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STABLE RINGS

BY
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I. **Introduction.** Let R be an associative ring with identity. If R is von-Neumann regular of a left v-ring, then for each left ideal, I, we have $I^2=I$. In this note we study rings such that for each left ideal L there exists an integer $n=n(L)>0$ such that $L^n=L^{n+1}$. We call such rings stable rings. We completely describe the stable commutative rings. These descriptions give rise to concepts related to, but more general than, finite Goldie dimension and T-nilpotence, and a notion of power pure.

We begin with an example of a commutative ring with the property that either $I^n=0$ for some n or $I^2=I$, and for each n there is an ideal, I, such that $I^n=0$ but $I^{n-1}\neq 0$.

Since Nakayama's lemma expresses the existence of maximal and minimal submodules we obtain an extended version of Nakayama's lemma for stable rings and concepts of depth and height formulated in terms of the integer n such that $J^n=J^{n+1}$, where J is the Jacobson radical.

II. **The general setting.** Throughout R will denote an associative ring with identity and all modules will be unitary. A left (right) ideal $L(H)$ is *stable* if there exists an integer $n>0$ such that $L^n=L^{n+1}$ ($H^n=H^{n+1}$). We call a ring left (right) *stable* if each left (right) ideal is stable. We call a set of left (right) ideals *bounded stable* if there exists an integer n_0 such that $L^{n_0}=L^{n_0+1}$ ($H^{n_0}=H^{n_0+1}$) for all left (right) ideals $L(H)$ in the set. A ring is left (right) *bounded stable* if its set of left (right) ideas is bounded.

We first note the following

PROPOSITION 1. *A ring is left (bounded) stable iff all the two sided ideals are (bounded) stable iff it is right (bounded) stable.*

Proof. Suppose all two sided ideals are stable. Let L be a left ideal. Choose n such that $(LR)^n=(LR)^{n+1}$. Then $L^{n+1}=(LR)^nL=(LR)^{n+1}\,1=L^{n+2}$ so L is stable. Clearly, if n_0 is a bound for the two sided ideals n_0+1 is a bound for the left ideals. Symmetry gives the conclusion of the proposition.

REMARK. Note in the above the bound for two sided ideals appears in general to be one less than the bound for one sided ideals. At present I know

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of no ring where this actually occurs. Is it possible that the bound for left ideals is the bound for right ideals and two sided ideals too?

With the above proposition in mind we shall speak of stable and bounded stable rings.

We make the following definition and record a proposition as a curiosity as much as anything.

DEFINITION. A left ideal L is called *power pure* if for every right ideal H there exists a positive integer n such that $(HL)^n = (H \cap L)^n$.

PROPOSITION 2. *If R is a stable ring each left (right) ideal is power pure.*

Proof. Let L be a left ideal. Then $(LH)^{n-1} = (LH)^n$ for some n. Then

$$(HL)^n = H(LH)^{n-1}L = H(LH)^nL = (HL)^{2n} \subseteq ((H \cap L)R(H \cap L))^n \subseteq (HL)^n = (HL)^{2n}$$

So $(HL)^{2n} = (H \cap L)^{2n}$.

REMARK. Notice in the above that we obtain $(HL)^n = (H \cap L)^{2n}$ for some n. If we use this to define power pure then all left ideals power pure would be equivalent to stability of the ring, for letting $H = R$ we have $L^n = L^{2n}$.

