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SOME ENGEL CONDITIONS ON INFINITE SUBSETS OF CERTAIN GROUPS

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Let k be a positive integer. We denote by $\mathcal{E}_k(\infty)$ the class of all groups in which every infinite subset contains two distinct elements x, y such that $[x,_k y] = 1$. We say that a group G is an $\mathcal{E}_k^*$-group provided that whenever X, Y are infinite subsets of G, there exists $x \in X$, $y \in Y$ such that $[x,_k y] = 1$. Here we prove that:

(1) If G is a finitely generated soluble group, then $G \in \mathcal{E}_3(\infty)$ if and only if G is finite by a nilpotent group in which every two generator subgroup is nilpotent of class at most 3.

(2) If G is a finitely generated metabelian group, then $G \in \mathcal{E}_k(\infty)$ if and only if $G/Z_k(G)$ is finite, where $Z_k(G)$ is the $(k+1)$-th term of the upper central series of G.

(3) If G is a finitely generated soluble $\mathcal{E}_k(\infty)$-group, then there exists a positive integer t depending only on k such that $G/Z_t(G)$ is finite.

(4) If G is an infinite $\mathcal{E}_k^*$-group in which every non-trivial finitely generated subgroup has a non-trivial finite quotient, then G is k-Engel. In particular, G is locally nilpotent.

1. INTRODUCTION AND RESULTS

Paul Erdös posed the following question [**16**]: Let G be an infinite group. If there is no infinite subset of G whose elements do not mutually commute, is there then a finite bound on the cardinality of each such set of elements?

The affirmative answer to this question was obtained by B.H. Neumann who proved in [**16**] that a group is centre-by-finite if and only if every infinite subset of the group contains two different commuting elements.

Further questions of a similar nature, with slightly different aspects, have been studied by many people (see [**1, 2, 3, 4, 5, 6, 7, 13, 14**]).

For a group G we denote by $Z_n(G)$ and $\gamma_n(G)$, respectively, the $(n+1)$-th term of the upper central series and the n-th term of the lower central series of G. For $x, y, x_1, \ldots, x_n \in G$ we write

$$[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2, \qquad [x_1, \ldots, x_n] = \big[[x_1, \ldots, x_{n-1}], x_n\big],$$

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$$[x,_0 y] = x. \qquad [x,_n y] = \big[[x,_{n-1} y], y\big].$$

Recall that a group G is said to be n-Engel if $[x,_n y] = 1$ for all x, y in G. For k a positive integer, let $\mathcal{N}_k$ be the class of nilpotent groups of class at most k; let $\mathcal{F}$ be the class of finite groups and $\mathcal{N}$ be the class of all nilpotent groups. We denote by $(\mathcal{N}, \infty)$ $((\mathcal{N}_k, \infty))$ the class of all groups in which every infinite subset contains two distinct elements x, y such that $\langle x, y\rangle$ is nilpotent (nilpotent of class at most k, respectively). We also denote by $\mathcal{E}_k(\infty)$ $\big(\mathcal{E}(\infty)\big)$ the class of all groups in which every infinite subset contains two distinct elements x, y such that $[x,_k y] = 1$ ($[x,_n y] = 1$ for some positive integer n depending on x, y, respectively) and denote by $\mathcal{N}_k^{(2)}$ the class of all groups in which every 2-generator subgroup is nilpotent of class at most k.

Lennox and Wiegold [**13**] proved that a finitely generated soluble group G is in $(\mathcal{N}, \infty)$ if and only if G is $\mathcal{FN}$. In [**3**] and [**4**] Delizia proved that a finitely generated soluble group or finitely generated residually finite group G is in $(\mathcal{N}_2, \infty)$ if and only if $G/Z_2(G)$ is finite. Longobardi and Maj [**14**] proved that a finitely generated soluble group G belongs to $\mathcal{E}(\infty)$ if and only if G is $\mathcal{FN}$. Also it is proved in [**14**] that a finitely generated soluble group G belongs to $\mathcal{E}_2(\infty)$ if and only if $G/R(G)$ is finite, where $R(G)$ denotes the characteristic subgroup of G consisting of all right 2-Engel elements of G. Abdollahi [**1**] improved the later result by proving that a finitely generated soluble group G belongs to $\mathcal{E}_2(\infty)$ if and only if $G/Z_2(G)$ is finite. In fact this result shows that on the class of finitely generated soluble groups, we have $\mathcal{E}_2(\infty) = (\mathcal{N}_2, \infty)$. Here we prove that on the class of finitely generated soluble groups, we also have $\mathcal{E}_3(\infty) = (\mathcal{N}_3, \infty)$, by proving

