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MODULES OVER BOUNDED HEREDITARY NOETHERIAN PRIME RINGS

BY

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Singh introduced two conditions on a module M_R in [7]. The author introduced the concept of h-neat submodule of such module in [3] and generalized some of the well known results of neat subgroups. A theorem of Erdelyi was also shown to be true for such modules in [4]. The main purpose of this paper is to generalize a well known result of K. M. Benabdallah and J. M. Irwin and M. Rafiq [2, Theorem 10]. If M is a torsion module over a bounded (*hnp*)-ring R then under some conditions we have obtained an h-pure submodule C of M such that M/C is divisible (Theorem 7). Proposition 10 gives a necessary and sufficient condition for a quotient submodule to be complement of some given submodule. If M is torsion module over bounded (*hnp*)-ring R and K is an h-neat submodule of M then the question: "under what conditions $M=K+H_n(M)$ for every $n\geq 0$"? is answered in Theorem 11.

Throughout this paper M will be taken to be torsion module over bounded hereditary noetherian prime ring R. For any uniform element $x\in M$ the composition length $d(xR)$ is called exponent of x and is denoted as $e(x)$; $\sup\{d(yR/xR)\}$ where y is uniform element of M such that $x\in yR$, will be called the height of x and denoted by $H_M(x)$ (or simply $H(x)$). For any $k\geq 0$, $H_k(M)$ will denote the submodule generated by uniform elements of M of height at least k. M^1 will denote the submodule generated by uniform elements of infinite height in M.

As defined in [7], a submodule N of M is called h-pure if $H_k(N)=N\cap H_k(M)$ for every $k\geq 0$.

As defined in [3] a submodule N of M is called h-neat if $N\cap H_1(M)=H_1(N)$. If M is a module satisfying conditions (I) and (II) as introduced in [7], then we call M an S_2-module.

Now we restate the following results proved in [3].

LEMMA 1([3,Prop. 1]). *If M is an S_2-module and N is a submodule of M then any complement of N is h-neat in M.*

LEMMA 2 ([3, Lemma 2]). *If M is an S_2-module and N is h-neat submodule of M with same socle then $N=M$.*

LEMMA 3 ([3, Lemma 3]). *If M is an S_2-module and N is h-neat submodule of M such that* $\mathrm{Soc}(N)\oplus\mathrm{Soc}(T)=\mathrm{Soc}(M)$ *then N is a complement of T.*

The following lemma is of set theoretic nature and hence is stated for arbitrary modules.

LEMMA 4. *If M is a right R-module and $U \subseteq V$ are submodules of M. Let K be a complement of U in M. Then every complement of $K \cap V$ in K is a complement of V in M.*

It is well known that the homomorphic image of divisible module is divisible. In view of the Lemma 4 the next result is valid for arbitrary modules but we state for torsion modules over bounded (hnp)-ring as needed in the sequel.

LEMMA 5. *Suppose M is a torsion module over bounded (hnp)-ring R and N is a submodule of M. Suppose M/K is divisible for every complement K of N in M. Then M/T is also divisible for any complement T of any submodule U of N.*

Now we have the following proposition which generalizes [2, Lemma 7]. The technique of the proof is same as in groups.

PROPOSITION 6. *If M is a torsion module over a bounded (hnp)-ring R and N is a submodule of M such that M/K is divisible for every complement K of N in M then* $\mathrm{Soc}(N) \subseteq M^1$.

Proof. Let x be a uniform element in $\mathrm{Soc}(N)$ and $x \notin M^1$. Then appealing to [5, Theorem 10] we get $M = yR \oplus T$ such that $\mathrm{Soc}(yR) = xR$ and yR is uniform submodule of finite length. It is easy to check that T is a complement of xR. Now by Lemma 5, we get M/T to be divisible which is not possible consequently we have $\mathrm{Soc}(N) \subseteq M^1$.

