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# MOURRE THEORY FOR TIME-PERIODIC SYSTEMS

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**Abstract.** Studies for A.C. Stark Hamiltonian are closely related to that for the self-adjoint operator $K = -i\frac{d}{dt} + H(t)$ on torus. In this paper we use Mourre's commutator method, which makes great progress for the study of time-independent Hamiltonian. By use of it we show the asymptotic behavior of the unitary propagator $e^{-i\sigma K}$ as $\sigma \to \pm\infty$.

## §1. Introduction

We consider the following Schrödinger equation with time-dependent Hamiltonian on $\mathbb{R}^\nu$,

$$i\frac{\partial}{\partial t}u(t,x) = H(t)u(t,x), \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^\nu, \tag{1.1}$$

$$H(t) = -\Delta_x + V(t), \tag{1.2}$$

where $V(t)$ is a multiplicative operator by a function $V(t,x)$ which is periodic in $t$ with period $2\pi$:

$$V(t+2\pi, x) = V(t,x). \tag{1.3}$$

As is well-known, with some suitable conditions on $V(t,x)$, $H(t)$ generates a unique unitary propagator $\{U_1(t,s)\}_{-\infty<t,s<\infty}$. For $H_0 = -\Delta_x$, the associated unitary propagator is denoted by $U_0(t,s) = e^{-i(t-s)H_0}$. A traditional way to study the temporal asymptotics as $t \to \pm\infty$ of $U_1(t,s)$ is to introduce a family of operators $\{\mathbb{U}(\sigma)\}_{\sigma\in\mathbb{R}}$ on $\mathbb{H} = L^2(\mathbb{T}\times\mathbb{R}^\nu)$ ($\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$) as follows and to investigate the asymptotic behavior of $\mathbb{U}(\sigma)$.

$$(\mathbb{U}(\sigma)f)(t,x) = (U_1(t,t-\sigma)f(t-\sigma,\cdot))(x), \quad \text{for} \quad f \in \mathbb{H}. \tag{1.4}$$

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We write the generator of this group as $-iK$. Then $K = -i\frac{d}{dt} + H(t)$ is a self-adjoint operator on $\mathbb{H}$. Let

$$K_0 = -i\frac{d}{dt} + H_0. \tag{1.5}$$

Then for short-range potentials, the wave operators

$$\Omega_\pm = s - \lim_{\sigma\to\pm\infty} e^{i\sigma K} e^{-i\sigma K_0} \quad \text{on} \quad L^2(\mathbb{T}\times\mathbb{R}^\nu)$$
$$W_\pm(s) = s - \lim_{t\to\pm\infty} U_1(t,s)^* U_0(t,s) \quad \text{on} \quad L^2(\mathbb{R}^\nu)$$

are known to exist, and $\Omega_\pm$ are asymptotically complete, namely

$$\operatorname{Ran}\Omega_\pm = \mathcal{H}_{ac}(K)$$

where $\mathcal{H}_{ac}(K)$ denotes the absolutely continuous subspace of a self-adjoint operator $K$. Moreover, the asymptotic completeness of $W_\pm(s)$ holds in the following sense.

$$\operatorname{Ran} W_\pm(s) = \mathcal{H}_{ac}(U_1(s, s+2\pi)) \quad \text{for all} \quad s\in\mathbb{R}$$

These facts were first proved by Howland [How] and Yajima [Ya] by using the smoothness theory of Kato [Ka]. These results were extended to the 3-body problem by Nakamura [Na]. Kuwabara-Yajima [Ku-Y] studied the limiting absorption principle for the long-range potentials by using the pseudo-differential calculus due to Agmon and Hörmander. The asymptotic completeness of modified wave operator for long-range potential was proved by Kitada-Yajima [Ki-Y].

The aim of this paper is to accommodate the commutator technique of E. Mourre [Mo], which has brought a big progress in the spectral and scattering theory to the time-periodic 2-body Schrödinger operators. It covers almost all known results by a simpler method with weaker assumption on the potential. More precisely, we establish the limiting absorption principle for $K$ and study propagation properties of $e^{-i\sigma K}$.

Let $S$ be the set of functions $f$ such that $f\in C^\infty(\mathbb{T}\times\mathbb{R}^\nu)$ and for all $\alpha$, $\gamma\in\mathbb{N}$ and multi index $\beta$, $|\langle x\rangle^\alpha \partial_x^\beta \partial_t^\gamma f(t,x)| \le C_{\alpha\beta\gamma}$ on $\mathbb{T}\times\mathbb{R}^\nu$ for some constant $C_{\alpha\beta\gamma} > 0$. Here $\langle\cdot\rangle = (1+|\cdot|^2)^{\frac{1}{2}}$. As the conjugate operator $A$, which plays an important role in the Mourre theory, we adopt the following one.

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DEFINITION 1.1.

$$A = \frac{1}{2}(L_D \cdot x + x \cdot L_D) \tag{1.6}$$

where $D_x = \frac{1}{i}\nabla_x$ and $L_D = (L_j)_{1\le j\le \nu}$ with $L_j = D_{x_j}\langle D_x\rangle^{-2}$.

$A$ is essentially self-adjoint on domain $D = D(|x|)$. (See Theorem **X**.36 in [R-S].)

The following assumption is imposed on $V(t)$.

ASSUMPTION 1.2. *Let $V$ be the operator of multiplication by the function $V(t,x)$ on $\mathbb{H}$. We assume that*

(i) *$V$, $[V,A]$ are extended to $K_0$-compact operators.*

(ii) *$[[V,A],A]$ is extended to a $K_0$-bounded operator.*

We denote the extension of the form $[K,A]$ as $[K,A]^0$. This assumption is satisfied in the following case. The proof is given in Lemma 2.4.

EXAMPLE 1. The potential $V(t,x)$ is split into two parts $V^L(t,x) + V^S(t,x)$ where $V^L(t,\cdot) \in C(\mathbb{T}; C^\infty(\mathbb{R}^\nu))$ and there exists $\delta > 0$ such that

$$|\partial_x^\alpha V^L(t,x)| \le C_\alpha \langle x\rangle^{-\delta-|\alpha|}, \quad {}^\forall\alpha. \tag{1.7}$$

$V^S(t,\cdot)$ is compactly supported and $V^S(t,\cdot) \in C(\mathbb{T}; L^p(\mathbb{R}^\nu))$ with $p > \max\{\nu/2, 1\}$.

Under Assumption 1.2, we have the following results.

THEOREM 1.3. *Suppose Assumption 1.2 is satisfied. For $\lambda \in \mathbb{R}\setminus\mathbb{Z}$, let $d(\lambda,\mathbb{Z})$ denote the distance from $\lambda$ to $\mathbb{Z}$. Then,*

(i) *For all $0 < \delta < d(\lambda,\mathbb{Z})$ and $f \in C_0^\infty([\lambda-\delta,\lambda+\delta])$, there exists a compact operator $\tilde{C}$ such that the following inequality holds:*

$$f(K)i[K,A]^0 f(K) \ge \frac{2d(I,\mathbb{Z})}{d(I,\mathbb{Z})+1} f(K)^2 + \tilde{C}, \tag{1.8}$$

*where $I = [\lambda-\delta,\lambda+\delta]$ and $d(I,\mathbb{Z})$ is the distance from $I$ to $\mathbb{Z}$.*

(ii) *Eigenvalues of* $K$ *(the set of which are denoted by* $\sigma_{pp}(K)$*) are discrete with possible accumulation points in* $\mathbb{Z}$.

