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CUMULANTS AND PARTITION LATTICES VI. VARIANCES AND COVARIANCES OF MEAN SQUARES

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Abstract

Formulae are given for the variances and covariances for mean squares in anova under the broadest possible assumptions. The results of other authors are obtained by specializing appropriately: these include ones concerning randomization and/or random sampling models, as well as additive (linear) models consisting of mutually independent sets of exchangeable effects. Although the illustrations given refer only to doubly and triply-indexed arrays, the approach is quite general. Particular attention is drawn to the generalized cumulants (and their natural unbiased estimators) which vanish when additive models are assumed.

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1. Introduction

In the earlier papers in this series, Speed (1986a,b), Speed and Silcock (1988), hereafter referred to as II, III and V, we have defined and studied certain expressions generalizing the classical cumulants and k-statistics. We refer particularly to the introduction of III for a general discussion of the background to this work. The aim of this paper is to present a number of calculations of variances and covariances of estimates of components of variance and other mean squares which demonstrate that the results of the earlier papers can be used to carry out these calculations quite straightforwardly and under the most general assumptions for which the various notions are defined. Our approach is a natural generalization of that introduced by Tukey (1950). We regard the methods illustrated and the results given as the most reasonable completion of the program undertaken by Dayhoff (1964) and Carney (1965) that seems possible with existing combinatorial theory. All our formulae are readily implemented on a computer and the only limitation to producing further results of the same kind would be the space it takes to write them down, for at the level of generality at which we are operating, they become lengthy very quickly. They do simplify, of course, for the special cases such as those arising from linear models with mutually independent sets of independent and identically distributed effects, and if it were desired expressions for variances and covariances could be computed for more linear models than those we consider below.

As we have already remarked, our formulae get complicated rather quickly, and so we would like to emphasize that our aim is not just to produce lengthy formulae, but to give insight into the assumptions underlying the standard variance component models and the usual analyses of standard designed experiments. In these contexts we demonstrate the role played by a number of generalized cumulants (and their associated generalized k-statistics) which measure various forms of non-additivity and inhomogeneity and which appear in expressions for the variances and covariances of mean squares under assumptions wider than the usual ones. The best unbiased estimates of these measures are all readily computed, and at a later date we hope to present a range of tests for additivity, homogeneity, etc. which will be able to supplement the usual analyses of variance.

Turning now to a description of the contents of this paper, we begin by pointing out that it relies almost entirely on results presented in earlier papers in this series. One exception to this concerns expressions for the products of generalized k-statistics, and we collect the results of this kind that we need in the Appendix to this paper.

There are two major classes of results given below. The first takes as its starting point the formulae for products of generalized k-statistics of order 2, for these are only a step from the (co)variances of the same k-statistics. Following this we use results from Sections 4, 5 and 6 of III to specialize the results to permutation distributions, on the one hand, and linear models consisting of sums of independent sets of exchangeable effects on the other. We draw particular attention to the generalized cumulants which vanish when additivity models are assumed. The second set of results consist of re-derivations of the randomization variances given by Pitman (1937) and Welch (1937), see also Ogawa (1974), for the treatment mean square in randomized block and Latin square designs. Not only are our derivations rather more compact than those in the references cited, using the machinery developed in this series of papers, but it is also of

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interest to see the same non-additivity and inhomogeneity measures appearing in this context as well. We close with the derivation of analogous results for the treatment mean-squares in the classical split-plot design.

2. (Co)variances of estimates of components of (co)variance

In the following three sections we will compute the variances and covariances of a selection of second order generalized k-statistics which, as symmetric unbiased estimators of generalized cumulants, are generalizations of the standard estimators of components of variance in linear models with random effects. The method we adopt is the natural extension of that introduced by Tukey (1950), who showed the power of polykays for deriving such results. We illustrated it briefly in II, Section 5, calculating the variance of the sample variance $s^2 = (n-1)^{-1}\sum_i (X_i - X.)^2$, and it is worthwhile giving a brief recapitulation of that discussion here.

As we have emphasized many times, one of the strengths of our approach is that we keep all random variables under discussion as distinct as possible, only identifying them at the last stage if that is desired. Accordingly, we do not calculate var(s^2) where $s^2 = [F_{12}|X \otimes X]$ (notation as in II, repeated in the Appendix to V), but rather $\mathrm{cov}(k_{12}, k_{34})$ where $k_{12} = [F_{12}|W \otimes X], k_{34} = [F_{34}|Y \otimes Z]$ and $((W_i, X_i, Y_i, Z_i), i \in \mathbf{n})$ is an exchangeable array of 4-vectors. At the end we can put $W_i = X_i = Y_i = Z_i$, $i \in \mathbf{n}$. Now we need $[F_{12}|W \otimes X][F_{34}|Y \otimes Z] = [F_{12} \otimes F_{34}|W \otimes X \otimes Y \otimes Z]$ to calculate $\mathsf{E}\{k_{12}k_{34}\}$ and so we are thrown back to Table I of II (repeated in the Appendix below) for an expansion of $F_{12} \otimes F_{34}$. Once we have that done the problem is essentially solved, for we find

$$(4.1) \qquad \mathrm{cov}(k_{12}, k_{34}) = \frac{1}{n} f_{1234} + \frac{1}{n-1}\{f_{13|24} + f_{14|23}\} + \{f_{1234} - f_{12|34}\}.$$

All that remains is to substitute expressions for the f's according to the underlying model being assumed; in this case we might suppose that the (W_i, X_i, Y_i, Z_i) are mutually independent and identically distributed, or that they arose by simple random sampling without replacement, or that they come from a specified urn model, etc.

The two steps in the line of reasoning just exhibited are: (a) the calculation of the tensor products of relevant generalized k-statistics $F_\sigma, \sigma \in \mathcal{P}(\mathbf{2})$; and (b) the evaluation of the relevant generalized cumulants f_τ, $\tau \in \mathcal{P}(\mathbf{4})$, under a range of assumptions of interest. We do this below for a selection of $\sigma \in \mathrm{Hom}(P, \mathcal{P}(\mathbf{2}))$ and the associated $\tau \in \mathrm{Hom}(P, \mathcal{P}(\mathbf{4}))$, for a number of small posets P. More fully, we consider all possibilities for doubly indexed arrays W_{ij}, X_{ij}, etc. where either i is crossed with j, or j is nested within i, and we also discuss the 'finest' component of variance (that is, the bottom line of the anova table) for some

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triply indexed arrays, namely, when i and j are crossed and k is nested within both i and j (two-way arrays with equal replication), when j and k are crossed and both nested within i (repeated two-way arrays), and where i nests j which in turn nests k (three factor hierarchical).

In the Appendix we have tabulated a range of tensor products of generalized k-statistics including all of those necessary for the illustrations we give. These were derived using the formula given in Proposition 3.1 of III and a computer to carry out the calculations. Thus step (a) mentioned above is completed. For step (b) we begin with the most general *exchangeability* model appropriate to the index set **I** under discussion, see III, Section 4 for the notion of $GW(\mathbf{I})$-invariance which defines this generalized exchangeability. The most important submodel here is what we term the *simple sampling model* (SSM) involving simple random sampling of all factor levels from finite populations, subject, of course, to any nesting constraints embodied in the poset P. As was explained in III, Section 4, this is essentially the framework of the Iowa school, Kempthorne (1952), Dayhoff (1964, 1966), Carney (1967), as well as that of Hooke (1956a,b) in the simple two-way (that is, crossed) set-up.

A much more dramatic specialization is to what we call the *general additive model* (GAM) described in III, Section 5, where each of the (mutually independent) sets of effects in the linear model is a fully exchangeable array in the classical sense. This specializes to the *additive sampling model* (ASM) where the effects in the additive model are all sampled without replacement from a finite population,Tukey (1956b, 1957), on the one hand, and to the *i.i.d. additive model* (IAM) on the other, where the effects are now assumed to be independent and identically distributed (i.i.d.).

As remarked in the Introduction above, our aim is not just to give (complex) formulae, but also to seek greater insight into the assumptions underlying the usual components of variance analyses, which invariably assume (IAM) with common normal distributions (=NAM). One way of gaining such insight is through noting the form and interpretation of the generalized cumulants which vanish as we specialize our models from the most general through (GAM) and (IAM) to (NAM). For example, it is well known that the kurtosis term f_{1234} in (4.1) generally exists under (IAM), and vanishes under (NAM). Similarly we will find and interpret various other generalized cumulants which are present in the most general exchangeability model, but which vanish under (GAM), and it will be natural to regard these as *measures of non-additivity.*

3. Two indices with nesting

Here we consider doubly-indexed arrays $X = (X_{ij}), Y, Z$ and W where the j is nested within i, and our interest is in the variances and covariances of the generalized k-statistics $k(12,12)$ and $k(12,1|2)$ where (when $W = X$):

$$k(12,12) = \frac{1}{m(n-1)} \sum_i \sum_j (X_{ij} - X_{i\cdot})^2,$$

$$k(12,1|2) = \frac{1}{n}\left[\frac{n}{m-1}\sum_i (X_{i\cdot} - X_{\cdot\cdot})^2 - \frac{1}{m(n-1)}\sum_i \sum_j (X_{ij} - X_{i\cdot})^2\right].$$

These expressions are the best quadratic estimators of the generalized cumulants $f(12,12)$ and $f(12,1|2)$ respectively, and it will be recalled from the Corollary to Proposition 5.1 of III and III, Section 6 that under (GAM) these generalized cumulants are the within and between class components of variance, respectively. We will continue a practice adopted in earlier papers in this series of switching between $k(\sigma), f(\sigma)$ and k_σ, f_σ as convenience dictates.