THEOREM 7. *Suppose M is a torsion module over a bounded (hnp)-ring R and S is a subsocle of M with $\mathrm{Soc}(M) = S + \mathrm{Soc}(H_k(M))$ for every $k \geq 0$. Then there exists an h-pure submodule C of M such that $S = \mathrm{Soc}(C)$ and M/C is divisible.*

Proof. Let C be maximal with respect to $\mathrm{Soc}(C) = S$ then we prove that $H_1(M) \cap C = H_1(C)$. Let x be a uniform element in $H_1(M) \cap C$ then there exists a uniform element $y \in M$ such that $x \in yR$ and $d(yR/xR) = 1$. If $y \in C$ then we are done. Let $y \notin C$ then $S < \mathrm{Soc}(C + yR)$; Hence there exists a uniform element $z \in \mathrm{Soc}(C + yR)$ such that $z \notin S$ and $z = u + yr$ for some $u \in C$, $r \in R$. As yR is totally ordered it is easy to check that $yrR = yR$, hence without any loss of generality we can assume that $z = u + y$. Now define $\eta : yR \to uR$ given as $yr \to ur$. Let $yr = 0$ then $zr = ur$. Now either $zrR = zR$ or $zr = 0$. If $zrR = zR$ then $z = zrr'$ for some $r' \in R$; hence $z = urr' \in S$ which is a contradiction. Consequently $zr = 0$ and we get $ur = 0$, therefore η is well defined. Trivially η is onto homomorphism and we get uR, being homomorphic image of yR, to be a uniform module.

Now let $P = \mathrm{ann}(yR/xR)$ then by Eisenbud and Griffith [1, Corollary 3.2] R/P is a generalized uniserial ring. Hence appealing to [6, Lemma 2.3] we get

$yP=xR$. Now $x=yr$ for some $r\in P$ and for every $r\in P$, $zr=ur+yr$. Trivially $zr=0$, hence $x=yr=-ur$. Now we assert that $urR<uR$. Suppose $urR=uR$ then $u=yr_1$ for some $r_1\in R$ and hence $z=yc_1$ for some $c_1\in R$. Trivially $yc_1R\subseteq yR$. Now either $yc_1R\subseteq xR$ or $xR<yc_1R$. If $yc_1R\subseteq xR$, then $z\in S$, which is not possible. Hence $xR<yc_1R$ and we get $yc_1R=yR=zR$ which is a contradiction. Therefore $urR<uR$ and we get $x\in H_1(C)$. Consequently $C\cap H_1(M)=H_1(C)$. Now suppose $H_n(C)=C\cap H_n(M)$ then we show that $H_{n+1}(C)=C\cap H_{n+1}(M)$. Let x be a uniform element in $C\cap H_{n+1}(M)$ then we can find a uniform element $y\in M$ such that $d(yR/xR)=n+1$. Let $\mathrm{Soc}(yR/xR)=zR/xR$. If $z\in C$ then there is nothing to prove. Let $z\notin C$. As $d(zR/xR)=1$, we can find a uniform element $u\in C$ such that $x\in uR$ and $d(uR/xR)=1$. Hence by [5, Lemma 2] there exists an isomorphism $\theta:zR\to uR$ such that θ is identity on xR. Choose θ such that $\theta(z)=u$. Now define $\eta:zR\to(z-\theta(z))R$ given as $zr\to(z-\theta(z))r$ then η is R-epimorphism with $xR\subseteq\ker\eta$. Hence $e(z-\theta(z))\le 1$ and we get $z-\theta(z)=z-u\in\mathrm{Soc}(M)$. Hence $z-u-s\in H_n(M)$ for some $s\in S$ and $z-u-s=t$ for some $t\in H_n(M)$. Now by supposition $z-t=u+s\in H_n(C)$. Now appealing to [5, Lemma 1] $(u+s)R=\oplus\Sigma b_iR$ where $b_i\in H_n(C)$. Trivially every b_i can not be of exponent 1. Similarly $sR=\oplus\Sigma t_iR$ where t_iR are simple modules. Let $P_i=\mathrm{ann}(t_iR)$ then $sP_1P_2\cdots P_q=0$. Let $P=\mathrm{ann}(uR/xR)$ then $uP=xR$. Let $b_1,\ldots,b_\alpha$ be uniform elements of exponent greater than 1 and $b_{\alpha+1},\ldots,b_n$ be uniform elements of exponent 1. Now we can find submodules d_jR such that $d(b_jR/d_jR)=1$. Let $Q_j=\mathrm{ann}(b_jR/d_jR)$ then $b_jQ_j=d_jR$ for $j=1,\ldots,\alpha$. Let $Q_i=\mathrm{ann}(b_iR)$, $i=\alpha+1,\ldots,n$ then $b_iQ_i=0$. Without any loss of generality we can assume $P_1,\ldots,P_q$, $Q_1,\ldots,Q_\alpha$, P to be distinct. Now

