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QUASI-SIMILARITY ORBIT OF A SUBCLASS OF COMPACT OPERATORS ON A HILBERT SPACE

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In this note we study certain properties of quasi-similarity orbit of a subclass of compact operators defined on a separable Hilbert space. This class and its quasi-similarity orbit were introduced and studied by Fialkow in *Pacific J. Math.* 70 (1977), 151-161.

Preliminaries and notations

We start with some notations and definitions. $B(H)$ will denote the Banach algebra of all bounded linear operators on a Hilbert space H which is taken to be separable. For $T \in B(H)$, $\sigma(T)$ is its spectrum, $\sigma_p(T)$ its point spectrum, $r(T)$ its spectral radius, $N(T)$ its null space, $R(T)$ its range and T^* is its adjoint. For a complex number λ, λ^* is its complex conjugate. The closed linear span of a family $\{M_i\}_{i=1}^{\infty}$ of subspaces of H will be denoted by $\bigvee_{i=1}^{\infty} M_i$.

A sequence $\{M_i\}_{i=1}^{\infty}$ of subspaces of H is said to be a basic sequence if M_i and $\bigvee_{k \neq i} M_k$ are complementary for each i.

A subspace M of H is called hyperinvariant under T if it is invariant under any operator which commutes with T.

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An invariant subspace M for T is called a spectral maximal subspace for T if the following condition is satisfied:

If N is another subspace invariant under T such that $\sigma(T/N) \subseteq \sigma(T/M)$, then $N \subseteq M$. Note that a spectral maximal subspace for T is hyperinvariant for T ([2], Proposition 3.2, p. 18).

For the properties of spectral and scalar type operators we refer to [4]. We note here that $T \in B(H)$ is spectral if and only if $T = S + Q$ where S is a scalar type operator and Q is a quasi-nilpotent operator which commutes with S ([4], Theorem 5, p. 1939). Also, any scalar type operator on a Hilbert space is of the form BNB^{-1} where N is a normal operator ([4], Theorem 4, p. 1947). Thus, any spectral operator on a Hilbert space is similar to an operator of the form normal plus a commuting quasi-nilpotent.

For the ideas of single valued extension property of operators, decomposable operators, quasi-nilpotent equivalence of operators and other related topics we refer to [2]. We note that compact and spectral operators are decomposable operators ([2], p. 33).

For an operator $T \in B(H)$ with single valued extension property, consider the subset $\zeta_T(x)$ of elements λ_0 of complex plane such that there exists an analytic function $\lambda \to x(\lambda)$ defined on a neighbourhood of λ_0 with values in H, which satisfies $(\lambda I-T)x(\lambda) = x$ for all λ in the neighbourhood.

Take $\sigma_T(x)$ to be the complement of $\zeta_T(x)$ in the complex plane and define

$$X_T(\delta) = \{x \in H \mid \sigma_T(x) \subseteq \delta\}$$

for any δ in complex plane.

For the properties of $X_T(\delta)$ and $\sigma_T(x)$ we again refer to [2].

If λ is an isolated point of $\sigma(T)$, then there is a projection (not necessarily self adjoint) associated with $\{\lambda\}$ ([2], p. 26). We denote this projection by $P_T(\{\lambda\})$. If H_1 and H_2 are two Hilbert spaces, then $A : H_1 \to H_2$ is called quasi-invertible if it is one to one and has dense range.

An operator $T_1 \in B(H_1)$ is called quasi-similar to $T_2 \in B(H_2)$ if there exist two quasi-invertible operators $A : H_1 \to H_2$, $B : H_2 \to H_1$ such that $T_2A = AT_1$ and $T_1B = BT_2$.

