DOUBLE SERIES OF ISOLS

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1. Introduction. It is assumed that the reader is familiar with the following notions: regressive function, regressive set, regressive isol, infinite series of isols, the minimum of two regressive isols, combinatorial function, and canonical extension. We shall use the slightly more general definition of a regressive function introduced in **(3)**. The next three notions are defined in **(2)**, the fifth in **(3)**, and the last two in **(7** and **8)**. Let

$$\epsilon = \text{the set of all non-negative integers } (\textit{numbers}),$$
$$\Lambda = \text{the collection of all isols},$$
$$\Lambda_R = \text{the collection of all regressive isols}.$$

Dekker **(2)** associated with every sequence $\{a_n\}$ of numbers a function $\sum_S a_n$ from Λ_R into Λ. In the special case that a_n is a recursive function it can be shown **(1)** that $\sum_S a_n$ maps Λ_R into Λ_R. The main object of this paper is to introduce and study a double series $\sum_{(S,T)} a_{ij}$ associated with any double sequence $\{a_{ij}\}$ of numbers (i.e., function from ϵ^2 into ϵ) and any ordered pair (S, T) of regressive isols. The principal definition is as follows:

DEFINITION 1. *Let $\{a_{ij}\}$ be any double sequence of numbers. Then for $(S, T) \in \Lambda_R^2$,*

$$\sum_{(S,T)} a_{ij} = \text{Req} \sum_{k \in \delta s} \sum_{l \in \delta t} j_3[s_k, t_l, \nu(a_{kl})],$$

where $\nu(a_{kl}) = \{y | y < a_{kl}\}$ and s_k and t_l are any two regressive functions such that $\rho s_k \in S$ and $\rho t_l \in T$.

Since any two regressive functions which range over sets belonging to the same isol are recursively equivalent, it easily follows that the sum of the double series given above is well defined. It will be shown that $\sum_{(S,T)} a_{ij}$ is always an isol. On the other hand, it need not be a regressive isol; this is true even if a_{ij} is a recursive function.

The principal result of this paper is the following

THEOREM. *Let a_{xy} be a recursive function. Let the recursive function $q(x, y)$ be defined by*

$$q_{xy} = \sum_{i<x} \sum_{j<y} a_{ij} \qquad (= 0 \textit{ if either } x = 0 \textit{ or } y = 0),$$

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and let $Q(X, Y)$ be its canonical extension to Λ^2. Then for all regressive isols S and T,

$$\sum_{(S,T)} a_{ij} = Q(S, T).$$

2. Preliminaries. Throughout this paper we shall use the notation and terminology introduced in **(2** and **3)**. In particular, we let

$$\begin{aligned} V &= \text{the class of all sets,}\\ Q &= \text{the class of all finite sets,}\\ \Lambda_1 &= \text{the collection of all cosimple isols,}\\ \Lambda_{CR} &= \text{the collection of all cosimple regressive isols,}\\ \Omega &= \text{the collection of all RET's.}\end{aligned}$$

The functions $j, j_3, k, k_{31}, k_{32}, k_{33}, l$ will denote the familiar primitive recursive functions defined by

$$\begin{aligned} j(x, y) &= x + (x + y)(x + y + 1)/2,\\ j_3(x, y, z) &= j(x, j(y, z)),\\ j(k(n), l(n)) &= n,\\ j_3(k_{31}(n), k_{32}(n), k_{33}(n)) &= n.\end{aligned}$$

The function j maps ϵ^2 one-to-one onto ϵ and the function j_3 maps ϵ^3 one-to-one onto ϵ. If n is any number, then $\nu(n) = \{y \mid y < n\}$. If α is any set, then α' denotes its complement, i.e., $\alpha' = \epsilon - \alpha$; $\rho_0, \rho_1, \ldots$ will denote the well-known canonical enumeration of the class Q defined by

$$\rho_0 = \emptyset,$$
$$\rho_{x+1} = \begin{cases} (y_1, \ldots, y_k) \text{ where } y_1, \ldots, y_k \text{ are the distinct}\\ \text{numbers such that } x + 1 = 2^{y_1} + \ldots + 2^{y_k}.\end{cases}$$

The cardinality of the set ρ_x is denoted by $r(x)$; it is easily seen that $r(x)$ is a recursive function. Finally, if t_n is a regressive function and p is a regressing function of t_n, then p^* will denote the partial recursive function associated with p **(3**, p. 348**)** having the property that $p^*(t_n) = n$ for $n \in \delta t$.

