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ON THE LATTICE OF PRIMITIVE CONVERGENCE STRUCTURES

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Introduction. Let S be any set and denote by $F(S)$ the collection of all filters on S. The collection $A(S)$ of all mappings from $F(S)$ to 2^S, 2^S being ordered by the dual of its usual ordering, may be regarded as a product of complete Boolean algebras and is, therefore, a complete atomic Boolean algebra [**4**]. $A(S)$ is called the *lattice of primitive convergence structures on* S. If $q \in A(S)$ and $\mathscr{F} \in F(S)$, then $\mathscr{F}$ is said to *q-converge* to a point $x \in S$ if $x \in q(\mathscr{F})$. The collection of all topologies on S may be identified with a subset of $A(S)$; this subset of $A(S)$ will be denoted by $T(S)$. A more specialized class of primitive convergence structures, and one which properly contains $T(S)$, is $C(S)$, the subcomplete lattice of all convergence structures on S. If $q \in A(S)$, then q is a *convergence structure* on S if (i) the principal ultrafilter $\dot{x}$, generated by x, q-converges to x for each $x \in S$, and (ii) whenever $\mathscr{G}$ and $\mathscr{H}$ are filters on S, then $\mathscr{H} \geqq \mathscr{G}$ implies that $q(\mathscr{H}) \supseteq q(\mathscr{G})$. $\mathscr{V}_q(x) = \bigcap\{\mathscr{F} \in F(S): x \in q(\mathscr{F})\}$ is called the *q-neighbourhood filter at x*. In general, $\mathscr{V}_q(x)$ does not *q-converge* to x; however, there is a set $P(S) \subseteq C(S)$ consisting precisely of those convergence structures q such that $\mathscr{V}_q(x)$ q-converges to x for every $x \in S$. The elements of $P(S)$ are called *pretopologies*. The collection of all pretopologies on S, as well as other subclasses of $A(S)$ such as the set of *limitierungs* and the set of pseudo-topologies, have been studied by Choquet [**5**], Fischer [**7**], Kent [**10**], and many others.

The property of being a regular topology may be generalized to apply to pretopologies, convergence structures, and so forth. Studies of regular convergence structures and related topics have been made by Cook and Fischer [**6**], Biesterfeldt [**2**; **3**], and Hearsey [**9**]. In fact, given any subset P of $A(S)$ whose elements have some property in common, it may be of interest to determine the closure of P with respect to various intrinsic lattice topologies on $A(S)$, generalize the property so as to enlarge P and include more elements of $A(S)$, and characterize elements of $A(S)$ which are in the lattice, subcomplete lattice, or subalgebra generated by P. This paper is devoted to the consideration of the closures of $P(S)$, $T(S)$, and certain subsets of $T(S)$ under ι, Frink's ideal topology [**8**].

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1. The ι-closures of $P(S)$ and $T(S)$. The ideal topology ι has, as an open subbase, the collection of all completely irreducible ideals and dual ideals. If L is a lattice and $X \subseteq L$, denote by $c(X)$ the closure of X under the ideal topology ι on L; also let $x^* = \{y \in L: y \geqq x\}$ and $x^+ = \{y \in L: y \leqq x\}$. Much use will be made of the following theorem, a complete proof of which may be found in [**1**, Lemma 1.1].

THEOREM 1.1. *Let B be a Boolean algebra and $A \subset B$.*

(i) *If A is closed under finite joins [meets], then x is in $c(A)$ if and only if x is in $c(A \cap x^*)$ $[c(A \cap x^+)]$.*

(ii) *If A is closed under finite joins [meets], then x is in $c(A \cap x^*)$ $[c(A \cap x^+)]$ only if x is a finite meet [join] of elements from $A \cap x^*$ $[A \cap x^+]$.*