If K is the ideal of compact operators on H then $B(H)/K$ is called the Calkin algebra and for $T \in B(H)$, $\hat{T}$ denotes the canonical image of T in $B(H)/K$. Define

$$\sigma_3(T) = \sigma(\hat{T}) ,$$

$$\sigma_4(T) = \cap\{\sigma(T+K) \mid K \text{ is compact and commutes with } T\}$$

and

$$\sigma_l(T) = \sigma_4(T) \cup \{\lambda \mid \lambda \text{ is a limit point of } \sigma(T)\} .$$

Note that $\sigma_3(T) \subseteq \sigma_4(T) \subseteq \sigma_l(T)$ and $\sigma_3(T) = \sigma_4(T) = \sigma_l(T) = \{0\}$ for a compact operator.

We reproduce below the definition of the subclass of compact operators defined by Fialkow.

Let K be a compact operator in $B(H)$ and λ be a nonzero scalar in $\sigma(K)$. Define

$$R(K, \lambda) = \{x \in H \mid (K-\lambda)^n x = 0 \text{ for some } n \geq 1\} .$$

Let C be the set of all compact operators K in $B(H)$ which satisfy the following properties:

(i) $\bigvee_{i=1}^{\infty} R_i^K = H$ (where $\{\lambda_i\}_{i=1}^{\infty}$ is the sequence of distinct nonzero members of $\sigma(K)$ and $R_i^K = R(K, \lambda_i)$);

(ii) $\bigcap_{i=1}^{\infty} \left\{\bigvee_{k\geq i} R_k^K\right\} = \{0\}$.

A compact normal injective operator is in C and is C closed under similarity [5].

The class of operators quasi-similar to operators in C is called quasi-similarity orbit of C . We shall denote it by C_{qs}

In [5] the following characterization of class C_{qs} appears.

THEOREM A ([5], Theorem 5.1). *An operator* $T \in B(H)$ *is quasi-similar to an operator in* C *if and only if* T *satisfies the following properties:*

(i) *there exists a basic sequence* $\{M_i\}_{i=1}^{\infty}$ *of finite dimensional hyperinvariant subspaces of* T ;

(ii) $\sigma(T/M_i) = \{\lambda_i\}$, $\lambda_i \neq 0$ *and* $\lambda_i \to 0$;

(iii) $\bigcap_{i=1}^{\infty} \left(\bigvee_{k \geq i} M_k \right) = \{0\}$.

We first give an alternative proof of the necessary part of Theorem A. Then we prove that the family $\{M_i\}$ obtained in the above theorem is in fact unique. Also we give a simple characterization of spectral operators in C_{qs} . We also study as consequences some other properties of operators in the orbit.

Main results

We first observe that for a compact operator K , $R_i^K = N\left[(K-\lambda_i)^n\right]$ for some n . Also $R_i^K = R(P_K\{\lambda_i\})$ ([3], p. 579) which implies that $R_i^K = X_K(\{\lambda_i\})$ ([2], Proposition 3.10, p. 26 and Theorem 1.5, p. 31).

We start with a lemma which will be useful in our work.

LEMMA 1. *If* T *has single valued extension property and* $TA = AT_1$ *with* A *injective, then* T_1 *has single valued extension property and* $AX_{T_1}(\delta) \subseteq X_T(\delta)$ *for any subset* δ *of the complex plane.*

The proof of Lemma 1 is routine.

THEOREM 1. *If* $T \in C_{qs}$ *, then there exists a family* $\{M_i\}_i^{\infty}$ *of finite dimensional spectral maximal subspaces of* T *which form a basic* *sequence such that*

$$\sigma(T/M_i) = \{\lambda_i\} , \quad \lambda_i \neq 0 , \quad \lambda_i \to 0$$

and

$$\bigcap_{i=1}^{\infty} \left(\bigvee_{k \geq i} M_k \right) = \{0\} .$$

Proof. Suppose T is quasi-similar to $K \in C$ so that

$$TA = AK$$

and

$$BT = KB$$

for some quasi-invertible operators A and B .

