \tag{29}$$

and, since $\Omega_N^J \Gamma_Q = \Omega_N^J$,

$$\Omega_N^J = \Gamma_Q P_N^J \Gamma_Q \tag{30}$$

and is the disjoint union of $f_n(N)$ right cosets of Γ_Q in G. It is also, of course, the disjoint union of $f_n(N)$ left cosets of Γ_Q in G. Also, by (28) and (29),

$$\Omega_N = \Gamma \Omega_N^J = \Omega_N^J \Gamma. \tag{31}$$

We now show that, for each $T \in P_N^J$, Theorem 2 applies to the double coset $\Gamma_Q T \Gamma_Q$. For this purpose we have to show that $n_1 = n_2$, where

$$n_1 = [\Gamma_Q : \Gamma_Q^1], \quad n_2 = [\Gamma_Q : \Gamma_Q^2]$$

and

$$\Gamma_Q^1 = T^{-1} \Gamma_Q T \cap \Gamma_Q, \quad \Gamma_Q^2 = T \Gamma_Q T^{-1} \cap \Gamma_Q.$$

Since $\Gamma_Q^2 = T \Gamma_Q^1 T^{-1}$, the groups Γ_Q^1 and Γ_Q^2 are isomorphic, but it does not follow immediately that they have the same index in Γ_Q.

However, since $T \in \Omega_N$, we can find S_3, $S_4 \in \Gamma$ and $H_1 \in \Sigma_N$ such that $T = S_3 H_1 S_4^{-1}$, and so

$$\Gamma_Q^1 = S_4 \Gamma_Q^3 S_4^{-1}, \quad \Gamma_Q^2 = S_3 \Gamma_Q^4 S_3^{-1},$$

where

$$\Gamma_Q^3 = H_1^{-1} \Gamma_Q H_1 \cap \Gamma_Q, \quad \Gamma_Q^4 = H_1 \Gamma_Q H_1^{-1} \cap \Gamma_Q$$

and $n_1 = [\Gamma_Q : \Gamma_Q^3]$, $n_2 = [\Gamma_Q : \Gamma_Q^4]$. Since H_1 is its own transpose, and since Γ_Q contains the transpose of each of its elements, it follows that Γ_Q^4 consists of the transposes of all elements in Γ_Q^3 and conversely. Hence $n_1 = n_2$; n_1 and n_2 are finite, of course, since the groups $\Gamma_Q^i (1 \leq i \leq 4)$ contain Γ_{QN} as a subgroup.

It follows that there is a common transversal for the right and left cosets of Γ_Q contained in each double coset $\Gamma_Q T \Gamma_Q (T \in P_N^J)$. Hence a subset C_N of Ω_N consisting of $f_n(N)$ elements exists such that

$$\Omega_N^J = \Gamma_Q . C_N = C_N . \Gamma_Q \tag{32}$$

and each member of C_N is congruent to J modulo Q. By (31),

$$\Omega_N = \Gamma C_N = C_N \Gamma$$

and (23) follows since $| C_N | = f_n(N)$.

In the case when $n = 2$ and

$$J = \begin{bmatrix} 1 & 0 \\ 0 & N \end{bmatrix}$$

the proof can be abbreviated.

We conclude by considering the case $n = 2$, $Q = 1$, and use Theorem 6 to give an alternative to prove that there exists a subset C_N of Ω_N such that (23) holds.

By (17), Ω_N is the union of $g_2(N)$ double cosets $\Gamma T\Gamma$, where $T \in \Sigma_N$. Now

$$T^{-1} = ATB$$

where

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1/N \\ -1/N & 0 \end{bmatrix},$$

so that both A and B belong to the normaliser of Γ in G. Also $[\Gamma : \Gamma \cap T^{-1}\Gamma T]$ is finite, and so the required conclusion follows from Theorem 6.

Note added 6 February, 1967. Since this paper was submitted, the comprehensive survey article (9) by Mirsky and Perfect has appeared. In §6.4 of this article references supplementing those mentioned in §2 of the present paper are given. In particular, double cosets have been used in a similar way by Ore (10) to prove results that include Theorem 3 above.

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