PROPOSITION 1.2. *$q \in c(T(S))$ only if $q \in P(S)$.*

Proof. Since $T(S)$ is closed under finite joins [**10**], it follows from Theorem 1.1 that $q = t_1 \wedge \ldots \wedge t_n$ for some finite collection of t_i in $T(S)$. $\mathscr{V}_q(x) = \bigcap_{i=1}^n \mathscr{V}_{t_i}(x)$ for any x in S. Therefore, q is a pretopology if and only if $\mathscr{V}_q(x) = \mathscr{V}_{t_i}(x)$ for some $i = 1, \ldots, n$. This is true for every x in S. Assume, on the contrary, that there exists a z in S such that $\mathscr{V}_q(z)$ is not equal to any of the $\mathscr{V}_{t_i}(z)$. If $t > q$, then $\mathscr{V}_t(z)$ q-converges to z implies that $\mathscr{V}_t(z) \geqq \mathscr{V}_{t_i}(z) > \mathscr{V}(z)$ for some $i = 1, \ldots, n$. Let D_1 be the dual ideal of $A(S)$ generated by the set of all topologies $t > q$ such that $\mathscr{V}_t(z) \geqq \mathscr{V}_{t_1}(z)$; similar dual ideals $D_2, \ldots, D_n$ may also be defined. q is not in any of the D_i and $q^* \cap T(S) \subset D_1 \cup \ldots \cup D_n$. Therefore q is not in $c(T(S))$ if q is not a pretopology.

COROLLARY 1.2.1. *A primitive convergence structure q is in $c(T(S))$ only if there does not exist a set $\{z_1, \ldots, z_n\}$ such that for each topology $t > q$, $\mathscr{V}_t(z_i) > \mathscr{V}_q(z_i)$ for some $i = 1, \ldots, n$.*

COROLLARY 1.2.2. $P(S) = c(P(S))$.

Consider the following condition:

($*$) For each $x \in S$ and any set $V \in \mathscr{V}_q(x)$, there exists a neighbourhood $\mathscr{U}$ in $\mathscr{V}_q(x)$ of V.

Then we can prove the following two propositions.

PROPOSITION 1.3. *Let q be a pretopology. If q satisfies ($*$), then q is a topology.*

PROPOSITION 1.4. *Let q be a pretopology. If q does not satisfy ($*$), then q is not in $c(T(S))$.*

Proof of Proposition 1.3. Define the operator Γ_q mapping 2^S into 2^S by

$$\Gamma_q(B) = \{x \in S: B \text{ is a member of a filter } \mathscr{F} \text{ which } q\text{-converges to } x\}.$$

To prove that a pretopology q is a topology, it suffices to show that Γ_q is a closure operator [**10**, § II, p. 130, Theorem 4]. $\Gamma_q(A) \subset \Gamma_q(\Gamma_q(A))$ is always true; thus let x be an element of $\Gamma_q(\Gamma_q(A))$. Then there exists a filter $\mathscr{F}$ containing $\Gamma_q(A)$ which q-converges to x. Therefore, $\mathscr{F} \geqq \mathscr{V}_q(x)$ and $\Gamma_q(A) \cap V$ is non-empty for all V in $\mathscr{V}_q(x)$. Given V in $\mathscr{V}_q(x)$, let y be an element of $\Gamma_q(A) \cap U$, where U is chosen so that V is a neighbourhood of U. The element y is in $\Gamma_q(A)$ only if there exists a filter $\mathscr{G}$ containing A such that y is in $q(\mathscr{G})$, in which case, $\mathscr{G} \geqq \mathscr{V}_q(y)$. y is in U; therefore V is in $\mathscr{V}_q(y)$. Since $A \cap V$ is non-empty, it follows that $A \cap V$ is non-empty for every V in $\mathscr{V}_q(x)$. Let $\mathscr{H}$ be the filter generated by $\{A \cap V\colon V \text{ is in } \mathscr{V}_q(x)\}$. Now $\mathscr{H} \geqq \mathscr{V}_q(x)$ and A is in $\mathscr{H}$; hence x is in $\Gamma_q(A)$. Thus $\Gamma_q(A) = \Gamma_q(\Gamma_q(A))$ for all sets $A \subset S$ and this proves Proposition 1.3.