$$(u+s)RP_1\cdots P_qQ_1\cdots Q_\alpha Q_{\alpha+1}\cdots Q_nP$$
$$=uP_1\cdots P_qQ_1\cdots Q_\alpha Q_{\alpha+1}\cdots Q_nP=uP=xR.$$

Also

$$(u+s)RP_1\cdots P_qQ_1\cdots Q_\alpha Q_{\alpha+1}\cdots Q_nP$$
$$=\textstyle\sum_1^\alpha b_iP_1\cdots P_qQ_1\cdots Q_\alpha Q_{\alpha+1}\cdots Q_nP,$$

but xR is uniform hence $xR=b_iP_1\cdots P_qQ_1\cdots Q_\alpha Q_{\alpha+1}\cdots Q_nP\subseteq d_jR<b_jR$ and we get $d(b_jR/xR)\ge 1$. Therefore, $x\in H_{n+1}(C)$. Hence C is h-pure submodule of M.

Now let $\bar{x}$ be a uniform element in $\mathrm{Soc}(M/C)$ then by [7, Lemma 2], there exists a uniform element $x'\in M$ such that $\bar{x}=\bar{x}'$ and $e(x')=1$. As $\mathrm{Soc}(M)=S+\mathrm{Soc}(H_k(M))$ for every k we get $\bar{x}\in H_k(M/C)$ for every k. Therefore $\bar{x}$ is of infinite height in M/C. Hence by [5, Lemma 8, Cor. 4], M/C is divisible.

Now an easy application of Lemma 1, Lemma 2, and Theorem 7, gives the following:

Corollary 8. *If M is a torsion module over a bounded (hnp)-ring R and N is*

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a submodule of M with $N\subseteq M^1$ then every complement U of N is h-pure and M/U is divisible.

Now appealing to proposition 6 and Corollary 8 we have the following:

COROLLARY 9. *If M is a torsion module over a bounded (hnp)-ring R and N is a submodule of M then M/K is divisible for every complement K of N if and only if* $\mathrm{Soc}(N)\subseteq M^1$.

Now we give a characterization for complement submodules which generalizes [2, Lemma 8].

PROPOSITION 10. *Let M be a torsion module over a bounded (hnp)-ring R and K be a submodule of M. If S is a sub socle of M with $S\subseteq \mathrm{Soc}(K)$ then K/S is a complement of $\mathrm{Soc}(M)/S$ in M/S if and only if $\mathrm{Soc}(K)=S$ and K is h-neat in M.*