*If* $\lambda \in \mathbb{R} \setminus (\mathbb{Z} \cup \sigma_{pp}(K))$, *for each* $\epsilon > 0$ *there exists* $0 < \delta < d(\lambda, Z)$ *such that*

$$f(K)i[K,A]^0 f(K) \geq (\frac{2d(I,\mathbb{Z})}{d(I,\mathbb{Z})+1} - \epsilon) f(K)^2 \tag{1.9}$$

*for all* $f \in C_0^\infty([\lambda - \delta, \lambda + \delta])$.

Let $\mathfrak{B}(\mathbb{H})$ be the set of bounded operators on $\mathbb{H}$.

THEOREM 1.4. *Suppose* $\alpha > 1/2$.

(i) *For each closed interval* $I \subset \mathbb{R} \setminus (\mathbb{Z} \cup \sigma_{pp}(K))$ *the following inequalities hold:*

$$\sup_{\operatorname{Im} z \neq 0, \operatorname{Re} z \in I} \| \langle A \rangle^{-\alpha} (K - z)^{-1} \langle A \rangle^{-\alpha} \|_{\mathfrak{B}(\mathbb{H})} < \infty, \tag{1.10}$$

$$\sup_{\operatorname{Im} z \neq 0, \operatorname{Re} z \in I} \| \langle x \rangle^{-\alpha} (K - z)^{-1} \langle x \rangle^{-\alpha} \|_{\mathfrak{B}(\mathbb{H})} < \infty. \tag{1.11}$$

(ii) *There exist the norm limits in* $\mathfrak{B}(\mathbb{H})$.

$$\lim_{\operatorname{Im} z \to \pm 0, \operatorname{Re} z \in I} \langle A \rangle^{-\alpha} (K - z)^{-1} \langle A \rangle^{-\alpha},$$

$$\lim_{\operatorname{Im} z \to \pm 0, \operatorname{Re} z \in I} \langle x \rangle^{-\alpha} (K - z)^{-1} \langle x \rangle^{-\alpha}.$$

$\langle A \rangle^{-\alpha} (K - \lambda \mp i0)^{-1} \langle A \rangle^{-\alpha}$ *and* $\langle x \rangle^{-\alpha} (K - \lambda \mp i0)^{-1} \langle x \rangle^{-\alpha}$ *are Hölder continuous with respect to* $\lambda \in \mathbb{R} \setminus (\mathbb{Z} \cup \sigma_{pp}(K))$.

Next we proceed to the propagation estimates. We need the following stronger assumption on the potential.

ASSUMPTION 1.5. *There exists* $\delta_0 > 0$ *such that*

$$V(t, \cdot) \in C(\mathbb{T}; C^\infty(\mathbb{R}^\nu)), \quad |\partial_x^\alpha V(t,x)| \leq C_\alpha \langle x \rangle^{-\delta_0 - |\alpha|}, \quad {}^\forall \alpha. \tag{1.12}$$

THEOREM 1.6. *Suppose Assumption 1.5 is satisfied. Let* $E \in \mathbb{R} \setminus (\mathbb{Z} \cup \sigma_{pp}(K))$, *and* $\epsilon > 0$ *be given. Then there exists a small open interval* $I$ *containing* $E$ *such that for any* $f \in C_0^\infty(I)$ *and* $s' > s > 0$,

$$\left\| \chi\left( \frac{|x|^2}{4\sigma^2} - \frac{d(I,\mathbb{Z})}{d(I,\mathbb{Z})+1} < -\epsilon \right) e^{-i\sigma K} f(K) \langle x \rangle^{-s'} \right\|_{\mathfrak{B}(\mathbb{H})} = O(\sigma^{-s}) \quad \text{as} \quad \sigma \to \infty \tag{1.13}$$

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*where* $\chi(x<a)$ *denotes the characteristic function of the interval* $(-\infty, a)$.

## §2. Conjugate operator

We shall assume Assumption 1.2 throughout this section. We prove the following Lemma at first.

LEMMA 2.1. *Let* $A$ *be as in* 1.6. *Then* $e^{iA\alpha}$ *leaves* $D(K)$ *invariant, i.e. for each* $\Psi \in \mathbb{H}$

$$\sup_{|\alpha|<1} \|Ke^{iA\alpha}(K+i)^{-1}\Psi\|_{\mathbb{H}} < \infty. \tag{2.1}$$

*Proof.* As $V$ is $K_0$-compact, it is sufficient to show $e^{iA\alpha}$ leaves $D(K_0)$ invariant. Let $\mathfrak{F}$ be the Fourier transformation with respect to $x$, and we define $\hat{A}$ by

$$\hat{A} = \mathfrak{F} A \mathfrak{F}^{-1}. \tag{2.2}$$

Then $e^{i\hat{A}\alpha}$ can be expressed as

$$(e^{i\hat{A}\alpha}\psi)(t,p) = |\det(\frac{\partial \Gamma_\alpha^l}{\partial p_j}(p))|^{\frac{1}{2}}\psi(t, \Gamma_\alpha(p)), \tag{2.3}$$

where $\Gamma_\alpha(p) = (\Gamma_\alpha^l(p))_{1\le l\le \nu}$ is the solution of the following differential equation

$$\begin{cases} \frac{d}{d\alpha}\Gamma_\alpha(p) = (1+|\Gamma_\alpha(p)|^2)^{-1}\Gamma_\alpha(p), \\ \Gamma_0(p) = p. \end{cases} \tag{2.4}$$

We note $-i\frac{d}{dt}$ on $L^2(\mathbb{T})$ has eigenvalues $k \in \mathbb{Z}$. Let $P_k$ be the associated eigenprojection. Then $K_0$ can be decomposed as

$$K_0 = \sum_{k\in\mathbb{Z}} (k+H_0)\otimes P_k.$$

And for each $\Psi \in \mathbb{H}$

$$K_0 e^{iA\alpha}(K_0+i)^{-1}\Psi$$
$$= \mathfrak{F}^{-1}\left(\sum_{k\in\mathbb{Z}} \left|\det(\frac{\partial\Gamma_l}{\partial p_j})\right|^{\frac{1}{2}}(k+|p|^2+i)(k+|\Gamma_\alpha(p)|^2+i)^{-1}\otimes P_k\mathfrak{F}\Psi\right)$$

From (2.4) it is easily seen that $|\,|\Gamma_\alpha(p)|^2 - |p|^2| \le 2|\alpha|$, which proves the Lemma. □

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Once we have proved Lemma 2.1, we can trace the Mourre theory in the same way.