Observe that $\sigma_p(T) = \sigma_p(K)$ and $\sigma_p(K)$ is countable, so that T has single valued extension property ([2], p. 22). Take $\lambda_i \neq 0$ in $\sigma(K)$. Then, by Lemma 1,

$$(1) \qquad \begin{cases} AX_K(\{\lambda_i\}) \subseteq X_T(\{\lambda_i\}) \\ \text{and} \\ BX_T(\{\lambda_i\}) \subseteq X_K(\{\lambda_i\}) . \end{cases}$$

Since $X_K(\{\lambda_i\}) = R(P_K(\{\lambda_i\}))$, which is finite dimensional and A and B are injective, we note that $X_T(\{\lambda_i\})$ is also finite dimensional and we actually have equality in the above two inclusions. Thus we have the family

$$\{X_T(\{\lambda_i\}) \mid 0 \neq \lambda_i \in \sigma_p(T)\}$$

of finite dimensional subspaces.

By applying Proposition 3.8 on p. 23 of [2], it can be seen that this is a family of spectral maximal subspaces of T (hence hyperinvariant subspaces of T). Further, this proposition states that

$$\sigma(T/X_T(\{\lambda_i\})) \subseteq \sigma(T) \cap \{\lambda_i\} = \{\lambda_i\} .$$

As $X_K(\{\lambda_i\}) \neq \{0\}$ (being the range of a nonzero projection),

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$X_T(\{\lambda_i\}) \neq \{0\}$ by (1). Hence

$$\sigma(T/X_T(\{\lambda_i\})) = \{\lambda_i\} .$$

To complete the proof, it remains to be shown that the family

$$\{X_T(\{\lambda_i\}) \mid 0 \neq \lambda_i \in \sigma_p(T)\}$$

is basic and

$$\bigcap_{i=1}^{\infty} \Big(\bigvee_{k \geq i} X_T(\{\lambda_k\}) \Big) = \{0\} .$$

Using the facts $X_T\{\lambda_i\} = AX_K(\{\lambda_i\})$, $\bigvee_{i=1}^{\infty} R_i^K = H$ and A has dense range, it can be easily checked that

$$X_T(\{\lambda_i\}) + \bigvee_{k \neq i} X_T(\{\lambda_K\}) = H$$

(note that this also gives us $\bigvee_{i=1}^{\infty} X_T(\{\lambda_i\}) = H$). If

$$x \in X_T(\{\lambda_i\}) \cap \Big(\bigvee_{k \neq i} X_T(\{\lambda_k\}) \Big) ,$$

then

$$Bx \in BX_T(\{\lambda_i\}) = X_k(\{\lambda_i\})$$

and

$$\begin{aligned} Bx \in B\Big(\bigvee_{k \neq i} X_T(\{\lambda_k\}) \Big) &\subseteq \bigvee_{k \neq i} BX_T(\{\lambda_k\}) \\ &= \bigvee_{k \neq i} X_K(\{\lambda_k\}) . \end{aligned}$$

Define

$$\delta = \{0\} \cup \Big\{ \bigcup_{k \neq i} \{\lambda_k\} \Big\} .$$

It is easily seen that δ is closed and $X_K(\{\lambda_k\}) \subseteq X_K(\delta)$ if $k \neq i$. Thus

$$\bigvee_{k \neq i} X_K(\{\lambda_k\}) \subseteq X_K(\delta)$$

so that

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$$\sigma_K(Bx) \subset \delta \cap (\{\lambda_i\}) = \emptyset$$

which implies that $x = 0$, B being injective. Thus the family $\{X_T(\{\lambda_i\}) \mid 0 \neq \lambda_i \in \sigma_p(T)\}$ is a basic family. Finally suppose

$$x \in \bigcap_{i=1}^{\infty} \left(\bigvee_{k \geq i} X_T(\{\lambda_k\})\right) .$$

Then

$$Bx \in \bigcap_{i=1}^{\infty} \left(\bigvee_{k \geq i} BX_T(\{\lambda_k\})\right)$$

$$= \bigcap_{i=1}^{\infty} \left(\bigvee_{k \geq i} X_K(\{\lambda_k\})\right) .$$

But $X_K(\{\lambda_k\}) = R_k^K$ and

$$\bigcap_{i=1}^{\infty} \left\{\bigvee_{k \geq i} R_k^K\right\} = \{0\} .$$

Hence $x = 0$. This completes the proof.