THEOREM 1. *Let G be a finitely generated soluble group. Then $G \in \mathcal{E}_3(\infty)$ if and only if G is $\mathcal{FN}_3^{(2)}$.*

Abdollahi and Taeri [**2**] studied the class $(\mathcal{N}_k, \infty)$ and proved that a finitely generated soluble group G is in $(\mathcal{N}_k, \infty)$ if and only if G is $\mathcal{FN}_k^{(2)}$. Also, they proved that a finitely generated metabelian group G is in $(\mathcal{N}_k, \infty)$ if and only if $G/Z_k(G)$ is finite. Here we extend the later result to the class of $\mathcal{E}_k(\infty)$ (Theorem 2, below). In [**2**] it is remarked that if $G/Z_k(G)$ is finite then G is $\mathcal{FN}_k^{(2)}$ but the converse is false for $k \geqslant 3$, even if G is finitely generated and soluble of derived length three. The examples cited, which are due to Newman [**17**], are torsion-free nilpotent.

THEOREM 2. *Let G be a finitely generated metabelian group. Then $G \in \mathcal{E}_k(\infty)$ if and only if $G/Z_k(G)$ is finite.*

By [**2**, Lemma 2], if G is a torsion-free nilpotent $(\mathcal{N}_k, \infty)$-group then G belongs to $\mathcal{N}_k^{(2)}$, and so G is k-Engel. By a result of Zel'manov [**20**], G is nilpotent of class at most $f(k)$, where $f(k)$ is a function of k and independent of the number of generators of G. We prove a similar result about the torsion-free nilpotent groups in the class $\mathcal{E}_k(\infty)$ (Lemma 4, below), from which we obtain

THEOREM 3. *Let G be a finitely generated soluble group which belongs to $\mathcal{E}_k(\infty)$. Then there exists a positive integer t depending only on k such that $G/Z_t(G)$ is finite.*

Let us recall that a group G is said to be locally graded whenever every finitely generated non-trivial subgroup of G has a non-trivial finite quotient. Delizia, Rhemtulla and Smith [5] recently showed that if G is a finitely generated locally graded group and $G \in (\mathcal{N}_k, \infty)$ then there is a positive integer c depending only on k such that $G/Z_c(G)$ is finite. We have been unable to prove a result similar to that of [5] about finitely generated locally graded $\mathcal{E}_k(\infty)$-groups, but we obtain a result as follows.

Let k be a positive integer. We say that a group G is an $\mathcal{E}_k^*$-group provided that whenever X, Y are infinite subsets of G, there exists $x \in X$, $y \in Y$ such that $[x,_k y] = 1$. In [18], Puglisi and Spiezia proved that every infinite locally finite or locally soluble $\mathcal{E}_k^*$-group is a k-Engel group. We improve this result as follows.

THEOREM 4. *Let G be an infinite locally graded $\mathcal{E}_k^*$-group. Then G is k-Engel. In particular, G is locally nilpotent.*

2. PROOFS

We need the following easy lemma in the proofs of both Theorems 1 and 2.

LEMMA 1. *Let G be a group. Suppose that $y, x_1, \dots, x_k \in Z_k(G)$ and $a, b, c, d \in Z_4(G)$. Then for all $i \in \{1, 2, \dots, k\}$ and for all integers n.*

(1) $[x_1 \dots, x_{i-1}, x_i y, x_{i+1}, \dots, x_k] = [x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_k] \times [x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_k]$,

(2) $[x_1, \dots, x_{i-1}, x_i^n, x_{i+1}, \dots, x_k] = [x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_k]^n$,

(3) $[a, b, c, d] = [b, a, c, d]^{-1}$.