LEMMA 2.2. *For $K$ and $A$ defined above, the following facts hold.*

(i) $(K-z)^{-1}$ *leaves* $D(A)$ *invariant for all* $z \in \mathbb{C} \setminus \sigma(K)$.

(ii) $(A+i\lambda)^{-1}$ *leaves* $D(K)$ *invariant for all* $\lambda \in \mathbb{R}$, *and* $\lim_{|\lambda|\to\infty}(K+i)\, i\lambda(A+i\lambda)^{-1}(K+i)^{-1}\Psi = \Psi$ *for all* $\Psi \in \mathbb{H}$.

COROLLARY 2.3. (the Virial theorem) *For all* $\Psi \in D(K)$, $\lim_{|\lambda|\to\infty} i[K, i\lambda A(A+i\lambda)^{-1}]\Psi = i[K,A]^0\Psi$.

For the proof of Lemma 2.2 and Corollary 2.3, see [Mo].

*Proof of Theorem* 1.3. By the symbol calculus we have

$$\begin{aligned} i[K,A] &= i[H_0,A] + i[V,A] \\ &= 2H_0(H_0+1)^{-1} + i[V,A]. \end{aligned}$$

Let us recall the well-known formula of functional calculus [H-S]. Let $f \in C^\infty(\mathbb{R})$ be such that for some $m_0 \in \mathbb{R}$

$$|f^{(k)}(t)| \le C_k(1+|t|)^{m_0-k}, \quad \forall k \in \mathbb{N} \cup \{0\}. \tag{2.5}$$

Then we can construct an almost analytic extension $\tilde{f}(z)$ of $f(t)$ satisfying

$$\begin{aligned} &\tilde{f}(t) = f(t), \quad t \in \mathbb{R}, \\ &|\partial_{\bar{z}}\tilde{f}| \le C_N |\operatorname{Im} z|^N \langle z\rangle^{m_0-1-N}, \quad {}^\forall N \in \mathbb{N}, \\ &\operatorname{supp} \tilde{f}(z) \subset \{z; |\operatorname{Im} z| \le 1 + |\operatorname{Re} z|\}. \end{aligned}$$

We remark that $\operatorname{supp} \tilde{f}$ is compact in $\mathbb{C}$ if $f \in C_0^\infty(\mathbb{R})$ (due to Appendix in [Gé1]).
Further, if (2.5) holds with $m_0 < 0$ we have

$$f(K) = \frac{1}{2\pi i}\int_{\mathbb{C}} \partial_{\bar{z}}\tilde{f}(z)(z-K)^{-1}\, dz \wedge d\bar{z}. \tag{2.6}$$

We assume $\lambda \in (l, l+1)$ with some $l \in \mathbb{N}$. From Assumption 1.2 and the above formula, $f(K) - f(K_0)$ is a compact operator. Therefore we have

$$\begin{aligned} f(K)i[K,A]^0 f(K) &= 2f(K)H_0(H_0+1)^{-1}f(K) + f(K)i[V,A]^0 f(K) \\ &= 2f(K_0)H_0(H_0+1)^{-1}f(K_0) + (\text{compact operator}). \end{aligned}$$

By decomposing $K_0$ as $\sum_{k\in\mathbb{Z}}(k+H_0)\otimes P_k$ again

$$2f(K_0)H_0(H_0+1)^{-1}f(K_0) = 2\sum_{k\in\mathbb{Z}} H_0(H_0+1)^{-1}f(k+H_0)^2 \otimes P_k \tag{2.7}$$

Since $\operatorname{supp} f(k+\cdot) \subset [\lambda-\delta-k, \lambda+\delta-k]$ and $\frac{t}{t+1}$ is a monotone increasing function for $t \geq 0$, we have the following inequality

$$\begin{aligned} f(K_0)H_0(H_0+1)^{-1}f(K_0) &\geq \sum_{k\leq l} \frac{\lambda-\delta-l}{\lambda-\delta-l+1} f(k+H_0)^2 \otimes P_k \\ &\geq \frac{d(I,\mathbb{Z})}{d(I,\mathbb{Z})+1} f(K_0)^2, \end{aligned}$$

which proves(1). By shrinking $\operatorname{supp} f$ we also obtain (2).

We omit the proof of Theorem 1.4. Since it follows from Theorem 1.3 by the well-known arguments.

LEMMA 2.4. *Let $V(t,x)$ be as in Example 1. Then as a multiplicative operator, $V = V(t,x)$ satisfies Assumption 1.2.*

*Proof.* As was proved by Yajima (Lemma 3.1 in [Ya]), if $W(t,x) \in C(\mathbb{T}; L^p(\mathbb{R}^\nu))$ with $p > \max\{\nu/2, 1\}$, $W$ is $K_0$-compact. $K_0$-compactness of $[V^s, A]$ and $K_0$-boundness of $[[V^s, A], A]$ also hold as we take $V^s(t,x)$ supported in a compact set. One can also see the following fact: For any $\delta > 0$, $\langle x\rangle^{-\delta}$ is a $K_0$-compact operator. In fact, we have only to approximate $\langle x\rangle^{-\delta}$ by a compactly supported function. It indicates that $V^L$ is $K_0$-compact. For the sake of convenience, we write $V$ and $D_j$ instead of $V^L$ and $D_{x_j}$. It is sufficient to show that $[V, X_jL_j]$ is $K_0$-compact, and $[[V, X_jL_j], X_kL_k]$ is $K_0$-bounded. Here $1 \leq j,\ k \leq \nu$ and $X_j$ is a multiplicative operator by a function $x_j$. We denote $x_jV(t,x)$ as $V_j(t,x)$. At first we split the commutator into two parts

$$\begin{aligned} [V, X_jL_j] &= [V_j, L_j] + [L_j, X_j]V \\ &\equiv I_1 + I_2. \end{aligned}$$

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From the assumption we assume in Example 1, we can easily see that $I_2\langle x\rangle^{\delta} \in \mathfrak{B}(\mathbb{H})$. For $I_1$, we split it again

$$\begin{aligned} I_1 &= \langle D_x\rangle^{-2}\{H_0V_jD_j - D_jV_jH_0\}\langle D_x\rangle^{-2} + \langle D_x\rangle^{-2}[V_j, D_j]\langle D_x\rangle^{-2}, \\ &\equiv I_3 + I_4. \end{aligned}$$

We use the Assumption for $V^L$ to see $I_4\langle x\rangle^{1+\delta} \in \mathfrak{B}(\mathbb{H})$. We can rewrite $I_3$ as

$$\langle D_x\rangle^{-2}\{(-\Delta V_j)D_j - 2((\nabla V_j)\cdot\nabla)D_j + [V_j, D_j]H_0\}\langle D_x\rangle^{-2}.$$

We use the Assumption for $V^L$ again to prove that $[V^L, A]$ is $K_0$-compact. As for the double commutator, we compute

$$[[V, X_jL_j], X_kL_k] = [I_2 + I_3 + I_4, X_kL_k].$$

We can easily obtain the following result by using the pseudo differential calculus, as we commute $X_kD_k$ with $V$ or another PsDO.

$$[I_\alpha, X_kL_k] \text{ is } K_0\text{-compact for } \alpha = 2, 3, 4.$$

□

## §3. Propagation estimate

We shall prove Theorem 1.6 in this section. We follow the abstract framework constructed by Skibsted [Sk]. In our case $K$ is not a semi-bounded operator, which introduces a slight difference in applying the method of [Sk]. From Assumption 1.5, it follows that $[K, A]$ is extended to a bounded operator. We add this condition as an additional assumption in the abstract framework.