It is immediate from Appendix B(i) that $\mathrm{Cov}(k(12,12), k(34,34))$ is

$$\begin{aligned}
&[f_{12|34,12|34} - f_{12,12}f_{34,34}] + \frac{1}{mn} f_{1234,1234} + \frac{1}{m} f_{1234,12|34} \\
&+ \frac{1}{m(n-1)}[f_{1234,13|24} + f_{1234,14|23}] \\
&+ \frac{1}{m(n-1)}[f_{13|24,13|24} + f_{14|23,14|23}]
\end{aligned} \tag{3.1}$$

where the f's are the fourth-order generalized cumulants $f_\sigma = f_\sigma^{WXYZ}$ of the array $((W_{ij}X_{ij}, Y_{ij}, Z_{ij}) \colon (i,j) \in \mathbf{m/n})$ whose first four moments (at least) are invariant under arbitrary finite permutations of i and, independently within each of a finite number of i, of arbitrary finite permutations of j. If this invariance holds for the entire joint distribution, we call it Between/Within Exchangeability (BWE).

Now let us suppose that we have a *population array* $((\tilde{W}_{IJ}, \tilde{X}_{IJ}, \tilde{Y}_{IJ}, \tilde{Z}_{IJ}) \colon (I,J) \in \mathbf{M/N})$, $m \le M < \infty$, $n \le N < \infty$, and that we sample m labels $I(1), \ldots, I(m)$ from $\mathbf{M}$ and, independently within each of the m sampled I's, n labels from $\mathbf{N}$: $J(1,1), \ldots, J(1,n), J(2,1), \ldots, J(2,n), \ldots, J(m,1), \ldots, J(m,n)$. Here the $\tilde{W}_{IJ}, \tilde{X}_{IJ}$, etc. may simply be real numbers or they may constitute a (BWE) array of random variables, and all sampling is without replacement. We can define an array $((W_{ij}, X_{ij}, Y_{ij}, Z_{ij}) \colon (i,j) \in \mathbf{m/n})$ by putting $W_{ij} = \tilde{W}_{I(i)J(i,j)}$, and similarly for X_{ij}, Y_{ij} and Z_{ij}, and this *sample* array clearly has the (BWE) property. Note that if we have a population of numbers with the corresponding permutation distribution, the random sampling is unnecessary; we can simply select the *first* m labels $I = 1, \ldots, I = m$, and, within each of these, the *first* n labels.

It is easy to see that the generalized cumulants f_σ of the *sample* coincide with the *population* generalized k-statistics $\tilde{k}_\sigma$ in the case where we are dealing with a permutation distribution over an array of numbers, and in this case—the

(SRS) model—the covariance of the *sample* generalized k-statistics $k(12,12) = k_{12,12}(W,X)$ and $k(34,34) = k_{34,34}(Y,Z)$ is just

$$\begin{aligned}(3.2)\qquad &\left(\frac{1}{mn}-\frac{1}{MN}\right)\tilde{k}_{1234}+\left(\frac{1}{m}-\frac{1}{M}\right)\tilde{k}_{1234,12|34}\\ &+\left(\frac{1}{m(n-1)}-\frac{1}{M(N-1)}\right)[\tilde{k}_{1234,13|24}+\tilde{k}_{1234,14|23}]\\ &+\left(\frac{1}{m(n-1)}-\frac{1}{M(N-1)}\right)[\tilde{k}_{13|24,13|24}+\tilde{k}_{14|23,14|23}].\end{aligned}$$

Note that this results from expanding $f_{12,12}f_{34,34} = \tilde{k}_{12,12}\tilde{k}_{34,34}$ in the first line of (3.1) by just the same rule that we used to get (3.1) in the first place. Expansion (3.2) appears in a rather different notation in Dayhoff (1966), where it was derived by a quite different method.

How can we interpret (3.2)? Using the expansion for (1234, 1234) given in terms of the T-tensors in Section 6 of paper V in this series, and interpreting it, we find that the first term is a measure of within class kurtosis, for when $\tilde{W}_{IJ} = \tilde{X}_{IJ} = \tilde{Y}_{IJ} = \tilde{Z}_{IJ}$ we have

$$\tilde{k}_{1234} = \frac{n^2(n+1)}{m(n)_4}\sum_I\sum_J(\tilde{X}_{IJ}-\tilde{X}_{I-})^4 - \frac{3n(n-1)}{m(n)_4}\sum_I\left(\sum_J(\tilde{X}_{IJ}-\tilde{X}_{I-})^2\right)^2$$

where $\tilde{X}_{I-} = M^{-1}\sum_J \tilde{X}_{IJ}$. Similarly the terms on the second line of (3.2) are linear combinations of $\sum_I\sum_J(\tilde{X}_{IJ}-\tilde{X}_{I-})^4$, $\sum_I(\sum_J(\tilde{X}_{IJ}-\tilde{X}_{I-})^2)^2$ and $(\sum_I\sum_J(\tilde{X}_{IJ}-\tilde{X}_{I-})^2)^2$, and the interpretation of the results, for example of $\tilde{k}(12|34, 12|34)$, as an inhomogeneity measure is evident. See also Section 6 below.

We will now specialize to the (GAM). This means that we are supposing that each of W_{ij}, X_{ij}, Y_{ij} and Z_{ij} have a representation of the form

$$X_{ij} = \mu + \alpha_i + \varepsilon_{ij} \tag{3.3}$$

where the μ's are constant, the $(\alpha_i(W), \alpha_i(X), \alpha_i(Y), \alpha_i(Z))$ are (jointly) classically exchangeable, as are the $(\varepsilon_{ij}(W), \varepsilon_{ij}(X), \varepsilon_{ij}(Y), \varepsilon_{ij}(Z))$, and the α_i are independent of the ε_{ij}. How does (3.1) simplify in this case? By applying the results of III, Section 5, especially Proposition 5.1, we find that $f(1234, 12|34)$, $f(1234, 13|24)$ and $f(1234, 14|23)$ all vanish under (3.3) and so it is natural to describe these generalized cumulants as measures of non-additivity. Expression (3.1) becomes

$$[f^{\varepsilon}_{1234} - f^{\varepsilon}_{12}f^{\varepsilon}_{34}] + \frac{1}{mn}f^{\varepsilon}_{1234} + \frac{1}{m(n-1)}[f^{\varepsilon}_{13|24} + f^{\varepsilon}_{14|23}] \tag{3.4}$$

where the superscript ε signifies the fact that these f's refer to the classically exchangeable array of ε's.

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Next, we assume the model (SAM): that the ε's are sampled without replacement from a population of size P. In this case (3.4) specializes to a result first proved by Tukey (1956b, Section 4) using a different approach:

$$(3.5) \qquad \left(\frac{1}{mn} - \frac{1}{P}\right) \tilde{k}^{\varepsilon}_{1234} + \left(\frac{1}{m(n-1)} - \frac{1}{P-1}\right) [\tilde{k}^{\varepsilon}_{13|24} + \tilde{k}^{\varepsilon}_{14|23}].$$

Finally, we specialize (3.3) to the (IAM). Then (3.4) reduces to

$$(3.6) \qquad \frac{1}{mn}\kappa_{1234}(\varepsilon) + \frac{1}{m(n-1)}[\kappa_{13}(\varepsilon)\kappa_{24}(\varepsilon) + \kappa_{14}(\varepsilon)\kappa_{23}(\varepsilon)]$$

where the κ's are (joint) cumulants of the random vector $(\varepsilon_{ij}(W), \varepsilon_{ij}(X), \varepsilon_{ij}(Y), \varepsilon_{ij}(Z))$. In this form the result may be compared with the single index analogue due to Fisher (1929). Of course the first term vanishes under (joint) normality of the ε's.

We have considered the various simplifications in (3.1) in some detail to demonstrate what can be learned from the various formulae. In the remainder of this section we wil be more brief, indicating only the new points that crop up.

We turn now to the variance of the generalized k-statistic $k(12, 1|2)$ and the covariance of $k(12, 12)$ and $k(12, 1|2)$. From Appendix B(ii) we have the following expression for $\operatorname{cov}(k(12, 1|2), k(34, 3|4))$:

$$(3.7) \qquad \begin{aligned} &[f_{12|34,1|2|3|4} - f_{12,1|2}f_{34,3|4}] + \frac{1}{mn(n-1)}[f_{1234,13|24} + f_{1234,14|23}] \\ &+ \frac{1}{mn}[f_{1234,1|3|24} + f_{1234,2|4|13} + f_{1234,1|4|23} + f_{1234,2|3|14}] \\ &+ \frac{1}{m}f_{1234,1|2|3|4} + \frac{mn-1}{mn^2(m-1)(n-1)}[f_{13|24,13|24} + f_{14|23,14|23}] \\ &+ \frac{1}{(m-1)n}[f_{13|24,13|2|4} + f_{13|24,24|1|3} + f_{14|23,14|2|3} + f_{14|23,23|1|4}] \\ &+ \frac{1}{m-1}[f_{13|24,1|2|3|4} + f_{14|23,1|2|3|4}]. \end{aligned}$$

Each of these terms except the first bracketed set has the form $c_\sigma(m,n)f_\sigma$ for $\sigma \in \operatorname{Hom}(P, \mathcal{P}(4))$, where P is the 2-element chain, and the line of reasoning demonstrated above tells us that the corresponding expression under the (SRS) model is the sum over all the σ included in (3.7) of expressions having the form $[c_\sigma(m,n) - c_\sigma(M,N)]\tilde{k}_\sigma$, where $f_\sigma = \tilde{k}_\sigma$ is the corresponding population generalized cumulant, here coinciding with a generalized k-statistic.