3. Elementary properties of double series. The main purpose of this section is to investigate which of the following elementary propositions concerning series of the form $\sum_T a_n$ can be generalized to double series.

Let $T \in \Lambda_R$ and let a_n, b_n be recursive functions. Then

$$(\alpha)\qquad \sum_T a_n \in \Lambda,$$
$$(\beta)\qquad \sum_T a_n \in \Lambda_R,$$
$$(\gamma)\qquad \sum_T k\cdot a_n = k\cdot\sum_T a_n, \qquad \text{for } k \in \epsilon,$$
$$(\delta)\qquad \sum_T a_n + \sum_T b_n = \sum_T (a_n + b_n),$$
$$(\epsilon)\qquad \sum_T 1 = T.$$

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Remark. The references for these five propositions are as follows: the first is **(2**, Theorem 1**)**, the second is **(1**, Theorem 1**)**, both propositions (γ) and (δ) follow from **(1**, Theorem 2**)**, and proposition (ϵ) appears on **(2**, p. 89**)**.

It is readily verified that Definition 1 is equivalent to

DEFINITION 2. *Let $\{a_{ij}\}$ be any double sequence of numbers and (S, T) any ordered pair of regressive isols. If both S and T are finite, say $S = p$ and $T = q$,*

$$\sum\nolimits_{(S,T)} a_{ij} = \sum\nolimits_{i<p} \sum\nolimits_{j<q} a_{ij} \quad (= 0 \textit{ if either } p = 0 \textit{ or } q = 0).$$

If S is finite and T is infinite, say $S = p$,

$$\sum\nolimits_{(S,T)} a_{ij} = \sum\nolimits_{T} (a_{0,j} + \ldots + a_{p-1,j}) \quad (= 0 \textit{ if } p = 0).$$

If S is infinite and T is finite, say $T = q$,

$$\sum\nolimits_{(S,T)} a_{ij} = \sum\nolimits_{S} (a_{i,0} + \ldots + a_{i,q-1}) \quad (= 0 \textit{ if } q = 0).$$

If both T and S are infinite,

$$\sum\nolimits_{(S,T)} a_{ij} = \operatorname{Req} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} j_3(s_k, t_l, \nu(a_{kl})),$$

where s_k and t_l are any two regressive functions ranging over sets belonging to S and T, respectively.

PROPOSITION 1. *For every double sequence $\{a_{ij}\}$ of numbers, $\sum_{(S,T)} a_{ij}$ is a function from Λ_R^2 into Λ.*

Proof. Let $\{a_{ij}\}$ be any double sequence of numbers and (S, T) any ordered pair of regressive isols. If both S and T are finite, then so is $\sum_{(S,T)} a_{ij}$ and our assertion is true. If exactly one of S and T is finite, then $\sum_{(S,T)} a_{ij}$ is an isol **(2**, Theorem 1**)**. Assume that both S and T are infinite regressive isols. Let s_n and t_n be regressive functions ranging over the sets $\sigma \in S$ and $\tau \in T$, respectively. We know that σ and τ and hence also $j(\sigma, \tau)$ are immune. We have to prove that

$$(1) \qquad \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} j_3[s_k, t_l, \nu(a_{kl})] \text{ is an isolated set.}$$

Assume that

$$\delta \subset \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} j_3[s_k, t_l, \nu(a_{kl})],$$

where δ is r.e. Then $j[k_{31}(\delta), k_{32}(\delta)]$ is an r.e. subset of the immune set $j(\sigma, \tau)$. Hence $j[k_{31}(\delta), k_{32}(\delta)]$ is finite, say

$$j[k_{31}(\delta), k_{32}(\delta)] = (j(s_{u(0)}, t_{v(0)}), \ldots, j(s_{u(r)}, t_{v(r)})).$$

Then

$$\delta \subset \sum_{k=0}^{r} j_3[s_{u(k)}, t_{v(k)}, \nu(a_{u(k),v(k)})] \in Q,$$

and δ is finite. This completes the proof.