III. **The main example.** We now construct a ring R with the following properties: (1) R is stable, (2) for each left ideal L properly contained in the radical there exists an integer n, depending on L, such that $L^n = 0$, (3) for J = Jacobson radical, $J^2 = J$, (4) for each $n > 1$ there exists an ideal L such that $L^n = 0$ but $L^{n-1} \neq 0$, (5) the ideals of R are linearly ordered. We begin with a field k and indeterminates $x_1, x_2, x_3, \ldots, x_n, \ldots$. Form the polynomial ring $h[x_1, x_2, \ldots]$. Let I be the ideal generated by $\{x_i - x_{i+1}^2\}_{i=1}^{\infty}$ and $\{x_1^2\}$. Let $R = h[x_1, x_2, \ldots]/I$. We claim R has the five properties listed above. We start with property 3 since it's the easiest. It is easy to see that the ideal, J, generated by the images of the x_i's is a nil ideal and $R/J \cong k$ so J must be the Jacobson radical. In order to verify the rest of the claims we first abuse the notation and let x_i be the image of x_i in R so that $x_{i+1}^2 = x_i$. Let $0 \neq y \in J$. Then there exists a smallest index i such that y can be expressed as a polynomial in x_i. Let p_y be this polynomial (clearly p_y uniquely depends on y). Let $\theta(y) = \text{degree } p_y/2^i$. Note that even if $j > i$ and we write y as a polynomial in x_j, p'_y say, then degree $p'_y/2^j = \theta(y)$. If x and y are in J and $xy \neq 0$ then we claim $\theta(xy) = \theta(x) + \theta(y)$. To see this, if $\theta(x) = K/2^j$, $\theta(y) = l/2^i$ and $j > i$, then $l/2^i = l2^{j-i}/2^j$ hence $\theta(x) + \theta(y) = K + l2^{j-i}/2^j$. We also have degree p_{xy} = degree p_x + degree $p_y/2^{j-i}$ so $\theta(xy) = K + l2^{j-i}/2^j = \theta(x) + \theta(y)$. Set $\theta(0) = 0$. A little computation shows that $\theta(x) + \theta(y) \geq 1$ implies $xy = 0$. θ defines a function of J into the non-negative reals with all numbers greater than one identified with zero. Next take any ideal, I, in R. Since R is local $I \subset J$. Let $x \in I$ and $\theta(x) = K/2^i$. Let j be such that $1/2^i > K/2^i$. Then $x_j \in I$. To see this first write $x = p(x_i)$. Then subtracting a suitable multiple of x gives $x_K^K \in I$. Then $(x^K)x_i^w = x_i^{2^{i-j}} = x_j$, where $w+K=2^{i-j}$. Now $w=2^{i-j}-K>0$ since $2^{i-j}>K$. It follows that if $I \neq J$, then $\inf\{\theta(x), x \in I\} = \varepsilon > 0$. So if $I \neq J$ for n such that $n\varepsilon > 1$ $I^n = 0$. By similar arguments we see that if I and H are ideals, then $I \subseteq H$ iff image I under θ is contained in image H under θ. Notice also, that J is the only infinitely generated ideal in R and the rest are principal, and J has no minimal or maximal submodules.

IV. **Commutative stable rings.** In this section all rings are commutative. We start with showing all semi-primitive stable rings are regular in the sense of von Neumann.

THEOREM 3. *Let R be a ring with zero Jacobson radical. If R is stable, then R is a regular ring.*

Proof. Let $x \in R$. Then there exists an integer n such $Rx^n = Rx^{n+1}$. In particular $x^n = rx^{n+1}$ for some $r \in R$. Choose the smallest n so that there is an $r \in R$ for which $x^n = rx^{n+1}$. Then $(1-rx)x^n = 0$ so $((1-rx)x^{n-1})^2 = 0$ if $n > 0$. But, since R is semi-primitive, R has no nilpotent elements, hence $n = 1$ and R is regular.

COROLLARY 3.1. *If R is a stable ring and J is the Jacobson radical of R, then R/J is regular.*

Proof. It is routine to check that if R is stable so is R/J so we may apply the above theorem.

Before proceeding we need to introduce some notation and terminology.

DEFINITION. Let R be a ring with radical J and $x \in J$. Set $i(x)$ equal to the *index of nilpotence of x.*

DEFINITION. Let $\{x_i\}_{i=1}^{\infty}$ be a sequence in the radical of a ring R. If there is a bound on the indices of nilpotence of the x_i, and if the sequence is T-nilpotent we say the sequence is *bounded T-nilpotent.* We say an *ideal is bounded T-nilpotent* if each sequence for which there is a bound on the indices of nilpotence is T-nilpotent.