Also

(i) *If G is metabelian then for all $x_1, \dots, x_k \in G$*

$$[x_1, x_2, \dots, x_k] = [x_2, x_1, x_3 \dots, x_k]^{-1}.$$

(ii) *For all permutation σ on the set $\{1, \dots, k\}$, for all $a \in \gamma_2(G)$ and $x_1, \dots, x_k$ in G,* $[a, x_1, x_2, \dots, x_k] = \left[a, x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(k)}\right]$.

PROOF: One can check the proofs of parts (1)-(3) of the lemma by some formulas of the commutator calculus. For the proofs of parts (i) and (ii) we note that $\gamma_2(G)$ is Abelian and $[a, b, c] = [a, c, b]$ for all $a \in \gamma_2(G)$ and $b, c \in G$. Thus we use the later equality to permute the symbols in positions 3 to k of the commutator in the left hand side of (ii). ∎

We use the following lemma for the proof of Theorem 1. In the proof of this lemma, we use a result of Gupta and Newman (see [10, Theorem 3.5]) which asserts that every n-generator 2-torsion-free third Engel group is nilpotent of class at most $2n - 1$.

Lemma 2. *Every torsion-free nilpotent $\mathcal{E}_3(\infty)$-group belongs to $\mathcal{N}_3^{(2)}$.*

Proof: Let H be a two-generator torsion-free nilpotent $\mathcal{E}_3(\infty)$-group. By induction on the nilpotency class of H, we may assume $H = Z_4(H)$. We prove that H is 3-Engel and so by [**10**, Theorem 3.5], H is nilpotent of class at most 3. Let a, b be non-trivial elements of H; we show that $[b,a,a,a] = 1$. Consider the infinite subset $\{ab, a^2b, a^3b, \ldots\}$. Since $G \in \mathcal{E}_3(\infty)$, there exist distinct positive integers i, j such that $[a^ib, a^jb, a^jb, a^jb] = 1$. Therefore by Lemma 1 (parts (1)-(3))

$$\begin{aligned}
1 &= [a^ib, a^jb, a^jb, a^jb] = [a^i, b, a^j, a^jb][a^i, b, b, a^jb][b, a^j, a^j, a^jb][b, a^j, b, a^jb] \\
&= [a,b,a,a]^{ij^2}[a,b,a,b]^{ij}[a,b,b,a]^{ij}[a,b,b,b]^{i} \\
&\quad \times [b,a,a,a]^{j^3}[b,a,a,b]^{j^2}[b,a,b,b]^{j}[b,a,b,a]^{j^2} \\
&= [b,a,a,a]^{-ij^2}[b,a,a,b]^{-ij}[a,b,b,a]^{ij}[a,b,b,b]^{i} \\
&\quad \times [b,a,a,a]^{j^3}[b,a,a,b]^{j^2}[a,b,b,b]^{-j}[a,b,b,a]^{-j^2} \\
&= [b,a,a,a^{j^3-ij^2}b^{j^2-ij}][a,b,b,a^{ij-j^2}b^{i-j}] \\
&= \left([b,a,a,a^{j^2}b^j][a,b,b,a^jb]^{-1}\right)^{j-i}.
\end{aligned}$$

Since H is torsion-free, $[b,a,a,a^{j^2}b^j][a,b,b,a^jb]^{-1} = 1$, and so $[a,b,b,a]^j[b,a,a,a]^{j^2}[b,a,a,b]^j = [a,b,b,b]^{-1}$. By arguing as above on the infinite set $\{a^{j+1}b, a^{j+2}b, \ldots\}$ we get

$$[a,b,b,a]^t[b,a,a,a]^{t^2}[b,a,a,b]^t = [a,b,b,b]^{-1},$$

for some positive integer $t > j$. Therefore, by the last two equalities, we have $[b,a,a,a]^{t+j} = [a,b,b,a]^{-1}[b,a,a,b]^{-1}$.

By considering the infinite set $\{a^{t+1}b, a^{t+2}b, \ldots\}$ and arguing as before, we obtain an integer $s > t$ such that $[b,a,a,a]^{s+j} = [a,b,b,a]^{-1}[b,a,a,b]^{-1}$.