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PROPOSITION 2. *Let a_{ij} be a recursive function. Then $\sum_{(S,T)} a_{ij}$ is a function from Λ^2_{CR} into Λ_1.*

Proof. Let (S, T) be any ordered pair of cosimple, regressive isols. If either S or T is zero, then so is $\sum_{(S,T)} a_{ij}$ and our statement is correct. Assume now that both S and T are positive. Let s_n and t_n be regressive functions ranging over the cosimple sets $\sigma \in S$ and $\tau \in T$, respectively. Let $p(x)$ and $q(x)$ be regressing functions of s_n and t_n, respectively. Then

$$\sum_{k \in \delta s} \sum_{l \in \delta t} j_3[s_k, t_l, \nu(a_{kl})] \in \sum_{(S,T)} a_{ij}. \tag{1}$$

To prove that $\sum_{(S,T)} a_{ij}$ is cosimple, let ζ denote the set appearing on the left-hand side of (1). Then

$$j_3(x, y, z) \in \zeta \Leftrightarrow x \in \sigma, y \in \tau, \quad \text{and} \quad z < a_{p^*(x), q^*(y)},$$

$$j_3(x, y, z) \in \zeta' \Leftrightarrow x \in \sigma' \vee y \in \tau' \vee [x \in \delta p^*, y \in \delta q^*, \quad \text{and} \quad z \geqslant a_{p^*(x), q^*(y)}].$$

Hence

$$z \in \zeta' \Leftrightarrow k_{31}(z) \in \sigma' \vee k_{32}(z) \in \tau' \vee [k_{31}(z) \in \delta p^*, k_{32}(z) \in \delta q^*, \text{and}$$
$$k_{33}(z) \geqslant a(p^* k_{31}(z), q^* k_{32}(z))].$$

Since σ' and τ' are r.e. sets and a_{ij} is a recursive function, it follows that ζ has an r.e. complement. This completes the proof.

DEFINITION 3. *Let, for all numbers i and j,*

$$e_{ij} = 1,$$
$$d_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

PROPOSITION 3. *For every ordered pair (S, T) of regressive isols,*

$$\sum_{(S,T)} e_{ij} = S \cdot T, \tag{a}$$

$$\sum_{(S,T)} d_{ij} = \min (S, T). \tag{b}$$

Proof. (a) Let S and T be any two regressive isols. If either both S and T are finite or at least one is zero, then (a) readily follows. Assume that at least one of S and T is infinite and that both are positive. First, assume that exactly one of S and T is infinite. We may suppose without loss in generality that $S = p > 0$ and T is infinite. Then

$$\begin{aligned} \sum_{(S,T)} e_{ij} &= \sum_T \underbrace{(1 + \ldots + 1)}_{p} && \text{by Definition 2,} \\ &= \sum_T p \\ &= p \cdot \sum_T 1 && \text{by } (\gamma), \\ &= p \cdot T && \text{by } (\epsilon), \\ &= S \cdot T. \end{aligned}$$

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Suppose now that both S and T are infinite regressive isols. Let s_n and t_n be regressive functions ranging over the sets $\sigma \in S$ and $\tau \in T$, respectively. Then

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} j_3(s_k, t_l, 0) \in \sum_{(S,T)} e_{ij}.$$