Definition 3.1. Given $\beta$, $\alpha \geq 0$ and $\epsilon > 0$, we denote by $\mathfrak{F}_{\beta,\alpha,\epsilon}$ as the set of function $g$ of the form, $g(x,\tau) = g_{\beta,\alpha,\epsilon}(x,\tau) = -\tau^{-\beta}(-x)^{\alpha}\chi(\frac{x}{\tau})$ defined for $(x,\tau) \in \mathbb{R}\times\mathbb{R}^+$, where $\chi \in C^\infty(\mathbb{R})$ and satisfies the following properties:

$$\chi(x) = 1 \text{ for } x < -2\epsilon,\ \chi(x) = 0 \text{ for } x > -\epsilon.$$
$$\frac{d}{dx}\chi(x) \leq 0 \text{ and } \alpha\chi(x) + x\frac{d}{dx}\chi(x) = \tilde{\chi}(x)^2 \text{ for some } \tilde{\chi} \in C^\infty(\mathbb{R}),\ \tilde{\chi} \geq 0$$

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It follows from the last equation that $(g^{(1)}(x,\tau))^{\frac{1}{2}}$ is smooth. Here $g^{(n)}(x,\tau) = (\partial/\partial x)^n g(x,\tau)$. For operators $P$ and $Q$, we define $\mathrm{ad}_Q^0(P) = P$ and for $m \in \mathbb{N}$, $\mathrm{ad}_Q^m(P) = [\mathrm{ad}_Q^{m-1}(P), Q]$ inductively.

LEMMA 3.2. *Let $A$ and $P$ be linear operators on $\mathbb{H}$. Suppose $A$ is self-adjoint and $P$-bounded. Suppose that the form $\mathrm{ad}_A^m(P)$ extends to a bounded operator for $1 \le m \le n$. Then for any $g \in C^\infty(\mathbb{R})$ satisfying* 2.5 *with $m_0 < n$*

(i)

$$Pg(A) = \sum_{m=0}^{n-1} \frac{g^{(m)}(A)}{m!}\,\mathrm{ad}_A^m(P) + \frac{1}{2\pi i}\int_{\mathbb{C}} \partial_{\bar z}\tilde g(z) R^r_{n,A,P}(z)\,dz\wedge d\bar z, \tag{3.1}$$

*where* $R^r_{n,A,P}(z) = (z-A)^{-n}\,\mathrm{ad}_A^n(P)(z-A)^{-1}$.

(ii)

$$g(A)P = \sum_{m=0}^{n-1} \mathrm{ad}_A^m(P)\frac{(-1)^m}{m!}g^{(m)}(A) + \frac{1}{2\pi i}\int_{\mathbb{C}} \partial_{\bar z}\tilde g(z) R^l_{n,A,P}(z)\,dz\wedge d\bar z, \tag{3.2}$$

*where $R^l_{n,A,P}(z) = (z-A)^{-1}\,\mathrm{ad}_A^n(P)(A-z)^{-n}$ and $\tilde g(z)$ denotes an almost analytic extension of $g(x)$.*

These formulas of asymptotic expansion are obtained by virtue of (2.6) and the calculus of the commutator $[(z-A)^{-1}, P]$. (See Lemma 3.3 in [Gé2].)

ASSUMPTION 3.3. *Let $n_0 \in \mathbb{N}$, $\sigma_0 > 0$, $n_0 - \frac{1}{2} > \alpha_0 > 0$. Let $f$, $f_2 \in C_0^\infty(\mathbb{R})$ be such that $f_2 f = f$ and $K$, $A(\tau)$, $B$ be self-adjoint operators on $\mathbb{H}$. Assume that $A(\tau)$ have common domain $D$ for $\tau = \sigma + \sigma_0$ $\sigma \ge 0$, $D(K)\cap D$ is dense in $D(K)$, $B \ge I$ and that $\langle A_0\rangle^{\frac{n_0}{2}}\langle B\rangle^{-\frac{n_0}{2}} \in \mathfrak{B}(\mathbb{H})$ with $A_0 = A(\sigma_0)$. Assume moreover*

(i) *With $1 \le n \le n_0$, $i^n\,\mathrm{ad}^n_{A(\tau)}(K)$ extends to a bounded self-adjoint operator, and $\mathrm{ad}^n_{A(\tau)}(K) = O(1)$ in $\mathfrak{B}(\mathbb{H})$ as $\tau \to \infty$.*

(ii) *If $A(\tau)$ is unbounded, $\sup_{|\alpha|<1}\|Ke^{iA(\tau)\alpha}\psi\|_{\mathbb{H}} < \infty$ for any $\psi \in D(K)$ and $\tau \ge \sigma_0$.*

(iii) *For each* $\tau_1$, $\tau_2 \geq \sigma_0$, $A(\tau_1) - A(\tau_2)$ *is a bounded operator, and the derivative* $d_\tau A(\tau) = \frac{d}{d\tau}A(\tau)$ *exists in* $\mathfrak{B}(\mathbb{H})$. *Further* $\mathrm{ad}^{n-1}_{A(\tau)}(d_\tau A(\tau)) = O(1)$ *in* $\mathfrak{B}(\mathbb{H})$ *as* $\tau \to \infty$ *for* $1 \leq n \leq n_0$.

(iv) *For* $n \leq n_0$ $\mathrm{ad}^n_{A(\tau)}(K)$ *and* $\mathrm{ad}^{n-1}_{A(\tau)}(d_\tau A(\tau))$ *are continuous* $\mathfrak{B}(\mathbb{H})$*-valued functions of* $\tau \geq \sigma_0$.

(v) *There exists* $\delta > 0$ *such that the following condition* $q(\beta_0, \alpha_0, \delta)$ *holds.* $q(\beta_0, \alpha_0, \delta)$: *Let* $DA(\tau)$ *denote the symmetric operator* $i[K, A(\tau)] + d_\tau A(\tau)$.
*There exist bounded operators* $B_1(\tau)$ *and* $B_2(\tau)$ *on* $\mathbb{H}$ *such that*

$$f_2(K)DA(\tau)f_2(K) \geq B_1(\tau) + B_2(\tau). \tag{3.3}$$

$\|B_1(\tau)\|_{\mathfrak{B}(\mathbb{H})} = O(\tau^{-\delta})$ *as* $\tau \to \infty$, *and for* $(\beta, \alpha) = (0, 1), \ldots, (0, \alpha_0')$, $(\beta_0, \alpha_0)$ $(\alpha_0' = \max\{m \in \mathbb{N}: m < \alpha_0\})$ $(= (\beta_0, \alpha_0)$ *if* $\alpha_0 < 1)$ *the following estimates holds:*
*Given* $\epsilon > 0$ *and* $g(x, \tau) \in \mathfrak{F}_{\beta,\alpha,\epsilon}$, *there exists* $C > 0$ *such that with* $\zeta(\sigma) = (g^{(1)}(A(\tau), \tau))^{\frac{1}{2}} e^{-i\sigma K} f(K) B^{-\frac{\alpha}{2}}\phi$

$$\int_0^\infty d\sigma |(\zeta(\sigma), B_2(\tau)\zeta(\sigma))_{\mathbb{H}}| \leq C\|\phi\|^2, \qquad {}^\forall\phi \in \mathbb{H}, \tag{3.4}$$

*where* $(\cdot,\cdot)_{\mathbb{H}} = (\cdot,\cdot)$ *is the inner product of* $\mathbb{H}$.