Which terms in (3.7) vanish under the (GAM) (3.3) and what do they mean? As before we use Proposition 5.1 of III, and we find that (3.7) reduces to

$$
\begin{aligned}
&[f^{\alpha}_{12|34} - f^{\alpha}_{12} f^{\alpha}_{34}] + \frac{1}{m} f^{\alpha}_{1234} \\
(3.8) \qquad &+ \frac{mn-1}{mn^2(m-1)(n-1)} [f^{\varepsilon}_{13|24} + f^{\varepsilon}_{14|23}] + \frac{1}{m-1} [f^{\alpha}_{13|24} + f^{\alpha}_{14|23}] \\
&+ \frac{1}{(m-1)n} [f^{\alpha}_{13} f^{\varepsilon}_{24} + f^{\varepsilon}_{13} f^{\alpha}_{24} + f^{\alpha}_{14} f^{\varepsilon}_{23} + f^{\alpha}_{23} f^{\varepsilon}_{14}]
\end{aligned}
$$

where f^{α} and f^{ε} are the generalized k-statistics of the exchangeable arrays of α's and ε's. If we specialize to the model (SAM) assuming that the α's and ε's are sampled from finite populations of size Q and P respectively, then (3.8) changes in the following way (cf. (3.5) above): the first bracketed term disappears, all f's are replaced by $\tilde{k}$'s with the same sub- and superscripts and the multipliers m^{-1} and $(m-1)^{-1}$ become $[m^{-1} - Q^{-1}]$ and $[(m-1)^{-1} - (Q-1)^{-1}]$ respectively, these arising from the expansion of the product $\tilde{k}^{\alpha}_{12} \tilde{k}^{\alpha}_{34}$ in the first bracket. Again we agree with Tukey (1956b, Section 4).

Turning to the terms which vanish in the passage from (3.7) to (3.8), we note that apart from terms of the form $f(1234, 12|34)$ which have already been discussed, there are four terms of the form $f(1234, 12|3|4)$. These must also be non-additively measures, and using $V(6.3)$ we can examine the form of the corresponding k-statistic $k(1234, 12|3|4)$. When $W = X = Y = Z$ it is a linear combination of the following sums:

$$
\sum_i \sum_j (X_{ij} - X_{i\cdot})^4, \quad \sum_i \Big(\sum_j (X_{ij} - X_{i\cdot})^2\Big)^2, \quad \sum_i \sum_j (X_{ij} - X_{i\cdot})^3 (X_{i\cdot} - X_{\cdot\cdot}),
$$

$$
\sum_i \sum_j (X_{ij} - X_{i\cdot})^2 (X_{i\cdot} - X_{\cdot\cdot})^2, \quad \left(\sum_i \sum_j (X_{ij} - X_{i\cdot})^2 \right)^2
$$

and

$$
\sum_h \sum_i \sum_j (X_{hj} - X_{h\cdot})^2 (X_{i\cdot} - X_{\cdot\cdot})^2.
$$

This indeed seems to be (an estimate of) a measure of non-additivity, although a fairly complicated one. In a later paper we will be examining such measures in more detail.

Under the (IAM) (3.8) simplifies by simply replacing the fourth-order terms of the form $f_{13|24}$ by products $f_{13} f_{24}$ of the corresponding second-order cumulants, that is, (co)variances.

Our final results in this section concern the covariance between $k(12, 12)$ and $k(34, 3|4)$. Once more referring to the Appendix, this time B(iii), we find that

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under the most general assumptions this covariance is

$$
\begin{aligned}
&[f_{12|34,12|3|4} - f_{12,12}f_{34,3|4}] + \frac{1}{mn}[f_{1234,123|4} + f_{1234,124|3}] \\
(3.9) \qquad &- \frac{1}{mn(n-1)}[f_{1234,13|24} + f_{1234,14|23} + f_{13|24,13|24} + f_{14|23,14|23}] \\
&+ \frac{1}{m} f_{1234,12|3|4}.
\end{aligned}
$$

As before, if we specialize to the (SRS) model, the first bracketed terms disappear, f's become $\tilde{k}$'s, and coefficients $c(m,n)$ of f's become coefficients $c(m,n) - c(M,N)$ of $\tilde{k}$'s. Again, if we drop down to the (GAM), we find that $f(1234, 123|4)$ and $f(1234, 124|3)$ vanish, as well as $f(1234, 12|3|4)$, $f(1234, 13|24)$ and $f(1234, 14|23)$ which we have already met. Furthermore, $f(12|34, 12|3|4)$ factorizes into $f^{\varepsilon}(12)f^{\alpha}(34)$ and cancels with the other product in the first bracket. Expression (3.9) then becomes

$$
(3.10) \qquad -\frac{1}{mn(n-1)}[f^{\varepsilon}_{13|24} + f^{\varepsilon}_{14|23}].
$$

Noting that this agrees with Tukey (1956b) when we assume a (SAM), and that there is no *finite population correction* in this case, we close with a brief mention of the measure $f(1234, 123|4)$ of non-additivity. The leading term in its corresponding k-statistic $k(1234, 123|4)$ is (when $W = X = Y = Z$) a multiple of $t(1234, 123|4) = m^{-1}n^{-1}\sum_i \sum_j (X_{ij} - X_{i\cdot})^3(X_{i\cdot} - X_{\cdot\cdot})$.

4. Two indices with crossing

Now that we have given such a thorough discussion of the reuslts for two indices with nesting, the form of the results and the methods of derivation are now quite clear. Our task in this section will be to comment upon the form of the variances and covariances of

$$
k(12,12) = \frac{1}{(m-1)(n-1)} \sum_i \sum_j (X_{ij} - X_{i\cdot} - X_{\cdot j} + X_{\cdot\cdot})^2,
$$

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$$k(12,1|2) = \frac{1}{n}\left[\frac{n}{m-1}\sum_i (X_{i\cdot} - X_{\cdot\cdot})^2 - \frac{1}{(m-1)(n-1)}\sum_i\sum_j (X_{ij} - X_{i\cdot} - X_{\cdot j} + X_{\cdot\cdot})^2\right],$$

$$k(1|2,12) = \frac{1}{m}\left[\frac{m}{n-1}\sum_j (X_{\cdot j} - X_{\cdot\cdot})^2 - \frac{1}{(m-1)(n-1)}\sum_i\sum_j (X_{ij} - X_{i\cdot} - X_{\cdot j} + X_{\cdot\cdot})^2\right].$$

Here $X = (X_{ij})$ is a two-way array of random variables whose moments of order up to two are invariant under independent finitary row and column permutations of the subscripts. For the derivation of our results, however, we will be using a similarly indexed array of vectors $((W_{ij}, X_{ij}, Y_{ij}, Z_{ij}) : (i,j) \in \mathbf{m}\times\mathbf{n})$ and assuming the invariance property for joint moments up to order four. If the same invariance holds for the whole joint distribution, the array is termed row and column exchangeable (RCE), see Aldous (1981).

The crossing rule of III, Proposition 3.2 gives us an easy method of expanding products of generalized k-statistics using single index results of this type, and for this reason we have not tabulated expansions of products in this case. Using this method we find that under the most general invariance assumptions $\operatorname{cov}(k(12,12), k(34,34))$ is the sum of the following 17 terms.

$$\begin{aligned}
&[f_{12|34,12|34} - f_{12,12}f_{34,34}] + \frac{1}{mn}f_{1234,1234} + \frac{1}{m}f_{1234,12|34} + \frac{1}{n}f_{12|34,1234}\\
&+ \frac{1}{m(n-1)}[f_{1234,13|24} + f_{1234,14|23}] + \frac{1}{n(m-1)}[f_{13|24,1234} + f_{14|23,1234}]\\
&+ \frac{1}{n-1}[f_{12|34,13|24} + f_{12|34,14|23}] + \frac{1}{m-1}[f_{13|24,12|34} + f_{14|23,12|34}]\\
&+ \frac{1}{(m-1)(n-1)}[f_{13|24,13|24} + f_{13|24,14|23} + f_{14|23,14|23} + f_{14|23,13|24}].
\end{aligned} \tag{4.1}$$

By now we should find it possible to interpret the various bracketed terms in (4.1) as ones which remain right down to the (IAM) and are in essence a kurtosis or product of variances within rows and columns, as ones which measure non-additivity, and the first which vanishes under all assumptions less general than the invariance model.

For the corresponding sampling model (SRS), termed bisampling by Tukey, suppose we are given an array $((\tilde{W}_{IJ}, \tilde{X}_{IJ}, \tilde{Y}_{IJ}, \tilde{Z}_{IJ}) : (I,J) \in \mathbf{M}\times\mathbf{N})$ of numbers, where $m \le M < \infty, n \le N < \infty$, and independently sample m row labels

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$I(1), \ldots, I(m)$, and n column labels $J(1), \ldots, J(n)$, thereby defining a jointly sampled array (W, X, Y, Z) by putting $W_{ij} = \tilde{W}_{I(i)J(j)}$, and similarly for X, Y and Z. The generalized cumulants f_σ of this bisampled array then coincide with the corresponding generalized k-statistics $\tilde{k}_\sigma$ of the population array and matters proceed as in Section 3 above. In particular, we can readily recover the results of Hooke (1956b, Section 7). Since this involves no more than replacing terms of the form $c_\sigma(m, n) f_\sigma$ by $[c_\sigma(m, n) - c_\sigma(M, N)]\tilde{k}_\sigma$ in (4.1), and dropping the first bracketed term, we do not give further details.