Clearly,

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} j_3(s_k, t_l, 0) = j_3(\sigma, \tau, 0) \in S\cdot T,$$

and therefore

$$\textstyle\sum_{(S,T)} e_{ij} = S\cdot T.$$

(b) Let S and T be any two regressive isols. If one of S and T is zero, then both sides of (b) are zero and we are through. In addition, if both S and T are finite and positive, then (b) readily follows by elementary algebra. Assume now that at least one of S and T is infinite and that both are positive. First, assume that exactly one of S and T is infinite. We may suppose without loss of generality that $S = p > 0$, while T is infinite. Then $\min(S, T) = p$ and

$$\begin{aligned}\textstyle\sum_{(S,T)} d_{ij} &= \textstyle\sum_T (d_{0,j} + \ldots + d_{p-1,j}) && \text{by Definition 2,}\\ &= \textstyle\sum_T d_{0,j} + \ldots + \sum_T d_{p-1,j} && \text{by } (\delta),\\ &= \underbrace{1 + \ldots + 1}_{p} = p = \min(S, T).\end{aligned}$$

Suppose now that both S and T are infinite regressive isols. Let s_n and t_n be regressive functions ranging over sets belonging to S and T, respectively. Then, by Definitions 2 and 3,

$$\sum_{k=0}^{\infty} j_3(s_k, t_k, 0) \in \sum_{(S,T)} d_{ij}.$$

Clearly,

$$\sum_{k=0}^{\infty} j_3(s_k, t_k, 0) \simeq \sum_{k=0}^{\infty} j(s_k, t_k) \in \min(S, T)$$

and therefore

$$\textstyle\sum_{(S,T)} d_{ij} = \min(S, T).$$

Remark. It follows from Proposition 3(a) that the conditional

$$\left.\begin{array}{l} S, T \in \Lambda_R \\ a_{ij} \text{ a recursive function}\end{array}\right\} \Rightarrow \textstyle\sum_{(S,T)} a_{ij} \in \Lambda_R$$

is false. For if we take $a_{ij} = e_{ij}$, this conditional reduces to

$$S, T \in \Lambda_R \Rightarrow ST \in \Lambda_R,$$

which is false by (**3**, Theorem T2). The conditional

$$\left.\begin{array}{l} S, T \in \Lambda_{CR} \\ a_{ij} \text{ a recursive function}\end{array}\right\} \Rightarrow \textstyle\sum_{(S,T)} a_{ij} \in \Lambda_{CR}$$

is also false, since Λ_{CR} is not closed under multiplication either (**3**, Theorem T2).

The identity of Proposition 3(a) represents a special case of the following more general result.

PROPOSITION 4. *Let a_i and b_i be any pair of recursive functions. Let S and T be any pair of regressive isols. Then*

$$\sum_S a_i \cdot \sum_T b_i = \sum_{(S,T)} a_i \cdot b_j.$$

The proof of this proposition can be readily established and we shall omit it here. One can also show the following.

PROPOSITION 5. *Let a_{ij} and b_{ij} be two recursive functions. Let S and T be any two regressive isols. Then*

(a) $\sum_{(S,T)} k \cdot a_{ij} = k \cdot \sum_{(S,T)} a_{ij}, \qquad$ *for* $k \in \epsilon$,

(b) $\sum_{(S,T)} a_{ij} + \sum_{(S,T)} b_{ij} = \sum_{(S,T)} (a_{ij} + b_{ij}).$

A useful consequence of this proposition is the following corollary.

COROLLARY. *Let a_{ij} and b_{ij} be two recursive functions such that $b_{ij} \leqslant a_{ij}$. Then for all regressive isols S and T,*

$$\sum_{(S,T)} a_{ij} - \sum_{(S,T)} b_{ij} = \sum_{(S,T)} (a_{ij} - b_{ij}).$$

4. Recursive functions and double series. In this section we shall consider some relationships between recursive functions, regressive isols, and double series. As a starting point we choose the following two theorems concerning combinatorial functions and infinite series of isols.