Hence $[b,a,a,a]^{s+j} = [b,a,a,a]^{t+j}$ and so $[b,a,a,a] = 1$, this completes the proof. ∎

Proof of Theorem 1: By the result of [**2**], it suffices to prove that every finitely generated soluble $\mathcal{E}_3(\infty)$-group G is $\mathcal{F}\mathcal{N}_3^{(2)}$. By [**14**, Theorem 1], there exists a finite normal subgroup H of G such that G/H is nilpotent. Let T/H be the torsion subgroup of G/H. Then T is finite and G/T is a finitely generated torsion-free nilpotent group. Thus by Lemma 2, $G/T \in \mathcal{N}_3^{(2)}$ and the proof is complete. ∎

Corollary 1. *Let G be an n-generator soluble $\mathcal{E}_3(\infty)$-group. Then $G/Z_{2n-1}(G)$ is finite. In particular, every two-generator soluble group G belongs to $\mathcal{E}_3(\infty)$ if and only if $G/Z_3(G)$ is finite.*

Proof: By Theorem 1, G has a finite normal subgroup T such that G/T is a torsion-free n-generator $\mathcal{N}_3^{(2)}$-group. Thus by [**10**, Theorem 3.5], G/T is nilpotent of class at most $2n-1$. Therefore $\gamma_{2n}(G)$ is finite, and hence $G/Z_{2n-1}(G)$ is finite [**11**]. ∎

We need the following key lemma in the proof of Theorem 2. In the proof of this lemma, we use a result of Gruenberg (see [8, Theorem 1.10] or Gupta and Newman [**9**]),

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which implies that every torsion-free metabelian k-Engel group is nilpotent of class at most k.

LEMMA 3. *Every torsion-free nilpotent metabelian group in $\mathcal{E}_k(\infty)$ is nilpotent of class at most k.*

PROOF: Let G be a torsion-free nilpotent metabelian $\mathcal{E}_k(\infty)$-group. By induction on the nilpotency class of G, we may assume $G = Z_{k+1}(G)$. We prove that G is a k-Engel group and then by a result of Gruenberg (see [8, Theorem 1.10] or Gupta and Newman [9]), G is nilpotent of class at most k. Let x and y be arbitrary non-trivial elements of G. Consider the infinite subset $\{x^n y \mid n \in \mathbb{N}\}$. Since $G \in \mathcal{E}_k(\infty)$ then there exist two distinct positive integers i, j such that $[x^i y,_k x^j y] = 1$. Then by Lemma 1 (parts (1), (2) and (i)) $1 = \left[[x^i, y],_{k-1} x^j y\right]\left[[y, x^j],_{k-1} x^j y\right] = \left[[x, y],_{k-1} x^j y\right]^{i-j}$, and so $\left[[x, y],_{k-1} x^j y\right] = 1$. Therefore, by Lemma 1 (parts (1), (2), (i) and (ii)), we get

$$\text{(I)} \qquad \prod_{r=0}^{k-1}\left[\left[[x,y],_r x\right],_{k-r-1} y\right]^{((k-1)!/r!(k-r-1)!)j^r} = 1.$$

Put $t_1 := j$, $K(k-1, r) = \left[\left[[x,y],_r x\right],_{k-r-1} y\right]$ and consider the infinite subset $\{x^n y \mid n > t_1\}$. Then by arguing as above, there exists a positive integer $t_2 > t_1$ such that

$$\text{(II)} \qquad \prod_{r=0}^{k-1} K(k-1, r)^{((k-1)!/r!(k-r-1)!)t_2^r} = 1.$$

Suppose that

$$M(r, a_1, \dots, a_s) = \frac{(k-1)!}{r!(k-r-1)!}\sum_{i_1=s-2}^{r}\sum_{i_2=s-3}^{i_1-1}\cdots\sum_{i_{s-1}=0}^{i_{s-2}-1} a_1^{r-i_1-1}a_2^{i_1-i_2-1}a_3^{i_2-i_3-1}\cdots a_{s-1}^{i_{s-2}-i_{s-1}-1}a_s^{i_{s-1}},$$

and $N(r, a_1, \dots, a_s) = (a_s - a_{s-1})M(r, a_1, \dots, a_s)$, for all integers $s > 1$, $s-2 \leqslant r \leqslant k-1$ and $a_1, \dots, a_s$. We note that

$$N(r, a_1, \dots, a_{s+1}) = M(r, a_1, \dots, a_{s-1}, a_{s+1}) - M(r, a_1, \dots, a_s),$$

for all integers $s > 1$, $s-2 \leqslant r \leqslant k-1$ and $a_1, \dots, a_{s+1}$.