Perhaps the main point of interest in this section is the wide variety of measures of non-additivity which appear. As in Section 3 above, we suppose that a generalized cumulant f_σ measures non-additivity if it vanishes from (4.1) when the (GAM) is assumed. By this definition, $f(1234, 12|34)$ and $f(12|34, 13|24)$ (and f's of similar form) are such measures, and as in Section 3 above, we examine the leading terms $t(1234, 12|34)$ and $t(12|34, 13|24)$ cf. V, (6.2), which go into their unbiased estimators $k(1234, 12|34)$ and $k(12|34, 13|24)$ respectively. Using the definitions given in V we find that when $W = X = Y = Z$, and putting $\Delta_{ij} = X_{ij} - X_{i\cdot} - X_{\cdot j} + X_{\cdot\cdot}$, we have $t(1234, 12|34) = m^{-1}n^{-2}\sum_i(\sum_j \Delta_{ij}^2)^2$ and $t(12|34, 13|24) = m^{-2}n^{-2}\sum_i\sum_{i'}(\sum_j \Delta_{ij}\Delta_{i'j})^2$. These are the first of eight kinds of measures of non-additivity, and are not especially easy to interpret. Others to be given below are more easy to interpret and further discussion of such measures will be deferred to the later paper.

We note in passing that a measure quite similar to Tukey's well known *degree of freedom for non-additivity* is a third-order expression of this kind, namely $t(12|3, 13|2) = m^{-1}n^{-1}\sum_i\sum_j(X_{i\cdot} - X_{\cdot\cdot})(X_{\cdot j} - X_{\cdot\cdot})(X_{ij} - X_{i\cdot} - X_{\cdot j} + X_{\cdot\cdot})$. As a component in a third-order generalized k-statistic, $t(12|3, 13|2)$ plays no role in our discussion of (co)variances of (co)variance components, but its similarity to expressions appearing here helps us to put such matters in a broader context. As already stated, we leave a study of these measures to another time.

The general additive model (GAM) for two-way arrays takes the form

$$X_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij} \tag{4.2}$$

with corresponding representations for W_{ij}, Y_{ij} and Z_{ij}, where the $(\alpha_i), (\beta_j)$ and (ε_{ij}) are mutually independent and classically exchangeable sets of 4-vectors $(\alpha_i) = (\alpha_i(W), \alpha_i(X), \alpha_i(Y), \alpha_i(Z))$, etc. Note that (4.2) is essentially the model of equation (10) of Hooke (1956b) with no interactions; the model in his equation (25) with "tied" interactions is best covered (as it is by Hooke) by our earlier discussion of bisampling. Under (GAM) the expression (4.1) for $\mathrm{cov}(k(12, 12), k(34, 34))$ simplifies dramatically to:

$$[f^\varepsilon_{12|34} - f^\varepsilon_{12}f^\varepsilon_{34}] + \frac{1}{mn}f^\varepsilon_{1234} + \frac{1}{(m-1)(n-1)}[f^\varepsilon_{13|24} + f^\varepsilon_{14|23}]. \tag{4.3}$$

We omit any further simplification as the results so obtained closely parallel those in Section 3 above and coincide (although they are slightly more general) with those given by Hooke (1956b) and Tukey (1956b), see also Arveson (1976).

Turning now to the (co)variance $\text{cov}(k(12,1|2), k(34,3|4))$ of the 'between rows' component of (co)variances, we observe from A(ii) and A(iii) of the Appendix that the product $(12,1|2) \otimes (34,3|4)$ has 28 terms. Their calculation (by the crossing rule) is completely straightforward and after subtracting off $f(12,1|2)f(34,3|4)$ we obtain a 29 term expression for the covariance in question which we will not write out in full. There is an initial term $[f(12|34,1|2|3|4) - f(12,1|2)f(34,3|4)]$; a 'between row' kurtosis term $m^{-1}f(1234,1|2|3|4)$, which reduces to $m^{-1}f^{\alpha}(1234)$ under the (GAM) (4.2); terms of the form $(r-1)^{-1}f(12|34,1|2|3|4)$, which reduces to $(r-1)^{-1}f^{\alpha}(12|34)$ for 'between rows' under (4.2); terms of the form $[n(n-1)(m-1)]^{-1}f(13|24,13|24)$, which reduces to $[n(n-1)(m-1)]^{-1}f^{\varepsilon}(13|24)$ under (4.2); and the 'mixed' terms like $[n(m-1)]^{-1}$ $f(13|24,13|2|4)$, which reduces to $[n(m-1)]^{-1}f^{\varepsilon}(13)f^{\alpha}(24)$ under (4.2); finally, there are four types of non-additivity measures, including two not yet met which are typified by $f(1234,13|2|4)$ and $f(12|34,13|2|4)$. The leading terms in the t-expressions of the corresponding k-statistics $k(1234,13|2|4)$ and $k(12|34,13|2|4)$ are multiples of

$$\sum_i \sum_j (X_{ij} - X_{i\cdot} - X_{\cdot j} + X_{\cdot\cdot})^2 (X_{i\cdot} - X_{\cdot\cdot})^2$$

and

$$\sum_j \left[\sum_i (X_{ij} - X_{i\cdot} - X_{\cdot j} + X_{\cdot\cdot})(X_{i\cdot} - X_{\cdot\cdot}) \right]^2$$

respectively. This brings to four the number of types of measures of non-additivity met so far and two further types appear below.

The covariances $\text{cov}(k(12,12), k(34,3|4))$ between the generalizations of the 'error' and 'between row' components of (co)variance has 21 terms which are easily calculated and interpreted, and they include amongst six different types of measures of non-additivity two new ones typified by $f(12|34,123|4)$ and $f(1234,123|4)$. The leading terms in the generalized k-statistics estimating these generalized cumulants are multiples of

$$\sum_i \sum_{i'} \sum_j (X_{ij} - X_{i\cdot} - X_{\cdot j} + X_{\cdot\cdot})^2 (X_{i'j} - X_{i'\cdot} - X_{\cdot j} + X_{\cdot\cdot})(X_{i'\cdot} - X_{\cdot\cdot})$$

and

$$\sum_i \sum_j (X_{ij} - X_{i\cdot} - X_{\cdot j} + X_{\cdot\cdot})^3 (X_{i\cdot} - X_{\cdot\cdot})$$

respectively.

Finally, we mention only the analogues of the leading terms in the estimators of the new types of measures of non-additivity which appear in the covariance $\mathrm{cov}(k(12,1|2), k(3|4,34))$ between the 'row' and 'column' k-statistics $k(12,1|2)$ and $k(3|4,34)$. They are (multiples of) $t(12|3|4,1|234)$ and $t(123|4,1|234)$ which are

$$m^{-1}n^{-1}\sum_i\sum_j(X_{i\cdot}-X_{\cdot\cdot})(X_{ij}-X_{i\cdot}-X_{\cdot j}+X_{\cdot\cdot})(X_{\cdot j}-X_{\cdot\cdot})^2$$

and

$$m^{-1}n^{-1}\sum_i\sum_j(X_{i\cdot}-X_{\cdot\cdot})(X_{ij}-X_{i\cdot}-X_{\cdot j}+X_{\cdot\cdot})^2(X_{\cdot j}-X_{\cdot\cdot}).$$

These two expressions bear a striking resemblance to Tukey's measure of non-additivity; they would be the un-normalized form if the 2 in the exponent were omitted. It is perhaps not surprising that such measures appear in the covariance between estimators of rows and column 'components of variance'.

5. Three indices: some examples

We now give brief discussions of the (co)variance associated with the 'bottom line' mean square of the anova table of some triply-indexed arrays. Following Nelder (1965) they can be written $((W_{ijk}, X_{ijk}, Y_{ijk}, Z_{ijk})\colon (i,j,k)\in \mathbf{I})$ where $\mathbf{I}$ is $\mathbf{m/n/p}$, $(\mathbf{m}\times\mathbf{n})/\mathbf{p}$ and $\mathbf{m}/(\mathbf{n}\times\mathbf{p})$.

(a) $\mathbf{m/n/p}$.

From Appendix D we find that the (co)variance $\mathrm{cov}(k(12,12,12), k(34,34,34))$ of

$$k(12,12,12)=\frac{1}{mn(p-1)}\sum_i\sum_j\sum_k(W_{ijk}-W_{ij\cdot})(X_{ijk}-X_{ij\cdot})$$

and

$$k(34,34,34)=\frac{1}{mn(p-1)}\sum_i\sum_j\sum_k(Y_{ijk}-Y_{ij\cdot})(Z_{ijk}-Z_{ij\cdot})$$

is given by

(5.1)

$$\begin{aligned}&[f(12|34,12|34,12|34)-f(12,12,12)f(34,34,34)]+\frac{1}{mnp}f(1234,1234,1234)\\ &\quad+\frac{1}{mn}f(1234,1234,12|34)+\frac{1}{m}f(1234,12|34,12|34)\\ &\quad+\frac{1}{mn(p-1)}[f(1234,1234,13|24)+f(1234,13|24,13|24)\\ &\qquad\qquad+f(13|24,13|24,13|24)+f(1234,1234,14|23)\\ &\qquad\qquad+f(1234,14|23,14|23)+f(14|23,14|23,14|23)].\end{aligned}$$

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Once more we concentrate upon the non-additivity measures as the general approach to interpreting such formulae is by now evident. These measures—which generalize $f(1234, 12|34)$ found in the case of $\mathbf{m}/\mathbf{n}$—are $f(1234, 1234, 12|34)$ and $f(1234, 12|34, 12|34)$. The leading terms in their symmetric unbiased estimates are (when $W = X = Y = Z$):

$$t(1234, 1234, 12|34) = (mn)^{-1}p^{-2}\sum_i\sum_j\left[\sum_k(X_{ijk} - X_{ij\cdot})^2\right]^2$$

and

$$t(1234, 12|34, 12|34) = m^{-1}(np)^{-2}\sum_i\left[\sum_j\sum_k(X_{ijk} - X_{ij\cdot})^2\right]^2 .$$

In a sense these cumulants (and $f(1234, 12|34)$ in Section 3 above) might be regarded as measuring the *inhomogeneity* (at different levels) of kurtosis which results when an additive model is not assumed.