THEOREM A (Sansone). *Let $f(n)$ be a combinatorial function with $F(X)$ as its canonical extension, and let $T \in \Lambda_R$. Then*

$$F(T) = \sum_{2^T} c_{\tau(n)},$$

where $\{c_j\}$ is the sequence of combinatorial coefficients of the function $f(n)$.

THEOREM B. *Let a_n be a recursive combinatorial function, and let*

$$s_n = \sum_{i<n} a_i \qquad (= 0 \text{ for } n = 0).$$

Then s_n is a recursive combinatorial function; moreover, for $T \in \Lambda_R$,

$$S(T) = \sum_T a_n,$$

where $S(X)$ is the canonical extension of $s(x)$.

Theorem A is (**12**, Lemma 1) if $T \in \epsilon$ and (**13**, Theorem 4.5) if $T \in \Lambda_R - \epsilon$, while Theorem B is (**2**, Theorem 2). We shall first prove an analogue of Theorem A for double series and then one for Theorem B. The following proposition will be useful.

PROPOSITION 7. *Let $f(m, n)$ be a combinatorial function with $\{c_{ij}\}$ as the double sequence of its combinatorial coefficients, i.e.,*

$$f(m, n) = \sum_{i\leqslant m} \sum_{i\leqslant n} c_{ij} \binom{m}{i} \binom{n}{j}.$$

Then

$$f(m, n) = \sum_{i=0}^{2^m-1} \sum_{j=0}^{2^n-1} c_{r(i),r(j)}.$$

Proof. We proceed along the lines of the proof of (**12**, Lemma 1). Since every n-element set has $\binom{n}{i}$ subsets of cardinality i, we have

(1) $\quad f(m, n) = \text{card}\, \{j_3(x, y, z)|\rho_x \subset \nu_m,\ \rho_y \subset \nu_n \text{ and } z < c_{r(x),r(y)}\}$.

It follows from the definition of ρ_x that

$$\rho_x \subset \nu_n \Leftrightarrow x \leqslant 2^0 + 2^1 + \ldots + 2^{n-1} \Leftrightarrow x \leqslant 2^n - 1.$$

Combining this with (1), we obtain

$$f(m, n) = \text{card}\, \{j_3(x, y, z)|x \leqslant 2^{m-1},\ y \leqslant 2^{n-1}, \text{ and } z < c_{r(x),r(y)}\}$$
$$= \sum_{i=0}^{2^m-1} \sum_{j=0}^{2^n-1} c_{r(i),r(j)}.$$

DEFINITION 4. *Let $a(n)$ be a one-to-one function from ϵ into ϵ. Then $a'(n)$ is that unique function with the property $\rho_{a'(n)} = a(\rho_n)$.*

It is readily seen that if a_n is any one-to-one function with range α, then a'_n is also one-to-one, and ranges over 2^α; also, $r(a'_n) = r(n)$. It is known (**12**, Lemma 2) that if a_n is a regressive function, then so is a'_n. We shall make use of these facts in the discussion which follows. We also need the following proposition.

PROPOSITION 9. *Let $f(x, y)$ be a combinatorial function and let $\{c_{ij}\}$ be the double sequence of its combinatorial coefficients. Let p be any number. Put $g(x) = f(p, x)$ and $h(x) = f(x, p)$. Then $g(x)$ and $h(x)$ are also combinatorial functions. Moreover, if $\{d_j\}$ and $\{e_j\}$ are the sequences of combinatorial coefficients of $g(x)$ and $h(x)$, respectively, then*

(1) $$d_{r(j)} = \sum_{k=0}^{2^p-1} c_{r(k),r(j)},$$

(2) $$e_{r(j)} = \sum_{k=0}^{2^p-1} c_{r(j),r(k)}.$$

Proof. Since $c_{ij} \geqslant 0$ for all i and j, it follows from (1) and (2) that d_x, $e_x \geqslant 0$ for all x .Thus (1) and (2) imply that $g(x)$ and $h(x)$ are combinatorial functions. In view of considerations of symmetry it suffices to prove (1). For all x, y,

$$f(x, y) = \sum_{i=0}^{x} \sum_{j=0}^{y} c_{ij} \binom{x}{i} \binom{y}{j};$$

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hence

$$g(x) = f(p, x) = \sum_{i=0}^{p} \sum_{j=0}^{x} c_{ij} \binom{p}{i} \binom{x}{j} = \sum_{j=0}^{x} \left[\sum_{i=0}^{p} c_{ij} \binom{p}{i} \right] \binom{x}{j}.$$