COROLLARY 1. *If T is a compact operator in C_{qs} then T is in C .*

Proof. Suppose T is quasi-similar to K in C . As in the proof of the theorem above,

$$\bigvee_{i=1}^{\infty} X_T(\{\lambda_i\}) = H$$

and

$$\bigcap_{i=1}^{\infty} \left(\bigvee_{k \geq i} X_T(\{\lambda_k\})\right) = 0 ,$$

where $\{\lambda_i\}_{i=1}^{\infty}$ is the set of nonzero points of $\sigma_p(T)$. This set is the same as the set of nonzero points of $\sigma(T)$, T being compact. Also $X_T(\{\lambda_k\}) = R_k^T$, T being compact. This implies T is in C .

COROLLARY 2. *Suppose T is in C_{qs} , T_1 has single valued*

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extension property and $X_T(\{\lambda\}) = X_{T_1}(\{\lambda\})$ *for all scalars* λ . *Then* T_1 *is in* C_{qs} .

Proof. As T is in C_{qs} , the family $\{X_T(\{\lambda_i\}) \mid 0 \neq \lambda_i \in \sigma_p(T)\}$ satisfies the conditions of Theorem 5.3 of [5] (by Theorem 1). Now consider the family $\{X_{T_1}(\{\lambda_i\}) \mid 0 \neq \lambda_i \in \sigma_p(T)\}$. This is a family of finite dimensional subspaces. As T_1 has single valued extension property, (by [2], Proposition 3.8, p. 23) all these subspaces are spectral maximal, hence hyperinvariant, for T and

$$\sigma\left(T_1/X_{T_1}(\{\lambda_i\})\right) = \{\lambda_i\}$$

(as in Theorem 1). As $X_{T_1}(\{\lambda_i\}) = X_T(\{\lambda_i\})$ for all λ_i , all the other requirements of Theorem A are satisfied. Thus T_1 is in C_{qs} .

COROLLARY 3. *Suppose* T *is in* C_{qs} *and* T *is quasi-nilpotent equivalent to* T . *Then* T *is also in* C_{qs} .

Proof. We have only to observe that T_1 has single valued extension property as T has and

$$X_T(\{\lambda\}) = X_{T_1}(\{\lambda\})$$

for all scalars λ ([2], Theorem 2.3, p. 14 and Theorem 2.4, p. 16).

Let us note that if T is in C_{qs} , then there exists a basic sequence $\{M_i\}_{i=1}^{\infty}$ of hyperinvariant finite dimensional subspaces such that $\sigma\left(T/M_i\right) = \{\lambda_i\}$. Hence $T/M_i = \lambda_i I + N_i$ where N_i is a nilpotent operator. Hence T/M_i is spectral. By applying Theorem 5.7 of [5], we get that T is quasi-similar to a spectral operator. Without any loss of generality we can take this spectral operator to be of the form normal plus a commuting quasi-nilpotent. This motivates the study of spectral operators in C_{qs} . First we start with a general result.

THEOREM 2. *Suppose* T *is a spectral operator quasi-similar to a* *compact operator* K *. Then the scalar part* S *of* T *is compact.*