By (I) and (II), we have $\prod_{r=1}^{k-1} K(k-1, r)^{N(r,t_1,t_2)} = 1$, and since G is torsion-free, $\prod_{r=1}^{k-1} K(k-1, r)^{M(r,t_1,t_2)} = 1$. We note that from arguing as before, there exists an integer $t_3 > t_2$ such that $\prod_{r=1}^{k-1} K(k-1, r)^{M(r,t_1,t_3)} = 1$. Now suppose, inductively, that there exists a sequence $t_1 < t_2 < t_3 < \cdots < t_{s-1} < t_s$ of positive integers such that

$$(*) \qquad \prod_{r=s}^{k-1} K(k-1, r)^{M(r,t_1,\dots,t_s)} = 1.$$

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Also, there exists an integer $t_{s+1} > t_s$ such that

$$(**) \qquad \prod_{r=s}^{k-1} K(k-1,r)^{M(r,t_1,\dots,t_{s-1},t_{s+1})} = 1.$$

By $(*)$ and $(**)$, we have $\prod_{r=s}^{k-1} K(k-1,r)^{N(r,t_1,\dots,t_s,t_{s+1})} = 1$. Since $N(r,t_1,\dots,t_{s+1})$ has a factor of the form $t_{s+1}-t_s$ and G is torsion-free, $\prod_{r=s+1}^{k-1} K(k-1,r)^{M(r,t_1,\dots,t_s,t_{s+1})} = 1$. Therefore, we have a sequence $t_1 < t_2 < \cdots < t_{k+1}$ of positive integers such that

$$\prod_{r=k-1}^{k-1} K(k-1,r)^{N(r,t_1,\dots,t_{k+1})} = K(k-1,k-1)^{N(k-1,t_1,\dots,t_{k+1})} = 1.$$

But $N(k-1,t_1,\dots,t_{k+1}) = t_{k+1}-t_k > 0$ and $K(k-1,k-1) = \big[[x,y]_{,k-1}\,x\big] = [y_{,k}\,x]^{-1}$. Hence $[y_{,k}\,x]^{-(t_{k+1}-t_k)} = 1$ and so $[y_{,k}\,x] = 1$. This completes the proof. □

PROOF OF THEOREM 2: If $G/Z_k(G)$ is finite then G is in $(\mathcal{N}_k,\infty)$ and so G belongs to $\mathcal{E}_k(\infty)$. Conversely, by [**14**, Theorem 1], there exists a finite normal subgroup H of G such that G/H is nilpotent. Let T/H be the torsion subgroup of G/H; then T is finite and G/T is a torsion-free nilpotent metabelian group. Thus by Lemma 3, $\gamma_{k+1}(G) \leqslant T$ and so $\gamma_k(G)$ is finite. Hence $G/Z_k(G)$ is also finite [**11**]. □

To prove Theorem 3, we need the following key lemma, whose proof is similar to [**15**, Proposition 5].

LEMMA 4. *Every torsion-free nilpotent $\mathcal{E}_k(\infty)$-group has nilpotent class bounded by a function of k.*

PROOF: Suppose that G is a torsion-free nilpotent $\mathcal{E}_k(\infty)$-group. Let G be nilpotent of class c. Then $\gamma_{[c/2]}(G)$ is Abelian, where $[c/2]$ equals $(c+2)/2$ if c is even and $(c+1)/2$ if c is odd. Let A denote the isolator of $\gamma_{[c/2]}(G)$ in G. Then A is also Abelian since G is torsion-free. For any $1 \neq x \in A$ and $y \in G$, consider the infinite subset $\{xy, x^2y, x^3y, \dots\}$. Since $G \in \mathcal{E}_k(\infty)$, there exists two distinct positive integers i, j such that $[x^iy_{,k}\,x^jy] = 1$. Since A is a normal Abelian subgroup of G, we have

$$1 = [x^iy_{,k}\,x^jy] = [x^i{}_{,k}\,y]\big[[y,x^j]_{,k-1}\,y\big] = [x_{,k}\,y]^i[x_{,k}\,y]^{-j} = [x_{,k}\,y]^{i-j}.$$