We also have

$$g(x) = \sum_{j=0}^{x} d_j \binom{x}{j}.$$

Since every function uniquely determines the sequence of its combinatorial coefficients, we obtain

$$d_j = \sum_{i=0}^{p} c_{ij} \binom{p}{i} \qquad \text{for every } j,$$

$$\text{(3)} \qquad d_{r(j)} = \sum_{i=0}^{p} c_{i,r(j)} \binom{p}{i} \qquad \text{for every } j.$$

Let k successively assume the values $0, \ldots, 2^p - 1$. Then ρ_k ranges without repetitions over the class of all subsets of a p-element set, namely $(0, \ldots, p - 1)$, and for $0 \leqslant i \leqslant p$,

$$r(k) \text{ assumes the value } i \text{ exactly } \binom{p}{i} \text{ times.}$$

This implies that

$$\text{(4)} \qquad \sum_{i=0}^{p} c_{i,r(j)} \binom{p}{i} = \sum_{k=0}^{2^p-1} c_{r(k),r(j)} \qquad \text{for every } j.$$

Relations (3) and (4) imply (1).

THEOREM 1. *Let $f(x, y)$ be a combinatorial function with canonical extension $F(X, Y)$ and let $S, T \in \Lambda_R$. Then*

$$\text{(1)} \qquad F(S, T) = \sum\nolimits_{(2^S, 2^T)} c_{r(i),r(j)}$$

where $\{c_{ij}\}$ is the double sequence of combinatorial coefficients of $f(x, y)$.

Proof. If both S and T are finite, then (1) follows from Proposition 7. Assume now that at least one of S and T is infinite. First, let us assume that exactly one of S and T is infinite. We may suppose without loss in generality that S is finite, say $S = p$, while T is infinite. Let $g(x) = f(p, x)$. By Proposition 8, $g(x)$ is a combinatorial function. Let $\{d_i\}$ be the sequence of combinatorial coefficients of $g(x)$ and let $G(X)$ be the canonical extension of $g(x)$. In view of Proposition 8 and Theorem A, we have

$$\text{(2)} \qquad d_{r(j)} = \sum_{k=0}^{2^p-1} c_{r(k),r(j)},$$

$$\text{(3)} \qquad G(T) = \sum\nolimits_{2^T} d_{r(j)} \qquad \text{for} \quad T \in \Lambda_R.$$

In addition, by **(11,** p. 107, line −9**)**, $g(x) = f(p, x)$ implies that

$$G(T) = F(p, T).$$

Hence

$$F(S, T) = F(p, T) = G(T) = \sum_{2^T} d_{r(j)} \qquad \text{by (3)},$$
$$= \sum_{2^T} (c_{r(0),r(j)} + \ldots + c_{r(2-1),r(j)} \qquad \text{by (2)},$$
$$= \sum_{(2^S,2^T)} c_{r(i),r(j)} \qquad \text{by Definition 2.}$$

Suppose now that both S and T are infinite regressive isols. Let s_n and t_n be regressive functions ranging over the sets $\sigma \in S$, $\tau \in T$, respectively. Then s'_n and t'_n are also regressive functions and

(4) $$\rho s' = 2^\sigma \in 2^S, \qquad \rho t' = 2^\tau \in 2^T.$$

Hence

(5) $$\sum_{j(k,l)=0}^{\infty} j[s'_k, t'_l, \nu(c_{r(k),r(l)})] \in \sum{}_{(2^S,2^T)} c_{r(i),r(j)}.$$