Proof. As both T and K are decomposable, $\sigma(T) = \sigma(K)$ ([2], Theorem 4.4, p. 55). Hence $\sigma_l(T) = \sigma_l(K)$ ([1], Corollary 2). But $\sigma_l(K) = \{0\}$, K being compact, which implies that $\sigma_3(T) = \{0\}$. Let $T = S + Q$ be canonical decomposition of T . Then taking the canonical image in the Calkin algebra, $\hat{T} = \hat{S} + \hat{Q}$. As $\hat{S}$ commutes with $\hat{Q}$, $\sigma(\hat{T}) = \sigma(\hat{S})$. But $\sigma(\hat{T}) = \sigma_3(T) = \{0\}$. Hence $\sigma(\hat{S}) = \{0\}$. Let $S = BNB^{-1}$ where N is normal. Then $\hat{N}$ is a normal element of Calkin algebra with $\{0\}$ as spectrum. Since $r(\hat{N}) = |\hat{N}|$ ([6], Theorem 11.28 (b)), $\hat{N} = 0$. Thus N is compact, which implies that S is also compact.

We note that if S is the scalar part of a spectral operator T , then S and T are quasi-nilpotent equivalent to each other ([2], Corollary 2.4, p. 43). Hence we obtain the following

THEOREM 3. *A spectral operator* T *is in* C_{qs} *if and only if its scalar part* S *is in* C .

Proof. Sufficiency is obvious by Corollary 3. To prove necessity, note that S is in C_{qs} but S is compact by Theorem 2. Hence by using Corollary 1, we get that S is in C .

THEOREM 4. *Every operator in* C_{qs} *is injective.*

Proof. Suppose $T \in C_{qs}$. Let T_1 be the spectral operator which is quasi-similar to T . We take $T_1 = N_1 + Q$ where N_1 is normal. By Theorem 3, N_1 is in C . Now $N(N_1) \perp N(N_1-\lambda_i)$ for all nonzero λ_i in $\sigma(N_1)$. But as N_1 is normal operator

$$N(N_1-\lambda_i) = R_i^N .$$

Hence $N(N_1) \perp \bigvee_{i=1}^{\infty} R_i^N$. As $N_1 \in C$, $\bigvee_{i=1}^{\infty} R_i^N = H$. Thus it follows that $N(N_1) = \{0\}$. Hence N_1 is injective. Now T_1 is also injective (by [4], Corollary 4, p. 1956). Hence quasi-similarity between T and T_1

implies that T is also one to one.

Now we are in a position to give another simple characterization of spectral operators in C_{qs}, the proof of which we omit.

COROLLARY 4. *If* T *is a spectral operator, then* T *is in* C_{qs} *if and only if its scalar part* S *is compact, injective with countably infinite spectrum.*

Next we prove the uniqueness of the family $\{M_i\}_{i=1}^{\infty}$ of hyper-invariant subspaces, obtained in Theorem A.

THEOREM 5. *Suppose* $T \in C_{qs}$. *Then there exists one and only one family* $\{M_i\}_{i=1}^{\infty}$ *of finite dimensional hyperinvariant subspaces for* T *which form a basic sequence, such that*

$$\sigma(T/M_i) = \{\lambda_i\}, \quad \lambda_i \neq 0, \quad \lambda_i \to 0$$

and

$$\bigcap_{i=1}^{\infty} \Big(\bigvee_{k\geq i} M_k \Big) = \{0\} .$$

Proof. Suppose $T \in C_{qs}$. By Theorem 1, one such family, with the properties stated in the theorem is

$$\{X_T(\{\theta\}) \mid 0 \neq \theta \in \sigma_p(T)\} .$$

Let $\{M\}_{i=1}^{\infty}$ be any other family with these properties. Then we will show that this family coincides with the family

$$\{X_T(\{\theta\}) \mid 0 \neq \theta \in \sigma_p(T)\} .$$

If $x \in M_i$, then

$$\sigma_T(x) \subseteq \sigma_{T/M_i}(x) \subseteq \sigma(T/M_i) = \{\lambda_i\} .$$