(b) $(\mathbf{m} \times \mathbf{n})/\mathbf{p}$.

In this model our 'bottom line' mean square takes the same form as in 5(a) above, and here the (co)variance $\operatorname{cov}(k(12, 12, 12), k(34, 34, 34))$ is

$$\begin{aligned}
&[f(12|34, 12|34, 12|34) - f(12, 12, 12)f(34, 34, 34)] + \frac{1}{mnp}f(1234, 1234, 1234)\\
&+ \frac{1}{mn}f(1234, 1234, 12|34) + \frac{1}{mn(p-1)}[f(1234, 1234, 13|24)\\
&\qquad\qquad + f(1234, 1234, 14|23)]\\
&+ \frac{1}{m}f(1234, 12|34, 12|34) + \frac{1}{n}f(12|34, 1234, 12|34)\\
&+ \frac{1}{mn(p-1)}[f(1234, 13|24, 13|24) + f(13|24, 1234, 13|24)\\
&\qquad + f(13|24, 13|24, 13|24) + f(1234, 14|23, 14|23)\\
&\qquad + f(14|23, 1234, 14|23) + f(14|23, 14|23, 14|23)].
\end{aligned}$$

As with (a) the non-linearity cumulants are natural extensions of ones already met above, but this time we have measures which combine features of those in both Section 3 and Section 4. Writing out just the leading terms in the corresponding k-statistics we find

$$t(1234, 1234, 13|24) = m^{-1}n^{-1}p^{-2}\sum_i\sum_j\left[\sum_k(X_{ijk} - X_{ij\cdot})^2\right]^2 ,$$

$$t(1234, 12|34, 12|34) = m^{-1}n^{-2}p^{-2}\sum_i\left[\sum_j\sum_k(X_{ijk} - X_{ij\cdot})^2\right]^2 .$$

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(c) $\mathbf{m}/(\mathbf{n}\times\mathbf{p})$.

For this, our final example, the 'bottom line' mean square is

$$k(12,12,12) = \frac{1}{m(n-1)(p-1)}\sum_i\sum_j\sum_k(W_{ijk} - W_{ij\cdot} - W_{i\cdot k} + W_{i\cdot\cdot}) \times (X_{ijk} - X_{ij\cdot} - X_{i\cdot k} + X_{i\cdot\cdot})$$

and the covariance $\operatorname{cov}(k(12,12,12), k(34,34,34))$ is given by an expression with 15 terms easily obtained from Appendix F below. We will not write it out, but simply list the types of non-additivity cumulants involved. These are typified by $f(1234, 1234, 12|34)$, $f(1234, 12|34, 12|34)$ and $f(1234, 12|34, 13|24)$, and the leading terms in the t-expansions of their corresponding k-statistics are (when $W = X = Y = Z$) multiples of

$$t(1234, 1234, 12|34) = m^{-1}n^{-1}p^{-2}\sum_i\sum_j\left[\sum_k(X_{ijk} - X_{ij\cdot} - X_{i\cdot k} + X_{i\cdot\cdot})^2\right]^2,$$

$$t(1234, 12|34, 12|34) = m^{-1}n^{-2}p^{-2}\sum_i\left[\sum_j\sum_k(X_{ijk} - x_{ij\cdot} - X_{i\cdot k} + X_{i\cdot\cdot})^2\right]^2,$$

and

$$t(1234, 12|34, 13|24) = m^{-1}n^{-1}p^{-2} \times\sum_i\sum_j\sum_{j'}\left[\sum_k(X_{ijk} - X_{ij\cdot} - X_{i\cdot k} + X_{i\cdot\cdot})(X_{ij'k} - X_{ij'\cdot} - X_{i\cdot k} + X_{i\cdot\cdot})\right]^2.$$

These non-additivity measures are readily seen to be combinations of ones for the simple nested and crossed arrays discussed above.

6. The randomization variances of some treatment mean squares

Our second application of the techniques and results developed in earlier paper is to the calculation of means and variances of treatment mean-squares under randomization in some standard designed experiments. The first work in this area was by Welch (1937) who considered the randomized complete block design (RCBD) and Latin square designs (LSDs), and Pitman (1935), who independently derived not just the first two but the first four moments of the permutation distribution of the treatment mean square in a RCBD. In a number of papers Ogawa has extended this work to partially balanced incomplete block designs, see Ogawa (1974) and references therein; indeed he goes further in computing an approximate permutation beta-distribution for the non-centrality parameter of the usual F-test for significance of treatments, integrating out this

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parameter, and obtaining the usual *central* F-test. Other discussions of the topic with RCBDs appear in Kempthorne (1952) and Wilk (1955).

In this section we will show how the notation, terminology and viewpoint adopted in these papers, together with some of the formulae derived in V, lead quickly to the known results for RCBDs and LSDs, and we show how the classical split-plot design (SPD) can be analysed in the same way.

Our problem is to compute the permutation means and (co)variances of certain quadratic forms Q_1, Q_2 etc. and in line with the approach adopted throughout this series, we take these forms to be quadratic functions of the elements of arrays W and X, and Y and Z, respectively, leaving any identifications until the end of the calculation. If s and t are arrays of coefficients of the quadratic forms, then we can put $Q_1 = [s|W \otimes X]$ and $Q_2 = [t|Y \otimes Z]$ and use the formulae $\mathsf{E}Q_1 = \mathsf{E}[s|W \otimes X] = [s|\mathsf{E}(W \otimes X)]$. Now $\mathsf{E}(W \otimes X)$ expands (III. Proposition 4.1) to give

$$\mathsf{E}Q_1 = \sum_{\pi} f_{\pi}^{WX}[s|R_{\pi}] \tag{6.1}$$

where $\{f_{\pi}^{WX} : \pi \in \mathrm{Hom}(P, \mathcal{P}(2))\}$ are the second-order generalized cumulants of the array $W \otimes X$. Similarly

$$\mathsf{E}Q_2 = \sum_{\pi} f_{\pi}^{YZ}[t|R_{\pi}] \tag{6.2}$$

where $\{f_{\pi}^{YZ}\}$ are the second order generalized cumulants of $Y \otimes Z$, and

$$\mathsf{E}(Q_1 Q_2) = \sum_{\rho} f_{\rho}^{WXYZ}[s \otimes t|R_{\rho}] \tag{6.3}$$

where $\{f_{\rho}^{WXYZ} : \rho \in \mathrm{Hom}(P, \mathcal{P}(4))\}$ are the fourth-order generalized cumulants of $W \otimes X \otimes Y \otimes Z$.

We now see that our problem reduces to the computation of the expressions $[s|R_{\pi}], [t|R_{\pi}]$ and $[s \otimes t|R_{\rho}]$ for suitable π and ρ, as the generalized cumulants $\{f_{\pi}^{WX}\}, \{f_{\pi}^{YZ}\}$ and $\{f_{\rho}^{WXYZ}\}$ of the permutation distributions for the random arrays W, X, Y and Z are simply the corresponding generalized k-statistics $\{k_{\pi}(\omega, \xi)\}, \{k_{\pi}(\eta, \varsigma)\}$ and $\{k_{\rho}(\omega, \xi, \eta, \varsigma)\}$ for the "populations" $\omega, \xi, \eta, \varsigma$ of numbers being permuted. It will become clear that when s and t correspond to the treatment sum squares in the RCBD, LSD and SPD, most of the inner products $[s|R_{\pi}], [t|R_{\pi}]$ and $[s \otimes t|R_{\rho}]$ are zero, and the remainder easily calculated, and the expressions for the k-statistics corresponding to the nonzero inner products obtained from Section 6 of V lead us quickly to the answers sought. From now on it is quite satisfactory to put $W = X = Y = Z$ if that is desired.

(a) *Randomized complete block designs.* If we have m blocks each of n plots, then any array $(\xi_{ij} : (i,j) \in \mathbf{m}/\mathbf{n})$ of mn numbers, thought of as plot yields, will suffice to define a random array (X_{ij}) with a permutation distribution. We simply write

$$\mathrm{pr}((X_{ij}) = (x_{ij})) = \frac{1}{m!(n!)^m}$$

if (x_{ij}) is an admissible permutation of the array (ξ_{ij}), and zero otherwise, where admissible permutations are those obtained by permuting block labels i and, independently within each block, permuting plot labels j. Similarly we could define random arrays W, Y and Z in terms of arrays ω, η and ς of numbers, but this will not be necessary.