THEOREM 4. *A commutative ring R with radical J is stable iff R and all homomorphic images satisfy the following* (i) *R/J is regular,* (ii) *every countable direct sum of finitely generated ideals contained in J is nilpotent,* (iii) *J is bounded T-nilpotent.*

Proof. Suppose R is stable. We've seen that R/J is regular. Also it is easy to see that every homomorphic image of R is also stable. Now suppose there exists a stable ring R for which J is not bounded T-nilpotent. Take R to be a minimal counter example in the following sense: Let $\{x\}_{i=1}^{\infty}$ be a sequence in J for which $x_1x_2 \cdots x_n \neq 0$ for all n; $i(x_i) \leq N$ for all i and N is as small as possible. The idea is to show $N=2$ and then to show that this is impossible as well. Clearly we can assume without loss of generality that $x_i^{N-1}\neq 0$ for all i, for all but a finite number of the x_i must have this property by the minimality of N. Let H be the ideal generated by $\{x_i^{N-1}\}_{i=1}^{\infty}$. Now take k so that $H^{k-1}\neq H^k = H^{k+1}$. Let I be the ideal generated by the sequence $\{x_i\}_{i=1}^{\infty}$. We will show that $x_1x_2\cdots x_l\notin IH^k$ for all l, unless $N=2$. Suppose $x_1x_2\cdots x_l\in IH^k$ and $l\geq \max(3,k)$. Then $x_1x_2\cdots x_l=\sum_{i=1}^m r_i(x_1^{N-1}\cdots x_{i_{l-1}}^{N-1})x_i$ where $x_i\neq X_{i_j}$ for $j=1,\ldots,i_{l-1}$, because $H^k=H^l$. Choose l so that the m in the above sum is minimal. Now let $h=\max\{l, i_j, j=1,\ldots l-1, i=1,\ldots m\}$. Consider

$$0\neq x_1x_2\cdots x_h=\sum_{i=1}^m r_i(x_{i_1}^{N-1}\cdots x_{i_{l-1}}^{N-1})x_ix_{l+1}\cdots x_h$$

where

$$\{i_1, i_2,\ldots .i_{l-1}\}\subset\{1,2,3,\ldots,l\}.$$

Therefore we can write

$$x_1\cdots x_h=\sum_{j=1}^m r_j\Big(\prod_{i=j}(x_i^{N-1})\Big)x_jx_l\cdots x_h.$$

Multiplying by x_1x_2, say, gives that $x_1^2x_2^2x_3\cdots x_n=0$. But then all terms except the first two in the sum must be zero. But multiplying by x_1x_3 says the second term is zero and multiplying by x_2x_3 gives the first term zero.

In case $N=2$ we claim the ideal I is not stable. To see this we claim $x_1\notin I^2$, $x_1x_2\notin I^3,\ldots$. If in fact $x_1x_2\cdots x_a\in I^{a+1}$ then, as before,

$$x_1x_2\cdots x_a=\sum_{i=1}^m r_ix_{i_1}x_{i_2}\cdots x_{i_{a+1}}.$$

But each term on the right must be a multiple of some x_j with $j>a$. Let $j_1, j_2,\ldots, j_w$ be those subscripts appearing on the right which are greater than a. Then $x_1\cdots x_ax_{j_1}x_{j_2}\cdots x_{j_w}=0$ a contradiction so $I^a\neq I^{a+1}$ for all a. Since R is stable, evidently $N\neq 2$ either, and so the sequence must have been T-nilpotent. This establishes (iii).

To establish (ii) if $\bigoplus\sum_{i=1}^{\infty} A_i$ is a countable direct sum with each A_i finitely generated and not nilpotent then we can find a sequence of integers $n_1, n_2\cdots$ such that there exists $x_i\in A_{n_i}$ with $i(x_i)>i(x_j)$ for $i>j$. Then let $I=\bigoplus\sum_{i=1}^{\infty} Rx_i$. I is not stable.