Thus $M_i \subseteq X_T(\{\lambda_i\})$. Suppose K is an operator in C which is quasi-similar to T and let $KA = AT$ for some quasi-invertible operator A. Then, by Lemma 1,

$$AX_T(\{\lambda_i\}) \subseteq X_K\{\lambda_i\}$$

so that

$$AM_i \subseteq X_K(\{\lambda_i\}) \ .$$

As $M_i \neq \{0\}$ and A is one to one, $X_K(\{\lambda_i\}) \neq \{0\}$. Hence $\lambda_i \in \sigma(K)$. Being nonzero, λ_i is in $\sigma_p(K) = \sigma_p(T)$. Thus we have $M_i \subseteq X_T(\{\lambda_i\})$ for some $0 \neq \lambda_i \in \sigma_p(T)$. Also since $X_K(\{\lambda_i\})$ is finite dimensional, so is $X_T(\{\lambda_i\})$. As T is in C_{qs}, it is quasi-similar to a spectral operator $T_1 = N_1 + Q_1$ where N_1 is normal, compact and injective. Suppose $T_1B = BT$ and $CT_1 = TC$, for some quasi-invertible operators B and C. Then, as before,

$$BX_T(\{\lambda_i\}) = X_{T_1}(\{\lambda_i\}) \ ,$$

$$CX_{T_1}(\{\lambda_i\}) = X_T(\{\lambda_i\}) \ .$$

If M_i is properly contained in $X_T(\{\lambda_i\})$, then BM_i is properly contained in $BX_T(\{\lambda_i\}) = X_{T_1}(\{\lambda_i\})$. As N_1 is quasi-nilpotent equivalent to T_1, $X_{T_1}(\{\lambda_i\}) = X_{N_1}(\{\lambda_i\})$ ([2], Theorem 2.1, p. 40). But as N_1 is normal

$$X_{N_1}(\{\lambda_i\}) = N(N_1-\lambda_i) \ .$$

Thus $BM_i \subseteq N(N_1-\lambda_i)$ (containment is proper). Therefore there exists a nonzero $x \in N(N_1-\lambda_i)$ such that $x \perp BM_i$. Similarly $BM_j \subseteq N(N_1-\lambda_j)$ $(j = 1, 2, 3, \ldots)$. As $x \in N(N_1-\lambda_i)$ and $N(N_1-\lambda_i) \perp N(N_1-\lambda_j)$ for $j \neq i$ (N_1 being normal), $x \perp BM_j$ $(j = 1, 2, \ldots)$. Thus

$$x \perp \bigvee_{j=1}^{\infty} BM_j \ .$$

$\{M_j\}_{j=1}^{\infty}$ being a basic sequence,

$$\bigvee_{j=1}^{\infty} M_j = H .$$

This gives us that $x = 0$. This contradiction proves that $M_i = X_T(\{\lambda_i\})$, $0 \neq \lambda_i \in \sigma_p(T)$. Thus

$$\{M_i\}_{i=1}^{\infty} \subseteq \{X_T(\{\theta\}) \mid 0 \neq \theta \in \sigma_p(T)\} .$$

To complete the proof, we must show that the above two families are in fact the same. Again we show this by contradiction.

Suppose that $X_T(\{\theta\})$, for some $0 \neq \theta \in \sigma_p(T)$, is not a member of the family $\{M_i\}_{i=1}^{\infty}$. Since $X_T(\{\theta\}) \neq \{0\}$, we can choose a nonzero x in $X_T(\{\theta\})$. As $\{X_T(\{\theta\}) \mid 0 \neq \theta \in \sigma_p(T)\}$ forms a basic sequence,

$$x \notin \vee\{X_T(\{\theta_i\}) \mid 0 \neq \theta_i \in \sigma_p(T) \text{ and } \theta_i \neq \theta\} .$$

But the family $\{M_i\}_{i=1}^{\infty}$ is contained in the family

$$\{X_T(\{\theta_i\}) \mid 0 \neq \theta_i \in \sigma_p(T) \text{ and } \theta_i \neq \theta\}$$

and $\bigvee_{i=1}^{\infty} M_i = H$, $\{M_i\}_{i=1}^{\infty}$ being a basic sequence. Thus we arrive at the contradiction that $x \notin H$. This completes the proof.