The treatment sum of squares in a randomized complete block experiment is $[s|X \otimes X]$ where $s_{ij,i'j'} = (n-1)/mn$ if plot(i,j) and plot(i',j') have the same treatment, and $s_{ij,i'j'} = -1/mn$ otherwise. The following calculations are easy to perform: $[s|R(12,12)] = \sum_i \sum_j s_{ij,ij} = n-1, [s|R(12,1|2)] = \sum_i \sum_j \sum_{j'} s_{ij,ij'} = mn[(n-1)/mn + (-1/mn)(n-1)] = 0$ and similarly $[s|R(1|2,12)] = 0$. Thus we can calculate $\mathsf{E}[s|X \otimes X] = (n-1)f(12,12)$ which is just $m^{-1}\sum_i \sum_j (\xi_{ij} - \xi_{i\cdot})^2$.

Turning now to var$[s|X \otimes X]$ we readily calculate that $[s \otimes s|R(1234,1234)] = (n-1)^2/mn$, $[s \otimes s|R(\rho(1),\rho(2)] = 0$ if $\rho(2)$ has any singleton blocks, $[s \otimes s|R(1234,12|34)] = (n-1)^2/m, [s \otimes s|R(1234,13|24)] = (n-1)/m = [s\otimes s|R(1234,14|23], [s\otimes s|R(12|34,12|34] = (n-1)^2, [s\otimes s|R(13|24,13|24] = n-1 = [s \otimes s|R(14|23,14|23]$; all other such terms are zero because they involve singletons in blocks of $\rho(2)$. Thus

$$\begin{aligned} mn\mathsf{E}[s|X \otimes X] = {} & (n-1)^2 f(1234,1234) + n(n-1)^2 f(1234,12|34) \\ & + (n)_2[f(1234,13|24) + f(1234,14|23)] \\ & + mn(n-1)^2 f(12|34,12|34) \\ & + m(n)_2[f(13|24,13|24) + f(14|23,14|23)]. \end{aligned} \tag{6.4}$$

Subtracting off $\{\mathsf{E}[s|X]\}^2$, and using the formulae expressing $f_\sigma = \tilde{k}_\sigma$ in terms of t_σ's given in V6.3, we find after a minor simplification that

(6.5)

$$\begin{aligned} \mathrm{var}[s|X \otimes X] &= \frac{2}{m^2(n-1)} \left\{ \left[\sum_i \sum_j (\xi_{ij} - \xi_{i\cdot})^2 \right]^2 - \sum_i \left[\sum_j (\xi_{ij} - \xi_{i\cdot})^2 \right]^2 \right\} \\ &= \frac{2(m-1)(n-1)}{m} f_{12|34,12|34} \end{aligned}$$

thus reinforcing our view in Section 3 that $f(12|34,12|34)$ is a measure of *inhomogeneity*, a type of non-additivity.

These expressions can also be rewritten as:

$$\mathrm{var}\left\{ \frac{[s|X \otimes X]}{\sum_i \sum_j (\xi_{ij} - \xi_{i\cdot})^2} \right\} = \frac{2(m-1)}{m^3(n-1)} \left[1 - \frac{(CV)^2}{m} \right]$$

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where $CV = s_\Delta/\overline{\Delta}, \Delta_i = \sum_j(\xi_{ij} - \xi_{i\cdot})^2$, $\overline{\Delta} = m^{-1}\sum_i \Delta_i$, and $s_\Delta^2 = (m-1)^{-1}\sum_i(\Delta_i - \overline{\Delta})^2$.

In this form this is just the result obtained by Pitman (1937) and Welch (1937); see also Kempthorne (1952), Wilk (1955) and Ogawa (1961, 1974).

(b) *Latin square designs.* The discussion for Latin square designs is equally straightforward although, as we will see, we do not get such a simple and interpretable final result. Nevertheless our approach, exploiting modern algebra and combinatorics, cuts through some heavy classical algebra, cf. Ogawa (1974, pages 125–160).

This time we suppose that the plot yields are $(\xi_{ij}: (i,j) \in \mathbf{n} \times \mathbf{n})$ and that $X = (X_{ij})$ is the random array obtained by independently permuting the two indices labelling the ξ's. Expressions for the generalized cumulants f_π and f_ρ of orders 2 and 4 of X are just the corresponding generalized k-statistics in the (ξ_{ij}), and these are most compactly expressed in terms of the t-expressions, see V6.2. The treatment sum of squares is $[s|X \otimes X]$ obtained using the coefficients $s_{ij,i'j'} = (n-1)/n^2$ if plot(i,j) has the same treatment as plot(i',j'), and $s_{ij,i'j'} = -1/n^2$ otherwise. As with the RCBD we find that the only term $[s|R(\pi)]$, $\pi \in \mathrm{Hom}(P, \mathcal{P}(\mathbf{2}))$ which is non-zero is $\pi = (12, 12)$, and $[s|R(12,12)] = n-1$, whence

$$\mathsf{E}[s|X \otimes X] = (n-1)f(12,12) = (n-1)^{-1}\sum_i\sum_j(\xi_{ij} - \xi_{i\cdot} - \xi_{\cdot j} + \xi_{\cdot\cdot})^2.$$

This holds for every $n \times n$ Latin square.

For the variance of $[s|X \otimes X]$ we need to calculate $[s \otimes s|R(\rho(1), \rho(2))]$ for all $\rho(1)$, $\rho(2) \in \mathcal{P}(\mathbf{2})$, and here these are zero if either $\rho(1)$ or $\rho(2)$ has a singleton block. The remaining ones are readily calculated: $[s \otimes s|R(1234, 1234)] = (n-1)^2/n^2, [s \otimes s|R(1234, 12|34)] = (n-1)^2/n, [s \otimes s|R(1234, 13|24)] = (n-1)/n, [s \otimes s|R(12|34, 12|34)] = (n-1)^2, [s \otimes s|R(13|24, 13|24)] = n-1$, $[s \otimes s|R(12|34, 13|24)] = (n-1)$, with the same answers being valid if the first and second partitions are interchanged, or if $14|23$ replaces $13|24$, and finally we need to calculate $[s \otimes s|R(13|24, 14|23]$ and its partner with the partitions interchanged.

For *fixed* i, i' and j with $i' \neq i$ let us evaluate the sum over j' of the terms $s_{ij,i'j'}s_{ij',i'j}$. To do this we need to look at all rectangles in the Latin square having two of their corners at (i,j) and (i',j'); these correspond to different values of j'. The Latin square property means that there exists one value of j' for which the treatments at $(i'j')$ and (i,j) are the same. If these coincide, we have a 'special' rectangle with the same treatments at diagonally opposite corners. Otherwise we have two rectangles each with one pair of identical treatments. In either case, all the rectangles corresponding to *other* values of j' have all four treatments distinct.

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Thus there are two cases: 'special' triples, which give a sum of $(n-1)^2/n^4 + (n-1)\times 1/n^4 = (n-1)/n^3$, and other triples which give a sum of $-2(n-1)/n^4 + (n-2)\times 1/n^4 = -1/n^3$.

When $i = i'$ we use the Latin square property to see that the sum over j' of $s_{ij,ij'}s_{ij',ij}$ is $(n-1)^2/n^4 + (n-1)\times 1/n^4 = (n-1)/n^3$. If we write n_{ij} for the number of 'special' triples (i, i', j) with fixed i, j then

$$\begin{aligned}
&[s\otimes s|R(13|24, 14|23)] \\
&= \sum_i \sum_j \sum_{i'} \left[\sum_{j'} s_{ij,i'j'}s_{ij',i'j}\right] \\
&= \sum_i \sum_j [n_{ij}(n-1)/n^3 + (n-1)/n^3 + (n - n_{ij} - 1)\times -1/n^3] \\
&= \left[\sum_i \sum_j n_{ij}\right]/n^2,
\end{aligned}$$

the inner sum on the second line being broken into sums over all special triples, the cases $i = i'$ and the rest, respectively. Similarly $[s\otimes s|R(14|23, 13|24)] = [\sum_i \sum_j n_{ij}]/n^2$. [Note that our n_{ij} is different from that of Welch (1937), but that the sum $\sum_i \sum_j n_{ij}$ coincides with his sum $\sum_{k\neq k'} n_{kk'}$.]

Putting the foregoing results together we obtain

$$\begin{aligned}
&n^2 \mathrm{var}[s|X\otimes X] \\
&\quad = (n-1)^2 f(1234, 1234) + n(n-1)^2[f(1234, 12|34) + f(12|34, 1234)] \\
&\qquad + (n)_2[f(1234, 13|24) + f(13|24, 1234) + f(1234, 14|23) + f(14|23, 1234)] \\
&\qquad + (n)_2^2 f(12|34, 12|34) + n(n)_2[f(13|24, 13|24) + f(14|23, 14|23)] \\
&\qquad + 2(n)_2[f(12|34, 13|24) + f(13|24, 12|34) \\
&\qquad\qquad + f(12|34, 14|23) + f(14|23, 12|34)] \\
&\qquad + \sum_i \sum_j n_{ij}[f(13|24, 14|23) + f(14|23, 12|34)] - [n(n-1)]^2 f(12, 12)^2.
\end{aligned} \tag{6.6}$$