Proof of Proposition 1.4. If $T = \{y_u \in u\colon u \subset v; u, v \in \mathscr{V}_q(x) \text{ and } v \notin \mathscr{V}_q(y_u)\}$ for some x in S, let D_1, D_2, and D_3 be the dual ideals of $A(S)$ generated by the sets $\{t > q\colon \mathscr{V}_t(x) > \mathscr{V}_q(x)\}$, $\{t > q\colon \mathscr{V}_t(x) = \mathscr{V}_q(x)$ and $\mathscr{V}_t(y) > \mathscr{V}_q(y)$ for all y in $T\}$, and $\{t > q\colon \mathscr{V}_t(x) = \mathscr{V}_q(x)$ and $\mathscr{V}_t(y) = \mathscr{V}_q(y)$ for some y in $T\}$, respectively. $q^* \cap T(S) \subset D_1 \cup D_2 \cup D_3$ and q is not in $D_1 \cup D_2$. Assume that $q = t_1 \wedge \ldots \wedge t_n$, the t_i being topologies in D_3, and let $T_i = \{y \in t\colon \mathscr{V}t_i(y) = \mathscr{V}_q(y)\}$. The set T, ordered in the natural way, forms a net (which is frequently in at least one of the T_i). In particular, suppose that for each $U \subset V$, such that U is in $\mathscr{V}_q(x)$, there exists a $W \subset U$ such that W is in $\mathscr{V}_q(x)$ and y_W is in T_k. Since V is not in $\mathscr{V}_q(y_W)$, then U not in $\mathscr{V}_{t_k}(y_W)$ implies that U is not t_k-open. Therefore V in $\mathscr{V}_q(x) = \mathscr{V}_{t_k}(x)$ contains no t_k-open set, i.e., $\mathscr{V}_{t_k}(x)$ does not have a base of open sets, thus contradicting the fact that t_k is a topology.

An immediate consequence of the two preceding propositions is the following result.

THEOREM 1.5. *The set $T(S)$ is closed under the ideal topology on $A(S)$, that is,* $c(T(S)) = T(S)$.

2. The ι closures of some subsets of $T(S)$. A convergence structure q is a T_1 *convergence structure on* S if $\Gamma_s(x) = \{x\}$ for every x in S.

THEOREM 2.1. *If $T_1(S)$ denotes the collection of all T_1 topologies on S, then* $c(T_1(S)) = T_1(S)$.

Proof. Kent has shown (an unpublished result) that the T_1 convergence structures form a closed set under the order topology on $A(S)$. Thus $c(T_1(S))$ is a subset of $c(T(S)) = T(S)$ and also a subset of the collection of all T_1 convergence structures; however, these are precisely the T_1 topologies.

THEOREM 2.2. *If $T_2(S)$ denotes the collection of all Hausdorff topologies on S, then* $T_2(S) = c(T_2(S))$.

Proof. The set $T_2(S)$ is closed under joins in $T(S)$. Let t_1 and t_2 be elements of $T_2(S)$ such that $t = t_1 \wedge t_2$ is not in $T_2(S)$. In this case, there is an ultrafilter $\mathscr{F}$ on S which t-converges to two distinct points x and y in S. Since

$t_1, t_2 \geqq t$, it follows that $t(\mathscr{F}) \supset t_1(\mathscr{F}) \cup t_2(\mathscr{F})$. If x (or y) is not in $t_1(\mathscr{F}) \cup t_2(\mathscr{F})$, then $\mathscr{F} \geqq \mathscr{V}_{t_1}(x)$. Therefore there exists a set V in $\mathscr{V}_{t_1}(x)$ which is not in $\mathscr{F}$, and similarly a set W in $\mathscr{V}_{t_2}(x)$ such that W is not in $\mathscr{F}$. However, $\mathscr{F} \geqq \mathscr{V}_t(x) \geqq \mathscr{V}_{t_1}(x) \cap \mathscr{V}_{t_2}(x)$ implies that $V \cup W$ is in $\mathscr{F}$. By a property of ultrafilters, either V or W is in $\mathscr{F}$; this contradiction shows that $t(\mathscr{F}) = t_1(\mathscr{F}) \cup t_2(\mathscr{F})$. Since both t_1 and t_2 are Hausdorff, it may be assumed that $\{x\} = t_1(\mathscr{F})$ and $\{y\} = t_2(\mathscr{F})$. Let D_1 be the dual ideal of $A(S)$ generated by all Hausdorff topologies h such that $h(\mathscr{F}) \subset \{x\}$, and define D_2 similarly in terms of y; then $h^* \cap T_2(S) \subset D_1 \cup D_2$. If t is in D_1, then there exist $h_1, \ldots, h_n$ in D_1 such that $t = h_1 \wedge \ldots \wedge h_n$ and it may be shown, as above, that $t(\mathscr{F}) = h_1(\mathscr{F}) \cup \ldots \cup h_n(\mathscr{F})$. This contradicts the fact that y is not in $s(\mathscr{F})$ for any topology s in D_1; therefore, t is not in D_1. Similarly, it can be shown that t is not in D_2. Let $I_1 = A(S) - D_1$ and $I_2 = A(S) - D_2$ be maximal ideals containing t, then $I_1 \cap I_2$ is an ι-open set about t, and $I_1 \cap I_2 \cap (T_2(S) \cap t^*)$ is empty. Therefore, by Theorem 1.1, t is not in $c(T_2(S))$, and so $c(T_2(S)) = T_2(S)$.