THEOREM 3.4. *Suppose Assumption* 3.3 *is satisfied and in addition,*

$$\begin{aligned} &\alpha_0 + 2 < \beta_0 + n_0, \\ &\alpha_0' + 2 < n_0, \qquad (\textit{if} \quad \alpha_0 > 1) \\ &\frac{\alpha_0}{2} + \frac{5}{2} < n_0 + \beta_0, \\ &\alpha_0' + \frac{5}{2} \leq n_0, \qquad (\textit{if} \quad \alpha_0 > 1). \end{aligned} \tag{3.5}$$

*Then for* $(\beta, \alpha) = (0, 1), \ldots, (0, \alpha_0'), (\beta_0, \alpha_0)$ $(= (\beta_0, \alpha_0)$ *if* $\alpha_0 < 1)$, *any* $\epsilon > 0$ *and* $g(x, \tau) \in \mathfrak{F}_{\beta,\alpha,\epsilon}$,

$$\|(-g_{\beta,\alpha,\epsilon}(A(\tau), \tau))^{\frac{1}{2}} e^{-i\sigma K} f(K) B^{-\frac{\alpha}{2}}\|_{\mathfrak{B}(\mathbb{H})} = O(1) \quad \textit{as} \quad \tau \to \infty \tag{3.6}$$

COROLLARY 3.5. *Under the same conditions in Theorem* 3.4, *we have the following result:*
*For* $(\beta, \alpha) = (0,1), \dots, (0, \alpha_0'), (\beta_0, \alpha_0)$, *any* $\epsilon > 0$, $g(x,\tau) \in \mathfrak{F}_{\beta,\alpha,\epsilon}$, *and* $1 \geq \theta \geq 0$

$$\begin{aligned}(3.7)\quad &\|(-g_{0,\alpha(1-\theta),\epsilon}(A(\tau),\tau))^{\frac{1}{2}} e^{-i\sigma K} f(K) B^{-\frac{\alpha}{2}}\|_{\mathfrak{B}(\mathbb{H})} \\ &\qquad = O(\tau^{(\beta-\alpha\theta)/2}) \quad \textit{as} \quad \tau \to \infty.\end{aligned}$$

We note that (3.7) is easily obtained by (3.6) and the inequality

$$-\tau^{-\beta}(\epsilon\tau)^{\alpha\theta} g_{0,\alpha(1-\theta),2\epsilon}(x,\tau) \leq -g_{\beta,\alpha,\epsilon}(x,\tau).$$

*Sketch of Proof.* The proof of Theorem 3.4 is almost the same as that of Theorem 2.4 in [Sk]. Let $f_1 \in C_0^\infty(\mathbb{R})$ be real valued and satisfy $f_1 f_2 = f_2$. We denote $\psi(\sigma) = e^{-i\sigma K} f(K) B^{-\alpha/2}\phi$, and $D_1 A(\tau) = d_\tau A(\tau) + i[f_1(K)K, A(\tau)]$. Then $(\psi(\sigma), g(A(\tau),\tau)\psi(\sigma))$ is continuously differentiable with

$$(3.8)\qquad \frac{d}{d\sigma}(\psi(\sigma), g(A(\tau),\tau)\psi(\sigma)) = (\psi(\sigma), Dg(A(\tau),\tau)\psi(\sigma)),$$

where

$$\begin{aligned}Dg(A(\tau),\tau) &= (\frac{\partial}{\partial\tau} g)(A(\tau),\tau) + \sum_{m=1}^{n_0-1} (m!)^{-1} g^{(m)}(A(\tau),\tau)\, \mathrm{ad}_{A(\tau)}^{m-1}(D_1 A(\tau)) \\ &+ \frac{1}{2\pi i}\int_{\mathbb{C}} \partial_{\bar z}\tilde g(z,\tau)(z - A(\tau))^{-n_0}\, \mathrm{ad}_{A(\tau)}^{n_0-1}(D_1 A(\tau))(z - A(\tau))^{-1}\, dz \wedge d\bar z.\end{aligned}$$

We can then prove that $(\psi(\sigma), Dg(A(\tau),\tau)\psi(\sigma))$ is integrable with respect to $\tau$, which indicates the assertion of Theorem 3.4. Corollary 3.5 follows from the same argument as in [Sk]. □

With these results, we proceed to prove the propagation estimate for operator $K$ with potential $V$ satisfying Assumption 1.5.

Suppose $E \in \mathbb{R} \setminus \mathbb{Z}$ and $0 < E' < \frac{d(E,\mathbb{Z})}{d(E,\mathbb{Z})+1}$. We choose $f$ and $f_2$ as in Assumption 3.3, with support in a small interval $I \subset \mathbb{R} \setminus \mathbb{Z}$. Put $\sigma_0 = 1$, $A_1(\tau) = A - 2E'\tau$ $(\tau = \sigma + 1)$, and $B = \langle A_1(1) \rangle$.

By virtue of Lemma 2.1 and some elementary calculus one can prove that $A_1(\tau)$ verifies Assumption 3.3 with arbitrary $n_0$, $\alpha_0$, $\beta_0$. By the same argument as in the proof of Corollary 3.5 we have that:

$$(3.9)\qquad \|(-\frac{A_1(\tau)}{\tau})^{\frac{1}{2}} \chi(\frac{A_1(\tau)}{\tau}) e^{-i\sigma K} f(K) B^{-s}\|_{\mathfrak{B}(\mathbb{H})} = O(\tau^{-s}) \text{ as } \tau \to \infty$$

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for $s \geq \frac{1}{2}$.

LEMMA 3.6. *Fix* $0 < E'' < E' < \frac{d(E,\mathbb{Z})}{d(E,\mathbb{Z})+1}$. *Let* $f_2$, $f$, $\sigma_0$, $\beta_0$ *and* $\alpha_0$ *as above. For an arbitrary fixed* $\epsilon'' > 0$ *we take* $g \in \mathfrak{F}_{0,1,\epsilon''}$ *satisfying* $(-g(x,\tau))^{\frac{1}{2}}$, $(-(\frac{\partial}{\partial\tau}g)(x,\tau))^{\frac{1}{2}} \in C^\infty(\mathbb{R}\times\mathbb{R}^+)$. *We put* $M(x,\xi) = (E'' - \frac{|x|^2}{4\langle\xi\rangle^2})^{\frac{1}{2}}$, $G = (-g(-\tau M(\frac{x}{\tau},\xi),\tau))^{\frac{1}{2}}_{|\xi=D_x}$ *and set* $A_2(\tau) = -G^*G$

*Then for all* $\beta_0$, $\alpha_0$, $n_0$, *there exists* $\delta > 0$ *such that* $A_2(\tau)$ *satisfies Assumption* 3.3.