Proof. Let K/S be complement of $\mathrm{Soc}(M)/S$ in M/S. Let $x\in K\cap H_1(M)$, then there exists a uniform element $y\in M$ such that $x\in yR$ and $d(yR/xR)=1$. If $y\in K$ we are done. Let $y\notin K$ then $(\bar{y}R+K/S)\cap \mathrm{Soc}(M)/S\neq 0$, hence for some uniform element $\bar{z}\in \mathrm{Soc}(M)/S$ we have $\bar{z}=\bar{y}r+\bar{k}$. It is trivial to see that $yrR=yR$, hence without any loss of generality we can assume $\bar{z}=\bar{y}+\bar{k}$. Define $\eta:\bar{y}R\to\bar{k}R$ given as $\bar{y}r\to\bar{k}r$ it is easy to check that η is a well defined onto homomorphism. Hence $\bar{k}R$ is uniform module. So we can take k to be uniform otherwise there will exist a uniform element k' such that $\bar{k}=\bar{k}'$. Trivially $e(k)>1$. Hence we can find a submodule $dR\subseteq kR$ such that $d(kR/dR)=1$. Let $Q=\mathrm{ann}(kR/dR)$ then $kQ=dR$. Let $P=\mathrm{ann}(yR/xR)$ then $yP=xR$. Now $z-y-k\in S$, so $z-y-k=s$ for some $s\in S$. Let $sR=\oplus\Sigma b_iR$ where b_iR are simple submodules. Let $P_i=\mathrm{ann}(b_iR)$ and $Q'=\mathrm{ann}(zR)$ then s $P_1P_2\cdots P_t=0$ and $zQ'=0$. Now $(y+s)RQQ'P_1\cdots P_tP=(-k+z)RQQ'P_1\cdots P_tP$. But $(y+s)RQQ'P_1\cdots P_tP=yQQ'P_1\cdots P_tP=yP=xR$ and $(-k+z)RQQ'P_1\cdots P_tP=-kQQ'P_1\cdots P_tP\subseteq dR$. Hence $xR\subseteq dR$ consequently $d(kR/xR)\geq 1$ and we have $x\in H_1(K)$, Therefore K is h-neat submodule of M.

Now let x be a uniform element of $\mathrm{Soc}(K)$ then as $K/S\cap \mathrm{Soc}(M)/S=0$, $x\in S$. Hence $\mathrm{Soc}(K)=S$. For the converse trivially $K\cap \mathrm{Soc}(M)=S$ and $\mathrm{Soc}(K/S)\cap \mathrm{Soc}(M)/S=0$. Now we show that $\mathrm{Soc}(M/S)=\mathrm{Soc}(M)/S\oplus \mathrm{Soc}(K/S)$. Let $\bar{x}$ be a uniform element in $\mathrm{Soc}(M/S)$. Let $P=\mathrm{ann}(\bar{x}R)$ then $\bar{x}P=0$, hence for every $r\in P$, $xr\in S$. If $xrR=xR$ then $x=xrr'$ for some $r'\in R$ hence $\bar{x}=(xr+S)r'=0$ which is a contradiction. Consequently $xrR<xR$. It is easy to check that $d(xR/xrR)=1$. By h-neatness of K there exists a uniform element $z\in K$ such that $xrR\subseteq zR$ and $d(zR/xrR)=1$. Appealing to [5, Lemma 2] we can find an isomorphism $\theta: xR\to zR$ which is identity on xrR. Let $\eta: xR\to(x-\theta(x))R$ be the natural epimorphism then $xrR\subseteq \ker\eta$ and $e(x-\theta(x))\leq d(xR/xrR)=1$. Therefore $x-\theta(x)\in \mathrm{Soc}(M)$ and $x-\theta(x)=v$ for some $v\in \mathrm{Soc}(M)$. This yields $\bar{x}\in \mathrm{Soc}(M)/S+\mathrm{Soc}(K/S)$. Hence $\mathrm{Soc}(M/S)=$

$\mathrm{Soc}(M)/S \oplus \mathrm{Soc}(K/S)$. Appealing to Lemma 3 we get K/S to be complement of $\mathrm{Soc}(M)/S$ in M/S.

Now we have the following main theorem which generalizes [2, Theorem 10], since the proof runs on similar lines it is omitted.

THEOREM 11. *Let M be a torsion module over a bounded (hnp)-ring R and K be a h-neat submodule of M such that* $\mathrm{Soc}(K)=S$ *where* $S \subseteq \mathrm{Soc}(M)$. Then $M=K+H_n(M)$ *for every* $n \geq 0$ *if and only if* $\mathrm{Soc}(M)=S+\mathrm{Soc}(H_n(M))$ for every $n \geq 0$.

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