THEOREM 2.3. *If $K(S)$ denotes the collection of all compact topologies on S, then $c(K(S)) = T(S)$.*

Proof. Let t be any topology on S. If the only t-open covers of S include S, then t is compact. Otherwise, there is a t-open cover of S which does not contain S, in which case there exist two t-open sets A and B, distinct from S and not necessarily in the cover of S, such that the union of A and B is S. In this case, there are two situations to consider:

(i) $S - A$ contains at least two points, and so there exist non-empty disjoint sets C_1 and C_2 contained in $S - A$;

(ii) $S - A = \{b\}$ for some b in S.

(i) If there exist t-open sets A and B such that $A \cup B = S$ and $S - A$ contains more than one point of S, define t_A to be the topology on S whose open sets are $\emptyset$, S, and all sets of the form $G \cap A$, where G is t-open; t_B is defined similarly. If A is empty, then t_A is the indiscrete topology i on S, and if $A = S$, then $t_A = t$; otherwise $i < t_A < t$. $t_A \vee t_B = t$ and t_A, t_B are compact since A and B are proper subsets of S. Let $D_1, \ldots, D_n$ be maximal proper dual ideals of $A(S)$ containing t. Since every maximal dual ideal of $A(S)$ is dual prime, $t_A \vee t_B = t$ shows that either t_A or t_B is in D_i for each $i = 1, \ldots, n$. Assume that t_A is in $D_1, \ldots, D_k$. $t_A \wedge t_C = i$ whenever $A \cap C$ is empty. Therefore t_C is in $D_{k+1}, \ldots, D_n$ for every $C \subset S - A$. If C_1 and C_2 are disjoint subsets of $S - A$, then t_{C_1}, t_{C_2} in $D_{k+1}, \ldots, D_n$ implies that i is in $D_{k+1}, \ldots, D_n$. This contradicts the choice of the D_i as proper dual ideals; thus $k = n$ and t_A is in all of the D_i, $i = 1, \ldots, n$. Therefore t is in $c(t^+ \cap K(S))$; by Theorem 1.1 it follows that t is in $c(K(S))$.

(ii) If there do not exist t-open sets A and B such that $A \cup B = S$, and $S - A$ or $S - B$ contains more than one point, then t consists of the open sets $\emptyset$, S, $S - \{a\}$, $S - \{b\}$, and $S - \{a, b\}$, where a and b are distinct points of S. Thus, assume that there is one more open set G in t. If a and b are in G, then $G \cup (S - \{a, b\}) = S$ and $S - (S - \{a, b\})$ contains two points, and case (i) applies. If a is in G and b is not in G, then there exists a point c distinct from b which is not in G; then $G \cup (S - \{a\}) = S$, and again case (i) applies since $S - G$ contains two points. If a and b are not in G and there exists c in G distinct from a and b, then any open cover of S must contain S, in which case t is compact.

THEOREM 2.4. *If $C_0(S)$ denotes the collection of all connected topologies on S, then $c(C_0(S)) = T(S)$.*

Proof. Given a topology t with proper open subsets A and B of S such that $A \cup B = S$, define t_A and t_B as in the proof of Theorem 2.3. The topologies t_A and t_B are connected, since if C is a proper non-empty subset of S, either $C \cap (S - A)$ is non-empty or $(S - C) \cap (S - A)$ is non-empty. In any case, either C or $S - C$ is not open. Thus C cannot be both open and closed. Therefore $t = t_A \vee t_B$, and the arguments found in the proof of Theorem 2.3 will complete the proof of this theorem.

Since $A(S)$ is an atomic Boolean algebra, the ideal topology ι is strictly finer than the order topology on $A(S)$. This proves the following corollaries to Theorems 2.3 and 2.4.

COROLLARY 2.3.1. *The order closure of $K(S)$ coincides with the order closure of $T(S)$.*

COROLLARY 2.4.1. *The order closure of $C_0(S)$ coincides with the order closure of $T(S)$.*

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