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ON THE NORMALISER OF A GROUP IN THE CAYLEY REPRESENTATION

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Let G be a p-group of order n and embed G into S_n by the Cayley representation. If X is a group such that $G \lneqq X \leq S_n$ and $C_X(G) = G$, then it is proved that G is properly contained in $N_X(G)$.

1.

Let R be the Cayley representation (that is, the right regular representation) of a group G given by $R(g) = \binom{x}{xg}$ for all $g \in G$ and $x \in G$. Under the mapping R, the group G is embedded into a subgroup $R(G)$ of the symmetric group S_n where n is the cardinality of G. We identify G with $R(G)$. It is not hard to see that the centraliser of G in S_n consists of precisely the elements of the form $\binom{x}{gx}$.

Suppose that the group G is non-abelian. If X is a group containing a permutation of the form $\binom{x}{gx}$ for some $g \in G \backslash Z(G)$ such that the property

$$(*) \qquad G \lneqq X \leq S_n$$

holds then it follows that $N_X(G)$ contains G properly. However, it is

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easy to see that any element of S_n which normalises G is not always a permutation of the form $\binom{x}{gx}$. For example, take $G = S_3$ and embed it into S_6 by the Cayley representation. If $\alpha = (12)$, $\beta = (13)$, $\gamma = (23)$ are elements of S_3 and $x = (\alpha\gamma\beta)$, then one can check that x lies outside S_3 (in its embedding) and $x^{-1}S_3x = S_3$ but x does not centralise S_3 .

When the group G is abelian, the permutations $\binom{x}{gx}$ all lie in G and so G is self-centralising in S_n . (This can also be seen by using the fact that G is transitive and applying Wielandt [3], Theorem 4.4.) So when G is abelian one cannot obtain by the above method a group X satisfying (*) such that $N_X(G)$ contains G property.

However, we have

THEOREM 1. *Let G be a finite p-group and X be such that*

$$(*) \qquad G \lneqq X \leq S_n$$

where $n = |G|$ and G is embedded in S_n by the Cayley representation. Assume that the centraliser of G in X is G itself (such a situation will happen when for example, G is abelian). Then G is properly contained in $N_X(G)$.

When G is an elementary abelian p-group then $N_X(G)$ is clearly the group of all "affine transformations" on G regarded as a vector space over $GF(p)$. So from Theorem 1 we derive

COROLLARY 2. *If G is an elementary abelian p-group then there is no subgroup of S_n containing G which fails to intersect $N_{S_n}(T)\backslash T$.*

It is also interesting to study the problem in the case when G is an infinite group. A celebrated theorem of Higman, Neumann and Neumann [2] states that if G is a group then there is a group H containing G properly such that any two elements of H of the same order are conjugate. The proof involves first embedding G into an uncountable group and then

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use the Cayley representation inductively.

2.

Proof of Theorem 1. Suppose that X is a group satisfying (*) such that $N_X(G) = G$. Then G must be the Sylow p-group of X because if G is contained properly in a Sylow p-subgroup of X then there would be an element of $X\backslash G$ that normalises G which contradicts the above assumption. By *Burnside transfer theorem* (see for example, Hall [1], Theorem 14.3.1) X has a normal p-complement H say. Now X operates transitively on the set G. Let X_0 be the stabiliser of some "point" of the set G. Then we have

$$(1) \qquad \bigcap_{x \in X} x^{-1}X_0x = \text{identity}$$

since the permutation action map $X \to \text{Perm}(G)$ is injective. Further we have $|X_0| = |X|/|G| = |H|$. But considering the composite of group homomorphisms,

$$X_0 \to X \to X/H \simeq G ,$$

where the first homomorphism is the natural embedding, we see that the image of X_0 must be the identity because $|X_0|$ and $|G|$ are co-prime. Thus $X_0 = H$ which contradicts (1). This completes the proof of Theorem 1.

REMARK. If G is any group (abelian or non-abelian) and X is a group satisfying the property (*) then it is not hard to see that a minimal such X with the property $N_X(G) = G$ must be of the form $X = GU$ where U is a perfect group.

References

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