In order to expand this let us use Welch's (1937) notation:

$$n^2 t(1234, 1234) = \sum_i \sum_j (\xi_{ij} - \xi_{i\cdot} - \xi_{\cdot j} + \xi_{\cdot\cdot})^4 = D,$$

$$n^4 t(12|34, 12|34) = \left(\sum_i \sum_j (\xi_{ij} - \xi_{i\cdot} - \xi_{\cdot j} + \xi_{\cdot\cdot})^2\right)^2 = F,$$

$$n^3 t(1234, 12|34) = \sum_i \left(\sum_j (\xi_{ij} - \xi_{i\cdot} - \xi_{\cdot j} + \xi_{\cdot\cdot})^2 \right)^2 = G',$$

$$n^3 t(12|34, 1234) = \sum_j \left(\sum_i (\xi_{ij} - \xi_{i\cdot} - \xi_{\cdot j} + \xi_{\cdot\cdot})^2 \right)^2 = G'',$$

$$n^4 t(12|34, 13|24) = \sum_i \sum_{i'} \left(\sum_j (\xi_{ij} - \xi_{i\cdot} - \xi_{\cdot j} + \xi_{\cdot\cdot})(\xi_{i'j} - \xi_{i'\cdot} - \xi_{\cdot j} + \xi_{\cdot\cdot}) \right)^2 = H.$$

We now make use of the method explained in V6.2 for putting $\{f_\rho\}$ in terms of $\{t_\rho\}$ and simplify (6.6). The answer found by Welch (1937) and verified by Pitman (1937) and Ogawa (1962, 1974) is the following:

$$\begin{aligned} n^2 \mathrm{var}[s|X \otimes X] &= \frac{1}{(n-1)(n-2)^2(n-3)} [2n^2(n-1)D + (n^4 - 4n^3 + 2n^2 + 6n - 6)F \\ &\qquad - 2n(n^2 - 3n + 3)G - 2(n^2 - 6n + 6)H] \\ &+ \frac{2\sum_i \sum_j n_{ij}}{(n)_4^2} [n^2(n-1)^2 D + (2n^2 - 6n + 3)F \\ &\qquad - n(n-1)(n^2 - 3n + 3)G \\ &\qquad + (n^4 - 6n^3 + 13n^2 - 12n + 6)H]. \end{aligned} \tag{6.7}$$

Here $G = G' + G''$. Rather than re-derive the entire result (6.7) from (6.6), let us be content with seeing how we can verify the coefficients of $D, F, \ldots, H$ in the two bracketed terms. Recalling the formulae in V(6.1) and the crossing rule V Proposition 6.2 we find that D will occur in every term on the RHS of equation (6.6) above. The coefficient of D which appears together with a multiplier $[\sum_i \sum_j n_{ij}]/n^2$ must be $1/n^2$ times that which arises in the expansion of $f(13|24, 14|23) + f(14|23, 13|24)$ in terms of t-tensors and from V(6.1) this will be just $2[n(n-1)/(n)_4]^2$. This checks part of (6.7) and the remainder can be checked in a similar manner.

(c) *Split-plot design.* This final example illustrates in as simple a way as is possible the impact on randomization variances of having treatment sums of squares in different *strata*, Nelder (1965). The block structure is $\mathbf{m}/\mathbf{n}/\mathbf{p}$, that is, our nesting poset P is the three element chain, and we have a crossed treatment structure $A \times B$ where A has n levels and B has p levels. The classical split-plot design varies the level of B across sub-plots of plots, holding the level of A constant, and then allocates levels of A to whole plots withing blocks, just as in a randomized complete block design. It is well known that the main effects sum of squares Q^A for A comes out in the whole plot stratum, whilst that for B, Q^B and the interaction sum of squares Q^{AB} come out in the sub-plot stratum.

The genralized k-statistics of the array $(\xi_{ijk}: (i,j,k) \in \mathbf{m}/\mathbf{n}/\mathbf{p})$ of plot yields give the generalized cumulants of the permutation distribution of (X_{ijk}) and those of order 2 and 4 will be denoted by $f(\pi)$ and $f(\rho)$ respectively, $\pi \in \mathrm{Hom}(P, \mathcal{P}(\mathbf{2}))$, $\rho \in \mathrm{Hom}(P, \mathcal{P}(\mathbf{4}))$. As in our earlier examples, the main task is the computaton of inner products $[s|R(\pi)]$ and $[s \otimes s|R(\rho)]$ where s is the tensor of coefficients. There are three such tensors here, denoted by s^A, s^B and s^{AB}, where

$$s^A_{ijk,i'j'k'} = \begin{cases} (n-1)/mnp & \text{if } A(ijk) = A(i'j'k'), \\ -1/mnp & \text{otherwise;} \end{cases}$$

$$s^B_{ijk,i'j'k'} = \begin{cases} (p-1)/mnp & \text{if } B(ijk) = B(i'j'k'), \\ -1/mnp & \text{otherwise;} \end{cases}$$

and

$$s^{AB}_{ijk,i'j'k'} = \begin{cases} (n-1)(p-1)/mnp & \text{if } A(ijk) = A(i'j'k') \& B(ijk) = B(i'j'k'), \\ -(n-1)/mnp & \text{if } A(ijk) = A(i'j'k') \& B(ijk) \neq B(i'j'k'), \\ -(p-1)/mnp & \text{if } A(ijk) \neq A(i'j'k') \& B(ijk) = B(i'j'k'), \\ 1/mnp & \text{otherwise.} \end{cases}$$

Here we have written $A(ijk)$ for the level of treatment A applied to the plot labelled ijk, and similarly for $B(ijk)$.

Let us deal with s^B and s^{AB} first, because their results parallel those obtained for the RCBD quite closely. If we denote the number of *degrees of freedom* associated with each sum of squares by d: $d^A = n-1, d^B = p-1$ and $d^{AB} = (n-1)(p-1)$, then for s^B and s^{AB} *only* we have (cf. the RCBD where $d^T = n-1$): $[s|R(12,12,12)] = d$, and $[s|R(\pi)] = 0$ for all other $\pi \in \mathrm{Hom}(P, \mathcal{P}(\mathbf{2}))$. Thus for these two,

$$\mathsf{E}[s|X \otimes X] = df(12,12,12) = \frac{d}{mn(p-1)} \sum_i \sum_j \sum_k (\xi_{ijk} - \xi_{ij\cdot})^2.$$

The non-zero terms $[s \otimes s|R(\rho)]$ for $\rho \in \mathrm{Hom}(P, \mathcal{P}(\mathbf{4}))$ are as follows:

$$\begin{aligned} &[s \otimes s|R(1234,1234,1234)] = d^2/mnp, && [s \otimes s|R(1234,1234,12|34)] = d^2mn, \\ &[s \otimes s|R(1234,1234,13|24)] = d/mn, && [s \otimes s|R(1234,12|34,12|34)] = d^2/m, \\ &[s \otimes s|R(1234,13|24,13|24)] = d/m, && [s \otimes s|R(12|34,12|34,12|34)] = d^2, \end{aligned}$$

and

$$[s \otimes s|R(13|24,13|24,13|24)] = d,$$

with the same results arising if 13|24 is replaced by 14|23 in the LHS of the 3rd, 5th and 7th expression.

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By making use of (6.3) and simplifying we find that

$$mn\ \mathrm{Var}[s^B|X\otimes X] = \frac{2}{mn(p-1)}\left\{\left[\sum_i\sum_j\sum_k(\xi_{ijk}-\xi_{ij\cdot})^2\right]^2 - \sum_i\sum_j\left[\sum_k(\xi_{ijk}-\xi_{ij\cdot})^2\right]^2\right\} + \frac{(p+1)(m-1)}{mn(p-1)(n-1)}\sum_i\sum_j\left[\sum_k(\xi_{ijk}-\xi_{ij\cdot})^2\right]^2$$

where the term in braces is similar to the one obtained in the RCBD. Clearly when $m=1$ the above expression assumes the same form as (6.5).

Turning now to $\mathrm{Var}[s^{AB}|X\otimes X]$ we find that fewer simplifications occur; it can be written as follows.

$$mnp\ \mathrm{Var}[s^{AB}|X\otimes X] = \theta\sum_{ijk}(\xi_{ijk}-\xi_{ij\cdot})^4 + \varphi\sum_{ij}\left[\sum_k(\xi_{ijk}-\xi_{ij\cdot})^2\right]^2 + \psi\left[\sum_{ijk}(\xi_{ijk}-\xi_{ij\cdot})^2\right]^2$$

where

$$\theta = \frac{2(p)_2^2(n)_3}{mn^2(p)_4},\quad \psi = \frac{2p(n-1)}{mn(p-1)},$$
$$\varphi = d(d+2)\left\{\left[\frac{p^2-3p+3-3(p-1)d/(d+2)}{mn(p)_4}\right] + \frac{[1-n/m]}{(n)_2(p-1)^2}\right\}$$

and $d=(n-1)(p-1)$. There does not seem to be any illuminating rearrangement of these expressions, although it is interesting to note that the sum of the fourth powers $(\xi_{ijk}-\xi_{ij\cdot})^4$ appears only in the variance of the interaction sum of squares.

Finally, let us consider the mean and variance of $Q^A = [s^A|X\otimes X]$. It is easy to check that $[s^A|R(12,12,12)] = n-1$, $[s^A|R(12,12,1|2)] = p(n-1)$ whilst $[s^A|R(\pi)] = 0$ for $\pi = (12,1|2,1|2)$ and $\pi = (1|2,1|2,1|2)$. Thus $\mathsf{E}[s^A|X\otimes X] = (n-1)f(12,12,12)+p(n-1)f(12,12,1|2)$ which simplifies to $pm^{-1}\sum_i\sum_j(\xi_{ij\cdot}-\xi_{i\cdot\cdot})^2$. Though straightforward, the variance calculation for Q^A is much lengthier than those for Q^B and Q^{AB} because $[s^A\otimes s^A|R(\rho)]$ is non-zero for many more $\rho\in\mathrm{Hom}(P,\mathcal{P}(4))$. Putting $d=d^A=n-1$, we can quickly build up the following table.