Before the proof of this Lemma, we introduce a symbol class and asymptotic expansion formulas.

DEFINITION 3.7. For $l$, $m \in \mathbb{R}$, let $S(\tau^l\langle\xi\rangle^m)$ be the set of functions $a_\tau(x,\xi) \in C^\infty(\mathbb{R}^\nu_x \times \mathbb{R}^\nu_\xi)$ such that

$$|\partial_x^\alpha\partial_\xi^\beta a_\tau(x,\xi)| \leq C_{\alpha\beta}\tau^{l-|\alpha|}\langle\xi\rangle^{m-|\beta|}, \qquad (x,\xi) \in \mathbb{R}^\nu_x \times \mathbb{R}^\nu_\xi$$

for all multi-indexes $\alpha$, $\beta$.

We write $a_\tau(x,D_x) \in \operatorname{Op} S(\tau^l\langle D_x\rangle^m)$, if $a_\tau(x,\xi) \in S(\tau^l\langle\xi\rangle^m)$.

LEMMA 3.8. *Suppose* $a_\tau(x,\xi) \in S(\tau^l\langle\xi\rangle^m)$, *and* $b_\tau(x,\xi) \in S(\tau^{l'}\langle\xi\rangle^{m'})$. *Then* $a_\tau(x,D_x)^* \in \operatorname{Op} S(\tau^l\langle D_x\rangle^m)$ *and* $a_\tau(x,D_x)b_\tau(x,D_x) \in \operatorname{Op} S(\tau^{l+l'}\langle D_x\rangle^{m+m'})$. *We have the following asymptotic formulas.*

$$(3.10)\ a_\tau(x,D_x)^* - \sum_{|\alpha|<N}\frac{1}{\alpha!}\overline{a_\tau}^{(\alpha)}_{(\alpha)}(x,\xi)_{|\xi=D_x} \in \operatorname{Op} S(\tau^{l-N}\langle D_x\rangle^{m-N}),$$

*where* $p^{(\alpha)}_{(\beta)}(x,\xi) = D_\xi^\alpha\partial_x^\beta p(x,\xi)$,

$$(3.11)\quad \begin{aligned} &a_\tau(x,D_x)b_\tau(x,D_x) - \sum_{|\alpha|<N}\frac{1}{\alpha!}a_\tau{}^{(\alpha)}(x,\xi)b_{\tau(\alpha)}(x,\xi)_{|\xi=D_x} \\ &\qquad\qquad \in \operatorname{Op} S(\tau^{l+l'-N}\langle D_x\rangle^{m+m'-N}). \end{aligned}$$

*Proof of Lemma* 3.6. We rewrite $DA_2(\tau) = -(d_\tau G)^*G - G^*(d_\tau G) + i[K, A_2(\tau)]$. Let $M$ denote $M(\frac{x}{\tau},\xi)$. It can be easily verified that $G \in S(\tau^{\frac{1}{2}})$ and

$$(3.12)\quad \begin{aligned} (d_\tau G)^*G = -\frac{1}{2}\Big\{(\frac{\partial}{\partial\tau}g)(-\tau M,\tau) - g^{(1)}(-\tau M,\tau)E''M^{-1}\Big\}_{|\xi=D_x} \\ + \operatorname{Op} S(\tau^{-1}). \end{aligned}$$

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Using the assumption for $g$ in Lemma 3.6, we have

$$(3.13)\quad -(\frac{\partial}{\partial\tau}g)(-\tau M,\tau)_{|\xi=D_x} = \left\{(-\frac{\partial}{\partial\tau}g)^{\frac{1}{2}}_{|\xi=D_x}\right\}^*\left\{(-\frac{\partial}{\partial\tau}g)^{\frac{1}{2}}_{|\xi=D_x}\right\} + \mathrm{Op}\,S(\tau^{-1}).$$

The last term $i[K, A_2(\tau)]$ has the following expression

$$(3.14)\quad i[K, A_2(\tau)] = \left\{g^{(1)}(-\tau M,\tau)M^{-1}\cdot\frac{1}{2\tau}\frac{x\cdot\xi}{\langle\xi\rangle^2}\right\}_{|\xi=D_x} + i[V, A_2(\tau)] + \mathrm{Op}\,S(\tau^{-1})$$

We denote $(g^{(1)}(-\tau M,\tau)M^{-1})^{\frac{1}{2}}{}_{|\xi=D_x}$ as $g_H(x, D_x) \in \mathrm{Op}\,S(1)$. We also remark that $\frac{1}{\tau}\frac{x\cdot\xi}{\langle\xi\rangle^2}g_H(x,\xi) \in S(1)$. We can rewrite the right hand side of (3.14) as

$$\frac{1}{2}g_H(x, D_x)^*(\frac{A_1(\tau)}{\tau} + 2E')g_H(x, D_x) + i[V, A_2(\tau)] + R_0(\tau),$$

where $\|R_0(\tau)\|_{\mathfrak{B}(\mathbb{H})} = O(\tau^{-1})$ as $\tau \to \infty$.
For $i[V, A_2(\tau)]$, we obtain $\|[V, A_2(\tau)]\|_{\mathfrak{B}(\mathbb{H})} = O(\tau^{-\delta_0})$ by computing $\nabla_x V^L \cdot \nabla_\xi(g(-\tau M,\tau))$.

Summing up, we have

$$(3.15)\quad DA_2(\tau) \geq \frac{1}{2}g_H(x, D_x)^*(\frac{A_1(\tau)}{\tau} + 2(E' - E''))g_H(x, D_x) + R_1(\tau),$$

where $\delta_1 = \min\{\delta_0, 1\}$ and $\|R_1(\tau)\|_{\mathfrak{B}(\mathbb{H})} = O(\tau^{-\delta_1})$ as $\tau \to \infty$.

Since

$$(3.16)\quad \frac{A_1(\tau)}{\tau} + 2(E' - E'') \geq \frac{A_1(\tau)}{\tau}\chi(\frac{A_1(\tau)}{\tau}),$$

we can replace $\frac{A_1(\tau)}{\tau} + 2(E' - E'')$ by $\frac{A_1(\tau)}{\tau}\chi(\frac{A_1(\tau)}{\tau})$ with $\epsilon = E' - E''$. Thus it suffices to prove $(-\frac{A_1(\tau)}{\tau})^{1/2}\chi(\frac{A_1(\tau)}{\tau})g_H(x, D_x)f_2(K)(g^{(1)}_{\beta,\alpha,\epsilon}(A_2(\tau),\tau))^{1/2}$ $e^{-i\sigma K}f(K)B^{-\alpha/2}$ is square integrable.