We close this paper by making one more observation on the operators of class C_{qs} .

THEOREM 6. *If K is in C , then K^* is also in C .*

Proof. Suppose $KB = BT_1$ and $AK = T_1A$ for some quasi-invertible operators A and B where T_1 is a spectral operator whose scalar part N_1 is normal compact and injective (such T_1 exists as $K \in C$). If $0 \neq \lambda_i \in \sigma_p(K^*)$ then $R_i^{K^*} = X_{K^*}(\{\lambda_i\})$ and $K^*A^* = A^*T_1^*$, $B^*K^* = T_1^*B^*$ imply that $R_i^{K^*} = A^*X_{T_1^*}(\{\lambda_i\})$ (note that A, B are also quasi-invertible). Now

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$$\begin{aligned}\chi_{T_1^*}(\{\lambda_i\}) &= \chi_{N_1^*}(\{\lambda_i\}) \\ &= N(N_1^*-\lambda_i) \\ &= N(N_1-\lambda_i^*) \\ &= \chi_{N_1}(\{\lambda_i^*\}) \\ &= \chi_{T_1}(\{\lambda_i^*\}) .\end{aligned}$$

Thus $R_i^{K^*} = A^*\chi_{T_1}(\{\lambda_i^*\})$.

As $\{\chi_{T_1}(\{\lambda_i^*\}) \mid 0 \neq \lambda_i \in \sigma_p(K^*)\}$ forms a basic sequence, we can show that

$$\bigvee_{i=1}^{\infty} R_i^{K^*} = H .$$

If $x \in \bigvee_{k=i}^{\infty} R_i^{K^*}$ $(i = 1, 2, 3, \ldots)$, then

$$\begin{aligned}B^*x \in \bigvee_{k=i}^{\infty} B^*R_k^{K^*} &= \bigvee_{k=i}^{\infty} B^*A^*\chi_{T_1^*}(\{\lambda_k\}) \\ &\subseteq \bigvee_{k=i}^{\infty} \chi_{T_1^*}(\{\lambda_k\}) \quad (i = 1, 2, \ldots)\end{aligned}$$

as B^*A^* commutes with T_1^* and $\chi_{T_1^*}(\{\lambda_k\})$ is hyperinvariant subspace for T_1^* .

Now

$$\chi_{T_1^*}(\{\lambda_k\}) = \chi_{T_1}(\{\lambda_k^*\}) .$$

So

$$B^*x \in \bigcap_{i=1}^{\infty} \left(\bigvee_{k=i}^{\infty} \chi_{T_1}(\{\lambda_k^*\})\right) .$$

This last intersection is $\{0\}$ as in Theorem 1. Hence $x = 0$. Therefore $K^* \in C$.

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COROLLARY 5. *It* $T \in C_{qs}$ *, then* T^* *also belongs to* C_{qs} .

Proof. Since T is quasi-similar to $K \in C$, T^* is quasi-similar to K^* but by Theorem 6, $K^* \in C$. This completes the proof.

References

[1] Nilima N. Chourasia and Kirti K. Oberai, "On Weyl's theorem", submitted.

[2] Ion Colojoară and Ciprian Foiaş, *Theory of generalized spectral operators* (Gordon and Breach, New York, London, Paris, 1968).

[3] Nelson Dunford and Jacob T. Schwartz, *Linear operators*. Part I: *General theory* (Pure and Applied Mathematics, 7. Interscience, New York, London, 1958).

[4] Nelson Dunford and Jacob T. Schwartz, *Linear operators*. PartIII: *Spectral operators* (Pure and Applied Mathematics, 7. Wiley-Interscience [John Wiley & Sons], New York, London, Sydney, 1971).

[5] Laurence Fialkow, "A note on quasisimilarity II", *Pacific J. Math.* 70 (1977), 151-162.

[6] Walter Rudin, *Functional analysis* (Tata McGraw-Hill, New Delhi, 1974).

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