TABLE 1
Non-zero Inner Products

$\rho \in \mathrm{Hom}(P, \mathcal{P}(4))$	$[s^A \otimes s^A \mid R(\rho)]$	Number of such terms
$(1234, 1234, 1234)$	d^2/mnp	1
$(1234, 1234, 123\|4)$	d^2/mn	4
$(1234, 1234, 12\|34)$	d^2/mn	3
$(1234, 1234, 12\|3\|4)$	pd^2/mn	6
$(1234, 1234, 1\|2\|3\|4)$	p^2d^2/mn	1
$(1234, 12\|34, 12\|34)$	d^2/m	1
$(1234, 12\|34, 12\|3\|4)$	pd^2/m	2
$(1234, 12\|34, 1\|2\|3\|4)$	p^2d^2/m	1
$(1234, 13\|24, 13\|24)$	d/m	2
$(1234, 13\|24, 13\|2\|4)$	pd/m	4
$(1234, 13\|24, 1\|2\|3\|4)$	p^2d/m	2
$(12\|34, 12\|34, 12\|34)$	d^2	1
$(12\|34, 12\|34, 12\|3\|4)$	pd^2	2
$(12\|34, 12\|34, 1\|2\|3\|4)$	p^2d^2	1
$(13\|24, 13\|24, 13\|24)$	d	2
$(13\|24, 13\|24, 13\|2\|4)$	pd	4
$(13\|24, 13\|24, 1\|2\|3\|4)$	p^2d	2

Once more making use of equations (6.1) and (6.3), we combine the results in Table 1 with the expansions of generalized k-statistics in terms of the T-tensors given in V, Section 6.4, perform some routine classical algebraic simplifications, and obtain

$$\mathrm{Var}[s^A \mid X \otimes X] = \alpha \sum_{ij} \left(\sum_{k} (\xi_{ijk} - \xi_{ij\cdot})^2 \right)^2 + \beta \sum_{i} \left(\sum_{jk} (\xi_{ijk} - \xi_{ij\cdot})^2 \right)^2 + \gamma \left(\sum_{ijk} (\xi_{ijk} - \xi_{ij\cdot})^2 \right)^2 .$$

Here α, β and γ are functions of m, n and p which, whilst not especially complicated, do not simplify into compact expressions and do not appear to be easily interpreted. However it is of interest to observe that although the mean of Q^A is a multiple of $\sum_i \sum_j (\xi_{ij\cdot} - \xi_{i\cdot\cdot})^2$, its variance only involves the sums of squares of the terms $\xi_{ijk} - \xi_{ij\cdot}$ from the bottom stratum.

Appendix: Tensor products of generalized k-statistics

In this appendix we list just those expansions of the tensor products of second order generalized k-statistics needed for calculations done in this paper. These are taken from tables we have formed of expansions of the products of all second order generalized k-statistics for the partially ordered sets cited.

A. *Single index:* **n**.

(i) $$(12)\otimes(34) = (12|34) + \frac{1}{n}(1234) + \frac{1}{n-1}\{(13|24)+(14|23)\},$$

(ii) $$\begin{aligned}&(12)\otimes(3|4)\\ &\quad = (12|3|4) + \frac{1}{n}\{(123|4)+(124|3)\} - \frac{1}{n(n-1)}\{(13|24)+(14|23)\},\end{aligned}$$

(iii) $$\begin{aligned}(1|2)\otimes(3|4) = (1|2|3|4) &+ \frac{1}{n(n-1)}\{(13|24)+(14|23)\}\\ &+ \frac{1}{n}\{(13|2|4)+(14|2|3)+(23|1|4)+(24|1|3)\}.\end{aligned}$$

B. *Two indices, the second nested within the first:* **m/n**.

(i) $$\begin{aligned}(12,12)\otimes(34,34) = {}&\frac{1}{mn}(1234,1234) + \frac{1}{m}(1234,12|34)\\ &+ \frac{1}{m(n-1)}\{(1234,13|24)+(1234,14|23)\\ &+ (13|24,13|24)+(14|23,14|23)\} + (12|34,12|34).\end{aligned}$$

(ii)

$$\begin{aligned}(12,1|2)\otimes(34,3|4) = {}&\frac{1}{mn(n-1)}\{(1234,13|24)+(1234,14|23)\}\\ &+ \frac{1}{mn}\{(1234,1|3|24)+(1234,1|4|23)+(1234,14|2|3)+(1234,13|2|4)\}\\ &+ \frac{mn-1}{mn^2(m-1)(n-1)}\{(13|24,13|24)+(14|23,14|23)\} + \frac{1}{m}(1234,1|2|3|4)\\ &+ \frac{1}{(m-1)n}\{(13|24,1|3|24)+(13|24,13|2|4)\\ &\qquad\qquad + (14|23,1|4|23)+(14|23,14|2|3)\}\\ &+ \frac{1}{m-1}\{(13|24,1|2|3|4)+(14|23,1|2|3|4)+(12|34,1|2|3|4)\}.\end{aligned}$$

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$$
\text{(ii)}\quad
\begin{aligned}
(12,12)\otimes(34,3|4) = {} & \frac{1}{mn}\{(1234,3|124)+(1234,4|123)\}\\
& -\frac{1}{mn(n-1)}\{(1234,13|24)+(1234,14|23)\\
& \qquad +(13|24,13|24)+(14|23,14|23)\}\\
& +\frac{1}{m}(1234,12|3|4)+(12|34,12|3|4).
\end{aligned}
$$

C. *Two crossed indices*: $\mathbf{m}\times\mathbf{n}$.

$$
\begin{aligned}
(12,12)\otimes(34,34) = {} & (12|34,12|34)+\frac{1}{mn}(1234,1234)\\
& +\frac{1}{m}(1234,12|34)+\frac{1}{m(n-1)}\{(1234,13|24)+(1234,14|23)\}\\
& +\frac{1}{n}(12|34,1234)+\frac{1}{n(m-1)}\{(13|24,1234)+(14|23,1234)\}\\
& +\frac{1}{m-1}\{(13|24,12|34)+(14|23,12|34)\}\\
& +\frac{1}{n-1}\{(12|34,13|24)+(12|34,14|23)\}\\
& +\frac{1}{(m-1)(n-1)}\{(13|24,13|24)+(13|24,14|23)\\
& \qquad +(14|23,13|24)+(14|23,14|23)\}.
\end{aligned}
$$

Such expansions are obtained using the crossing rule of III, Section 3 and the results in **A** above.

D. *Three nested indices*: $\mathbf{m}/\mathbf{n}/\mathbf{p}$.

$$
\begin{aligned}
(12,12,12)\otimes(34,34,34) = {} & (12|34,12|34,12|34)+\frac{1}{mnp}(1234,1234,1234)\\
& +\frac{1}{mn}(1234,1234,12|34)+\frac{1}{m}(1234,12|34,12|34)\\
& +\frac{1}{mn(p-1)}\{(1234,1234,13|24)+(1234,1234,14|23)\\
& \qquad +(1234,13|24,13|24)+(1234,14|23,14|23)\\
& \qquad +(13|24,13|24,13|24)+(14|23,14|23,14|23)\}.
\end{aligned}
$$

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E. $(\mathbf{m} \times \mathbf{n})/\mathbf{p}$.

$$
\begin{aligned}
(12,12,12) \otimes (34,34,34) &= (12|34, 12|34, 12|34) \\
&+ \frac{1}{nmp}(1234, 1234, 1234) + \frac{1}{mn}(1234, 1234, 12|34) \\
&+ \frac{1}{mn(p-1)}\{(1234, 1234, 13|24) + (1234, 1234, 14|23)\} \\
&+ \frac{1}{m}(1234, 12|34, 12|34) + \frac{1}{n}(12|34, 1234, 12|34) \\
&+ \frac{1}{mn(p-1)}\{(1234, 13|24, 13|24) + (1234, 14|23, 14|23) \\
&\qquad + (13|24, 1234, 13|24) + (14|23, 1234, 14|23) \\
&\qquad + (13|24, 13|24, 13|24) + (14|23, 14|23, 14|23)\}.
\end{aligned}
$$

F. $\mathbf{m}/(\mathbf{n} \times \mathbf{p})$.

$$
\begin{aligned}
(12,12,12) \otimes (34,34,34) &= (12|34, 12|34, 12|34) \\
&+ \frac{1}{mnp}(1234, 1234, 1234) + \frac{1}{mn}(1234, 1234, 12|34) + \frac{1}{mp}(1234, 12|34, 1234) \\
&+ \frac{1}{mn(p-1)}\{(1234, 1234, 13|24) + (1234, 1234, 14|23)\} \\
&+ \frac{1}{mp(n-1)}\{(1234, 13|24, 1234) + (1234, 14|23, 1234)\} \\
&+ \frac{1}{m}(1234, 12|34, 12|34) \\
&+ \frac{1}{m(n-1)}\{(1234, 13|24, 12|34) + (1234, 14|23, 12|34)\} \\
&+ \frac{1}{m(p-1)}\{(1234, 12|34, 13|24) + (1234, 12|34, 14|23)\} \\
&+ \frac{1}{m(n-1)(p-1)}\{(1234, 13|24, 13|24) + (1234, 13|24, 14|23) \\
&\qquad + (1234, 14|23, 13|24) + (1234, 14|23, 14|23) \\
&\qquad + (13|24, 13|24, 13|24) + (14|23, 14|23, 14|23)\}.
\end{aligned}
$$

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