For the converse suppose R has the properties (i)–(iii) and is not stable. Let L be an ideal such that $L\supsetneqq L^2\supsetneqq L^3\supset\cdots$. Let J be the Jacobson radical. Then take $H=J\cap L$. We claim $H\supsetneqq H^2\supsetneqq H^3\supsetneqq\cdots$. Suppose $k>0$ and $x_k\in L^k$ with $x_k\notin L^{k+1}$. If $x_k\notin J$ then there exists $a_k\in R$ such that $x_ka_kx_k-x_k\in J$, since R/J is von Neumann regular. Also, $x_ka_kx_k-x_k\notin L^{k+1}$ for if it did then $x_k\in L^{k+1}$

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which it doesn't. Now $x_k=\sum_{i=1}^{m} r_i l_{i_1}\cdots l_{ik}$ where for $i=1,2,\ldots m_0\le m$, $l_{i_j}\in L$, $l_{i_j}\notin L^2$ for all i_j and if $i>m_0$ at least one $l_{i_j}\in L^2$. We have that $m_0\ge 1$. Consider each term separately. For each $i\le m_0$ there exists $a_{i_1}, a_{i_2},\ldots, a_{i_k}$ such that $l_{i_j}a_{i_j}l_{i_j}-l_{i_j}\in H$. For each $i\le m_0$, we have

$$\prod_{j=1}^{k}(l_{i_j}a_{i_j}l_{i_j}-l_{i_j})\in L^{k+1} \quad \text{iff} \quad \prod_{j=1}^{k} l_{i_j}\in L^{k+1}.$$

Since $x_k\notin L^{k+1}$ it follows that $\prod_{j=1}^{k} l_{i_j}a_{i_j}l_{i_j}-l_{i_j}\notin H^{K-1}$ for at least one $i\le m_0$. In this manner we can construct a sequence $y_k\in H^k$ and $y_k\notin H^{k+1}$ and hence $H^k\supsetneqq H^{k+1}$ for all k. Now suppose $H-H^2$ is not of bounded index. Choose $h_1\in H-H^2$ with $i(h_1)=n_1>3$. Take $h_2\in H-H^2$ with $i(h_2)=n_2>n_1^2+n_1$. Then h_1 and $h_2^{n_1+1}$ do not belong to the ideal generated by h_1h_2. Now choose $h_3\in H-H^2$ so that $i(h_3)=n_3>(n_1+n_2)^2+(n_1+n_2)$. Then $h_3^{n_2+1}$ does not belong to the ideal generated by (h_1h_2, h_2h_3, h_1h_3). In general choose h_k so that $i(h_k)=h_k$ is large enough so that $h_k^{n_{k-1}+1}$ does not belong to the ideal generated by $A_k=\{h_ih_j\}_{i=1,j=i}^{k-1}$. This can be done since each A_k is finite and hence generates a nilpotent ideal. Let

$$B_0=\bigcup_{k=1}^{\infty} A_k.$$

Let B be the ideal generated by B_0 and take R/B. Letting $\bar{h}_i$ be the image of h_i in R/B gives the sequence $\{\bar{h}_i\}$ where $\oplus\sum(R/B)\bar{h}_i$ is direct and there is no bound on the index of nilpotence violating (ii). Consequently we can assume the set $H-H^2$ is of bounded index.

If $H-H^2$ is of bounded index and $H^k\ne 0$ for all k using (iii) there exists in $H-H^2$ subsets $N_1, N_2,\ldots$ such that $|N_i|=n_i$, $N_i\cap N_j=\varnothing$, $n_i<n_j$ if $i<j$ and $\prod_{m=1}^{i} h_{i_m}\ne 0$ where $N_i=\{h_{i,1}, h_{i,2},\ldots, h_{i,n_i}\}$. Let K be the ideal generated by $\{h_{i,j}h_{l,g}\colon i\ne l, j=1,\ldots,n_i, g=1,\ldots,n_l, i=1,2,\ldots, l=1,2,\ldots\}$. Then in the ring R/K the ideals $\bar{N}_i$ generated by the images of the N_i's are an independent set but $(\bar{N}_i)^{n_i}\ne 0$ which violates (ii) in R/K. Therefore H must be stable and the proof of the theorem is complete.

