BULLETIN 13 (1), 1970

A NOTE ON A THEOREM OF H. L. ABBOTT

BY

ROBERT J. DOUGLAS[1]

Let I^n be the graph of the unit n-dimensional cube. Its 2^n vertices are all the n-tuples of zeros and ones, two vertices being adjacent (joined by an edge) if and only if they differ in exactly one coordinate. A path P in I^n is a sequence $x_1, \ldots, x_m$ of distinct vertices in I^n where x_i is adjacent to x_{i+1} for $1 \leq i \leq m-1$; P is a circuit if it is also true that x_m and x_1 are adjacent. A path is *Hamiltonian* if it passes through all the vertices of I^n. Finally, for vertices x and y in I^n, we define $d(x, y)$ to be the graph theorectic distance between x and y, i.e., the number of coordinates in which x and y differ.

A problem studied by H. L. Abbott [1] (also see E. N. Gilbert [2]) is to determine the number $h(n)$ of distinct Hamiltonian circuits in I^n. Abbott proved for $n \geq 2$ that

$$h(n) > c(\sqrt[7]{6})^{2^n} \tag{1}$$

where c is a constant. Here, by modifying Abbott's argument, we shall prove for $n \geq 2$ that

$$h(n) > c(\sqrt[7]{18})^{2^n} \tag{2}$$

for some constant c. We also will prove the following result about Hamiltonian paths in I^n, which will be useful in establishing (2).

THEOREM 1. *If $x, y \in I^n$, then $d(x, y)$ is odd if and only if there exists a Hamiltonian path from x to y.*

Proof. Assume there exists a Hamiltonian path P from x to y. Then the length of P is $2^n - 1$ which is an odd number, and since $d(x, y)$ must have the same parity as the length of P we are done.

Now we will prove the converse. The proof will be by induction on n. Obviously the theorem holds for $n=2, 3$. Assuming the theorem for dimension n, consider $x, y \in I^{n+1}$ where $d(x, y)$ is odd. Pick opposite n-dimensional faces I^n and I^n_* of I^{n+1} so that $x \in I^n$ and $y \in I^n_*$. Then pick any $z \neq x$ where $z \in I^n$ and $d(y, z) = 2$. Hence $d(x, z)$ is odd. Letting z^* be the vertex in I^n_* opposite z, we have $d(z^*, y) = 1$. By the induction hypothesis, there is a Hamiltonian path

$$x = x_1, x_2, \ldots, x_{2^n} = z$$

Received by the editors June 27, 1969.

[1]Research partially sponsored by the United States Air Force Office of Scientific Research, Office of Aerospace Research, Under Grant AFOSR-68-1406 with the Department of Statistics, University of North Carolina at Chapel Hill.

https://doi.org/10.4153/CMB-1970-016-1 Published online by Cambridge University Press

in I^n joining x to z, and a Hamiltonian path

$$z^* = y_1, \ldots, y_{2^n} = y$$

in I^n_* joining z^* to y. We then get that

$$x_1, \ldots, x_{2^n}, y_1, \ldots, y_{2^n}$$

is a Hamiltonian path in I^{n+1} joining x to y, *proving the theorem.*

From Theorem 1, we immediately get the following well-known fact:

COROLLARY. *I^n admits a Hamiltonian circuit for all n.*

Proof. Join any two adjacent vertices by a Hamiltonian path and then add the edge joining them.

Define a *proper path* to be a path that is not a circuit. Let $l(n)$, ($l_p(n)$) be the number of (proper) Hamiltonian paths in I^n, and let $l^0(n)$ ($l_p^0(n)$) be the number of (proper) Hamiltonian paths in I^n having the origin as the initial or terminal vertex.

For vertices P and Q in I^n such that $d(P, Q)$ is odd, let $\sigma(P, Q)$ be the number of distinct Hamiltonian paths from P to Q, and define $M_n = \min\{\sigma(P, Q) : P, Q \in I^n$ and $d(P, Q)$ is odd$\}$. Finally, if $P \in I^m$ and $Q \in I^n$, let $P+Q$ be the vertex in I^{m+n} whose first m coordinates are those of P and whose last n coordinates are those of Q.