For $l \in \mathbb{N}\cup\{0\}$, we put $g_l(x,\tau) = (\frac{\partial}{\partial x})^l((-\frac{x}{\tau})^{\frac{1}{2}}\chi(\frac{x}{\tau}))$ and we write the

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almost analytic extension of $g_l(x,\tau)$ as $\tilde{g}_l(z,\tau)$. From (3.2)

$$
\begin{aligned}
&(-\frac{A_1(\tau)}{\tau})^{\frac{1}{2}}\chi(\frac{A_1(\tau)}{\tau})g_H(x,D_x) \\
&\quad = \sum_{m=0}^{n_0-1}\frac{(-1)^m}{m!}\,\mathrm{ad}^m_{A_1(\tau)}(g_H(x,D_x))g_0^{(m)}(A_1(\tau),\tau) \\
&\qquad + \frac{1}{2\pi i}\int_{\mathbb{C}}\partial_{\bar z}\tilde{g}_0(z,\tau)R^l_{n_0,A_1(\tau),g_H(x,D_x)}(z)\,dz\wedge d\bar{z}.
\end{aligned}
\tag{3.17}
$$

By the symbol calculus of PsDO, we have

$$
\|\,\mathrm{ad}^m_{A_1(\tau)}(g_H(x,D_x))\|_{\mathfrak{B}(\mathbb{H})} = O(1) \quad \text{as} \quad \tau\to\infty \quad \text{for all} \quad 0\le m\le n_0.
$$

The last term in the right hand side of (3.17) is dominated from above by

$$
\begin{aligned}
&\int_{|z|\ge\epsilon''\tau}\tau^{-n_0-1}\langle\frac{z}{\tau}\rangle^{-3/2-n_0}|\,dz\wedge d\bar{z}|\cdot\|\,\mathrm{ad}^{n_0}_{A_1(\tau)}(g_H(x,D_x))\| \\
&\qquad = O(\tau^{1-n_0})
\end{aligned}
\tag{3.18}
$$

So it remains to prove that for $0\le m\le n_0$

$$
g_m(A_1(\tau),\tau)f_2(K)\left(g^{(1)}_{\beta,\alpha,\epsilon}(A_2(\tau),\tau)\right)^{\frac{1}{2}}e^{-i\sigma K}f(K)B^{-\frac{\alpha}{2}}
\tag{3.19}
$$

is square integrable. We apply (3.2) again to see that this is equal to

$$
\begin{aligned}
&\left\{\sum_{l=0}^{n_0-1}\frac{(-1)^l}{l!}\,\mathrm{ad}^l_{A_1(\tau)}(f_2(K))g_m^{(l)}(A_1(\tau),\tau)+O_{\mathfrak{B}(\mathbb{H})}(\tau^{1-n_0})\right\} \\
&\qquad \times(g^{(1)}_{\beta,\alpha,\epsilon}(A_2(\tau),\tau))^{\frac{1}{2}}e^{-i\sigma K}f(K)B^{-\frac{\alpha}{2}}.
\end{aligned}
\tag{3.20}
$$

Here we note that $\tau^{1-n_0}(g^{(1)}_{\beta,\alpha,\epsilon}(A_2(\tau),\tau))^{\frac{1}{2}}$ is square integrable with respect to $\tau$ because of the assumption (3.5) and the fact $\sup_{\tau\ge 1}\|\frac{A_2(\tau)}{\tau}\|_{\mathfrak{B}(\mathbb{H})} = L < \infty$. Again using (3.2) we have

$$
\begin{aligned}
&g_{m+l}(A_1(\tau),\tau)(g^{(1)}_{\beta,\alpha,\epsilon}(A_2(\tau),\tau))^{\frac{1}{2}} \\
&\quad = \sum_{j=0}^{n_0-1}\frac{(-1)^j}{j!}\,\mathrm{ad}^j_{A_1(\tau)}((g^{(1)}_{\beta,\alpha,\epsilon}(A_2(\tau),\tau))^{\frac{1}{2}})g^{(j)}_{m+l}(A_1(\tau),\tau) \\
&\qquad + \frac{1}{2\pi i}\int_{\mathbb{C}}\partial_{\bar z}\tilde{g}_{m+l}(z,\tau)R^l_{n_0,A_1(\tau),g^{(1)}}(z)\,dz\wedge d\bar{z}
\end{aligned}
\tag{3.21}
$$

We rewrite $(g^{(1)}_{\beta,\alpha,\epsilon}(x,\tau))^{\frac{1}{2}}$ as $\tau^{\frac{1}{2}(-\beta+\alpha-1)}(-\frac{x}{\tau})^{\alpha/2-1/2}\tilde{\chi}(\frac{x}{\tau})$ and put

$$h_\tau(x) = \tau^{\frac{1}{2}(-\beta+\alpha-1)}(-x)^{\alpha/2-1/2}\tilde{\chi}(x). \tag{3.22}$$

Let $\rho(x) \in C_0^\infty(\mathbb{R})$ be real valued and satisfies $\rho(x) \equiv 1$ on $|x| \leq L+1$. By constructing an almost analytic extension of $h_\tau(x)\rho(x)$, which we denote by $\tilde{h}_\tau(z)$, we have

$$(g^{(1)}_{\beta,\alpha,\epsilon}(A_2(\tau),\tau))^{\frac{1}{2}} = \frac{1}{2\pi i}\int_{\mathbb{C}} \partial_{\bar{z}}\tilde{h}_\tau(z)(z-\frac{A_2(\tau)}{\tau})^{-1}\, dz\wedge d\bar{z}, \tag{3.23}$$

$$\begin{aligned} &\mathrm{ad}^j_{A_1(\tau)}((g^{(1)}_{\beta,\alpha,\epsilon}(A_2(\tau),\tau))^{\frac{1}{2}}) \\ &\qquad = \frac{1}{2\pi i}\int_{\mathbb{C}} \partial_{\bar{z}}\tilde{h}_\tau(z)\,\mathrm{ad}^j_{A_1(\tau)}((z-\frac{A_2(\tau)}{\tau})^{-1})\, dz\wedge d\bar{z}. \end{aligned} \tag{3.24}$$

By induction, we can see that for $\mathrm{Im}\, z \neq 0$

$$\|\,\mathrm{ad}^j_{A_1(\tau)}((z-\frac{A_2(\tau)}{\tau})^{-1})\| \leq C_j|\,\mathrm{Im}\, z|^{-j-1}, \tag{3.25}$$

where $C_j$ is independent of $\tau$. Combining (3.24) and (3.25)

$$\|\,\mathrm{ad}^j_{A_1(\tau)}((g^{(1)}_{\beta,\alpha,\epsilon}(A_2(\tau),\tau))^{\frac{1}{2}})\|_{\mathfrak{B}(\mathbb{H})} = O(\tau^{(-\beta+\alpha-1)/2}) \quad \text{as} \quad \tau\to\infty. \tag{3.26}$$