Proposition 5. *Let R be a stable ring and $\{I_\alpha\}_{\alpha\in A}$ be a set of indempotent ideals contained in the radical of R. Then the set is independent only if A is finite.*

Proof. We proceed in the manner as we did to prove (ii) for if H is contained in J, with $H=H^2$, and n is any integer we will show that H contains an element x such that $i(x)>n$. To see this if H is of bounded index it must be T-nilpotent. But if $\{m_i\}_{i\in c}$ generate H then $m_1=\sum r_{ij}(m_im_j)$ so for some $j\in c$ there exists an i so that $m_im_j\ne 0$. But for this j, since m_i is in $H^2=H$ there exists h and l so that $m_lm_hm_j\ne 0$, and again since $m_l\in H^2$ we can find g and f

so that $m_g m_f m_h m_l \neq 0$. Since $m_g \in H^2$ we can continue and in this manner we construct a non T-nilpotent sequence, a contradiction. Hence H is not of bounded index. The rest follows as did (ii).

To show the independence of (ii) and (iii) in Theorem 4 let k be a field of characteristic two and take indeterminates $\{x_1, x_2, \ldots\}$. Form $k[x_1, x_2, \ldots] = R$. Let I be the ideal generated by the set $\{x_i^2\}_{i=1}^{\infty}$. Let $\bar{R} = R/I$. Then $\bar{R}$ has property (ii) but not (iii). To construct a ring satisfying (iii) but not (ii) simply take H to be the ideal generated by $\{x_i x_j, x_i^i, i = 1, \ldots, j = 1, 2, \ldots, j \neq i\}$. Then R/H is a ring with the desired property.

PROPOSITION 6. *If R is stable and A a set of bounded index, then the ideal generated by A is nilpotent.*

Proof. If H is the ideal generated by A and H is not nilpotent let $H^k = H^{k+1} \neq 0$ for some k. From this it is easy to see we can assume $H = H^2$. Now proceed as in the proof of Proposition 5.

PROPOSITION 7. *Let R be a stable ring.* If $\oplus\sum_{\alpha\in A} H_\alpha$ *is a direct sum of ideals in J, then there is a finite subset A' of such that $\sum_{\alpha\in A-A'} H_\alpha$ is nilpotent.*

Proof. If for infinitely many α, $H_\alpha^k \neq 0$ for all k, we would contradict proposition 5 so let A' be the finite subset of A such that $\alpha \in A'$ iff $H_\alpha^K \neq 0$ for all K. If for each integer $n > 0$, there exists an $\alpha \in A - A'$ such that $H_\alpha^n \neq 0$ we can easily construct an ideal $L \subset \sum_{\alpha\in A} H_\alpha$ such that $L \supsetneqq L^2 \supsetneqq L^3 \cdots$ so there exist a integer N such that $H_\alpha^N = 0$ for all $\alpha \in A - A'$ which proves the proposition.

REMARK. This says that if R is stable and of infinite Goldie dimension it must be "almost everywhere" of bounded index.

PROPOSITION 8. *Let R be a stable ring with Jacobson radical J. If $J^{n-1} \neq J^n = J^{n+1}$ then, for each module M, $M \supset JM \supset J^2M \supset \cdots J^nM \supset M_1 \supset M_2 \cdots \supset M_n = 0$ where $M_i/M_{i+1} =$ socle M/M_{i+1}* unless $M = 0$.

Proof. This is merely a restatement of Nakayama's lemma. For stable rings "one can apply Nakayama's lemma n-times from the top or bottom", with the middle term having no maximal or minimal submodules.

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