LEMMA. *For all positive integers m, $n \geq 2$,*

$$h(m+n) \geq 2^n M_n (l^0(n))^{2^m-1} h(m). \tag{3}$$

Also

$$l^0(n) = 2h(n) + l_p^0(n) = 2h(n) + \frac{l_p(n)}{2^{n-1}}. \tag{4}$$

Proof. Let $\mathscr{P} = \{P_1, \ldots, P_{2^m}\}$ be a Hamiltonian circuit in I^m, and fix $S_1^1 \in I^n$. Pick any Hamiltonian path $\mathscr{S}^1 = \{S_1^1, S_2^1, \ldots, S_s^1\}$ in I^n having S_1^1 as an end point ($s = 2^n$). Then, for $i = 2, \ldots, 2^m - 1$, pick any Hamiltonian path $\mathscr{S}^i = \{S_s^{i-1} = S_1^i, S_2^i, \ldots, S_s^i\}$ in I^n having S_s^{i-1} as an end point. Finally, pick any Hamiltonian path $\mathscr{S}^{2^m} = \{S_s^{2^m-1} S_1^{2^m}, = S_2^{2^m}, \ldots, S_s^{2^m} = S_1^1\}$ in I^n whose end points are $S_s^{2^m-1}$ and S_1^1. The last choice can be made as $d(S_s^{2^m-1}, S_1^1)$ is an odd number (for $d(S_s^{2^m-1}, S_1^1)$ has the same parity as $\sum_{i=1}^{2^m-1} d(S_1^i, S_s^i)$ which is odd because $d(S_1^i, S_s^i) \equiv 2^n - 1 \pmod 2$ for $i = 1, \ldots, 2^m - 1$). Thus the following is a Hamiltonian circuit in I^{m+n}:

$$\begin{array}{llll}
P_1 + S_1^1, & P_1 + S_2^1, & \ldots, P_1 + S_s^1 & = P_1 + S_1^2 \\
P_2 + S_1^2, & P_2 + S_2^2, & \ldots, P_2 + S_s^2 & = P_2 + S_1^3 \\
P_3 + S_1^3, & P_3 + S_2^3, & \ldots, P_3 + S_s^3 & = P_3 + S_1^4 \\
 & & & \vdots \\
P_{2^m-1} + S_1^{2^m-1}, & P_{2^m-1} + S_2^{2^m-1}, & \ldots, P_{2^m-1} + S_s^{2^m-1} & = P_{2^m-1} + S_1^{2^m} \\
P_{2^m} + S_1^{2^m}, & P_{2^m} + S_2^{2^m}, & \ldots, P_{2^m} + S_s^{2^m} & = P_{2^m} + S_1^1.
\end{array}$$

https://doi.org/10.4153/CMB-1970-016-1 Published online by Cambridge University Press

Now each $\mathscr{S}^i$, $i=1,\ldots,2^m-1$, can be chosen in $l^0(n)$ ways, $\mathscr{P}$ can be chosen in $h(m)$ ways, there are 2^n possibilities for S_1^1, and at least $M_n \geq 1$ possibilities for $\mathscr{S}^{2^m}$. Thus the total number of Hamiltonian circuits that can be chosen in the above fashion is at least

$$2^n \cdot M_n \cdot (l^0(n))^{2^m-1} \cdot h(m),$$

proving (3).

Clearly each Hamiltonian circuit in I^n yields two Hamiltonian paths in I^n each having the origin as an end point. (Simply omit one or the other of the edges in the circuit that has the origin as end point.) Hence $l^0(n)=2\cdot h(n)+l_p^0(n)$. But $2^n l_p^0(n)/2 = l_p(n)$, *which proves* (4) *and the lemma.*

Direct computation shows that $h(3)=6$, and that there are exactly six distinct Hamiltonian paths in I^3 from (0, 0, 0) to (1, 1, 1). (See Abbott [1].) Setting $n=3$, we get $M_3=6$, $l_p^0(3)=6$, and $l^0(3)=18$. Hence

$$h(m+3) \geq \frac{8}{3}\, 18^{2^m} \cdot h(m).$$

Pick $c>0$ so that $h(n)>c(\sqrt[7]{18})^{2^n}$ for $n=2, 3, 4$. Then for $n \geq 5$,

$$h(n) = h(n-3+3) \geq \frac{8}{3}(18)^{2^{n-3}} \cdot h(n-3) \geq \frac{8}{3}(18)^{2^{n-3}} \cdot c(\sqrt[7]{18})^{2^{n-3}} > c(\sqrt[7]{18})^{2^n},$$

and we have proved (2).

We note in closing that the following theorem, the statement of which was contained in a written communication from Abbott (and is an improvement on a result of his in [1]), can be proved very similarly to the lemma.

THEOREM. $l_p(m+n) \geq 2^n(2h(n)+l_p(n)/2^{n-1})^{2^m} l_p(m)$ *for all positive integers* m, $n \geq 2$; *hence* $l_p(n) > c(\sqrt[7]{18})^{2^n}$ *for all* $n \geq 2$.

Proof. Use the argument in the proof of the lemma, but replace the Hamiltonian circuit $\mathscr{P}$ by a proper Hamiltonian path, and only require $\mathscr{S}^{2^m}$ to have $S_s^{2^m-1}$ as an end point.

REFERENCES

1. H. L. Abbott, *Hamiltonian circuits and paths on the n-cube*, Canad. Math. Bull. (5) **9**, (1966), 557–562.

2. E. N. Gilbert, *Gray codes and paths on the n-cube*, Bell Syst. Tech. J. **37** (1958), 815–826.

UNIVERSITY OF WASHINGTON,
SEATTLE, WASHINGTON