Therefore $[x_{,k}\,y] = 1$, since G is torsion-free and $i-j \neq 0$. Hence, we have $[A_{,k}\,y] = 1$. Since G is torsion-free, it follows from a result of Zel′manov (see [**20**] p. 166) that A lies in $Z_{f(k)}(G)$, where $f(k)$ is a function of k and independent of the number of generators of G. Thus the nilpotency class of G is at most $[c/2] + f(k)$ and hence $c \leqslant 2\big(f(k)+1\big)$. □

PROOF OF THEOREM 3: By [**14**, Theorem 1], there exists a finite normal subgroup H of G such that G/H is a torsion-free nilpotent group. Thus by Lemma 4, there exists a positive integer t depending only on k such that $\gamma_{t+1}(G) \leqslant H$ and so $\gamma_{t+1}(G)$ is finite. Hence $G/Z_t(G)$ is also finite [**11**]. □

Following [**12**], we say that a group G is restrained if $\langle x\rangle^{\langle y\rangle} = \left\langle x^{y^i} \mid i \in \mathbb{Z}\right\rangle$ is finitely generated for all x, y in G.

REMARK 1. Note that an $\mathcal{E}_k^*$-group G with infinite centre Z is k-Engel. For, consider the infinite subsets xZ, yZ for any $x, y \in G$. There exist $z, t \in Z$ such that $[xz,_k yt] = 1$ and so $[x,_k y] = 1$.

LEMMA 5. *Every $\mathcal{E}_k^*$-group is restrained.*

PROOF: Let G be an $\mathcal{E}_k^*$-group and x, y in G. We must show that $H = \langle x\rangle^{\langle y\rangle}$ is finitely generated. Assume that y is of infinite order. Consider the two subsets $X = \{x^{y^n} \mid n \in \mathbb{N}\}$ and $Y = \{y^m \mid m \in \mathbb{N}\}$. If X is finite then the centre of $K := \langle x, y\rangle$ is infinite and so by Remark 1, K is k-Engel. Therefore, by [**12**, Lemma 1(i)], H is finitely generated. Thus, we may assume that X is infinite. Since $G \in \mathcal{E}_k^*$, there exist $n, m \in \mathbb{N}$ such that $\left[x^{y^n},_k y^m\right] = 1$ and so $[x,_k y^m] = 1$. Thus, arguing as in [**12**, Lemma 1(i)], $\langle x\rangle^{\langle y^m\rangle}$ is finitely generated. Therefore $H = \left\langle x^{y^i} \;:\; |i| \leqslant km\right\rangle$. This completes the proof. ∎

REMARK 2. We note that by [**18**, Remark 1.2], every infinite residually finite $\mathcal{E}_k^*$-group is k-Engel.

We are now ready to prove Theorem 4.

PROOF OF THEOREM 4: Let G be an infinite locally graded $\mathcal{E}_k^*$-group and suppose that $x, y \in G$. We must prove that $[x,_k y] = 1$. Assume that there exists an infinite finitely generated subgroup H of G which contains x, y. Let R be the finite residual of H. Then H/R is a finitely generated residually finite group in $\mathcal{E}_k^*$ and so, by Remark 2, H/R is k-Engel. Thus by a theorem of Wilson (see [**19**, Theorem 2]) H/R is nilpotent. By Lemma 5, H is restrained, therefore by repeated application of [**12**, Lemma 3], R is finitely generated. If R is finite then H is residually finite and so is k-Engel. Suppose, for a contradiction, that R is infinite. Since G is locally graded, R has a normal proper subgroup of finite index in R, so the finite residual subgroup T of R is proper in R. Therefore R/T is residually finite k-Engel group and so H/T is nilpotent-by-finite. Thus H/T is residually finite and $R \subsetneq T$, a contradiction.

We may assume that every finitely generated subgroup of G containing x, y is finite. Thus there exists an infinite locally finite subgroup L which contains x, y and so by [**18**, Theorem B], L is k-Engel. Therefore in any case, $[x,_k y] = 1$ and this completes the first part of Theorem 4. By a result of Kim and Rhemtulla (see [**12**, Corollary 6]) which asserts that every locally graded bounded Engel group is locally nilpotent, G is locally nilpotent. ∎

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