Using (3.2) we compute

$$\begin{aligned} &g_{m+l}(A_1(\tau),\tau)(g^{(1)}_{\beta,\alpha,\epsilon}(A_2(\tau),\tau))^{\frac{1}{2}}e^{-i\sigma K}f(K)B^{-\frac{\alpha}{2}} \\ &\qquad = O(\tau^{\frac{1}{2}(-\beta+\alpha-1)})\sum_{j=0}^{n_0-1} g^{(j)}_{m+l}(A_1(\tau),\tau)e^{-i\sigma K}f(K)B^{-\frac{\alpha}{2}} \\ &\qquad\qquad + O(\tau^{1-n_0+(-\beta+\alpha-1)/2})e^{-i\sigma K}f(K)B^{-\frac{\alpha}{2}}. \end{aligned} \tag{3.27}$$

Here we apply (3.9) with $B = \langle A_1(1)\rangle^{1+\kappa}$ ($\kappa > 0$). Then

$$g^{(j)}_{m+l}(A_1(\tau),\tau)e^{-i\sigma K}f(K)B^{-\frac{\alpha}{2}} = O(\tau^{-\alpha(1+\kappa)/2}) \tag{3.28}$$

So we have proved

$$\begin{aligned} &(-\frac{A_1(\tau)}{\tau})^{\frac{1}{2}}\chi(\frac{A_1(\tau)}{\tau})g_H(x,D_x)f_2(K)(g^{(1)}_{\beta,\alpha,\epsilon}(A_2(\tau),\tau))^{\frac{1}{2}}e^{-i\sigma K}f(K)B^{-\frac{\alpha}{2}} \\ &\qquad = O(\tau^{-\frac{1}{2}-\frac{\alpha\kappa}{2}}) \end{aligned}$$

is square integrable in $\tau$. ∎

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Hence the conclusions of Theorem 3.4 and Corollary 3.5 hold. i.e.

$$\|\chi(\frac{A_2(\tau)}{\tau})e^{-i\sigma K}f(K)\langle A\rangle^{-s'}\|_{\mathfrak{B}(\mathbb{H})} = O(\tau^{-s}) \tag{3.29}$$

for all $0 < s < s'$ as $\tau \to \infty$

Our final aim is to change the weight in (3.29) by functions of $x$.

*Proof of Theorem* 1.6. It follows from (3.29) that

$$\|\chi(\frac{A_2(\tau)}{\tau} < -\epsilon)e^{-i\sigma K}f(K)\langle x\rangle^{-s'}\|_{\mathfrak{B}(\mathbb{H})} = O(\sigma^{-s}) \quad \text{as} \quad \sigma \to \infty. \tag{3.30}$$

Therefore Theorem 1.6 is proved if we show for any $N \in \mathbb{N}$,

$$\begin{aligned} &\chi(\frac{|x|^2}{4\tau^2} - \frac{d(I,\mathbb{Z})}{d(I,\mathbb{Z})+1} < -\epsilon)\chi(\frac{A_2(\tau)}{\tau} < -\epsilon) \\ &\quad = \chi(\frac{|x|^2}{4\tau^2} - \frac{d(I,\mathbb{Z})}{d(I,\mathbb{Z})+1} < -\epsilon) + O_{\mathfrak{B}(\mathbb{H})}(\tau^{-N}) \quad \text{as} \quad \sigma \to \infty. \end{aligned} \tag{3.31}$$

Again we use an almost analytic extension of $\chi\rho$(denoted by $\tilde{\chi}$) and

$$\chi(\frac{A_2(\tau)}{\tau}) = \frac{1}{2\pi i}\int_{\mathbf{C}} \partial_{\bar{z}}\tilde{\chi}(z)(z - \frac{A_2(\tau)}{\tau})^{-1}\, dz \wedge d\bar{z}. \tag{3.32}$$

We denote the symbol of $\frac{A_2(\tau)}{\tau}$ as $a_\tau(x,\xi)$.
Then

$$R_\tau(x,\xi) = a_\tau(x,\xi) - \frac{1}{\tau}g(-\tau M,\tau) \in S(\tau^{-1}\langle\xi\rangle^{-1}). \tag{3.33}$$

We construct a parametrix of $(z - \frac{A_2(\tau)}{\tau})$ by putting

$$\begin{cases} q_0(x,\xi) = (-\frac{1}{\tau}g(-\tau M,\tau) + z)^{-1} \\ q_j(x,\xi) = -\sum_{\substack{j'+|\alpha|=j \\ j'<j}} \frac{1}{\alpha!}(-\frac{1}{\tau}g(-\tau M,\tau) + z)^{(\alpha)} q_{j'(\alpha)}q_0 \\ \qquad\qquad - \sum_{j'+|\alpha|=j-1} \frac{1}{\tau}R_\tau^{(\alpha)} q_{j(\alpha)}q_0 \quad (j \geq 1) \end{cases} \tag{3.34}$$

Then

$$(z - \frac{A_2(\tau)}{\tau})\sum_{j=0}^{N} q_j(x,D_x) - \mathbf{I} \in \operatorname{Op} S(\tau^{-N}). \tag{3.35}$$

Moreover we have the following estimates: There exists $l \gg 1$ such that

$$\|(z - \frac{A_2(\tau)}{\tau}) \sum_{j=0}^{N} q_j - \mathbf{I}\|_{\mathfrak{B}(\mathbb{H})} \leq C\tau^{-N} |\operatorname{Im} z|^{-N-l}. \tag{3.36}$$

So replacing the resolvent by the parametrix $\sum q_j(x, D_x)$ we have

$$\begin{aligned}\chi(\frac{A_2(\tau)}{\tau}) &= \frac{1}{2\pi i} \int_{\mathbb{C}} \partial_{\bar z} \tilde\chi(z)(z - \frac{A_2(\tau)}{\tau})^{-1}\, dz \wedge d\bar z \\ &= \sum_{j=0}^{N} \frac{1}{2\pi i} \int_{\mathbb{C}} \partial_{\bar z} \tilde\chi(z) q_j(x, D_x)\, dz \wedge d\bar z \\ &\qquad + \tau^{-N} \int_{\mathbb{C}} \partial_{\bar z} \tilde\chi(z) O(|\operatorname{Im} z|^{-N-l-1})\, dz \wedge d\bar z\end{aligned}$$

Combined with the fact that

$$\chi(E'' - \epsilon > \frac{|x|^2}{4\tau^2}) \chi(-g(-\tau M(\frac{x}{\tau}, \xi), \tau)/\tau) = \chi(E'' - \epsilon > \frac{|x|^2}{4\tau^2}),$$

this shows

$$\begin{aligned}&\chi(E'' - \epsilon > \frac{|x|^2}{4\tau^2}) \chi(\frac{A_2(\tau)}{\tau}) \\ &\qquad = \chi(E'' - \epsilon > \frac{|x|^2}{4\tau^2}) + O_{\mathfrak{B}(\mathbb{H})}(\tau^{-N}) \quad \text{as} \quad \tau \to \infty.\end{aligned} \tag{3.37}$$

Since $N$ is arbitrary, we take $N > s$ and obtain Theorem 1.6. □

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