Also, for $\alpha, \beta \in V$, let

$$\Phi(\alpha, \beta) = \{j_3(x, y, z) | \rho_x \subset \sigma, \rho_y \subset \beta, \text{ and } z < c_{r(x),r(y)}\};$$

or equivalently,

(6) $$\Phi(\alpha, \beta) = \sum\{j_3[x, y, \nu(c_{r(x),r(y)})] | x \in 2^\alpha \text{ and } y \in 2^\beta\}.$$

Then $\Phi(\alpha, \beta)$ denotes the *normal* mapping from V^2 into V which induces the combinatorial function $f(x, y)$; cf. **(4** and **7)**. Therefore

(7) $$\Phi(\sigma, \tau) \in F(S, T).$$

In view of (5) and (7), it suffices to prove that

(8) $$\Phi(\sigma, \tau) \simeq \sum_{j(k,l)=0}^{\infty} j_3[s'_k, t'_l, \nu(c_{r(k),r(l)})].$$

Combining (4) and (6), we obtain

(9) $$\Phi(\sigma, \tau) = \sum_{j(k,l)=0}^{\infty} j_3[s'_k, t'_l, \nu(c_{r(s'k),r(t'l)})].$$

Moreover, $r(s'_n) = r(n)$ and $r(t'_n) = r(n)$; cf. the remark after Definition 4. Combining this fact with (9), we have

$$\Phi(\sigma, \tau) = \sum_{j(k,l)=0}^{\infty} j_3[s'_k, t'_l, \nu(c_{r(k),r(l)})].$$

This proves (8) and completes the proof of (1).

DEFINITION 5. *Let a_{ij} be any function from ϵ^2 into ϵ. The function $q(x, y)$ defined by*

$$q(x, y) = \sum_{i<x} \sum_{j<y} a_{ij} \qquad (= 0 \text{ if } x = 0 \text{ or } y = 0),$$

is the partial sum-function of a_{ij}.

We state without proof

PROPOSITION 9. *If the function a_{ij} is combinatorial, so is its partial sum-function $q(x, y)$. In fact, if*

$$a_{xy} = \sum_{i=0}^{x} \sum_{j=0}^{y} c_{ij} \binom{x}{i} \binom{y}{j}, \tag{1}$$

then

$$q(x, y) = \sum_{i=0}^{x} \sum_{j=0}^{y} c_{i-1,j-1} \binom{x}{i} \binom{y}{j}, \tag{2}$$

where $c_{k,-1} = c_{-1,k} = 0$, for all $k \geqslant -1$.

COROLLARY. *Let a_{xy} be a combinatorial function and $q(x, y)$ its partial sum-function. Let $\{c_{ij}\}$ and $\{d_{ij}\}$ be the double sequences of combinatorial coefficients of a_{xy} and $q(x, y)$, respectively. Then for all numbers i and j,*

$$d_{0,j} = d_{i,0} = 0 \quad \text{and} \quad d_{i+1,j+1} = c_{i,j}.$$

Remark. It is easily seen from Definition 5 and Proposition 9 that if a_{xy} is a recursive and combinatorial function, so is its partial sum-function $q(x, y)$. In addition, if $\{c_{ij}\}$ and $\{d_{ij}\}$ denote the double sequences of combinatorial coefficients of a_{xy} and $q(x, y)$, respectively, then c_{ij} and d_{ij} are recursive functions.

THEOREM 2. *Let a_{xy} be a recursive combinatorial function. Let $q(x, y)$ be the partial sum-function of a_{xy} and $Q(X, Y)$ its canonical extension. Let $S, T \in \Lambda_R$. Then*

$$\textstyle\sum_{(S,T)} a_{ij} = Q(S, T).$$

Proof. If both S and T are finite, then (1) follows from Definitions 2 and 5. Also, if either S or T is zero, then both sides of (1) are zero and our assertion is obvious. Assume now that both S and T are positive and at least one is infinite. First, assume that exactly one of S and T is infinite. We may suppose without loss in generality that S is finite, say $S = p > 0$, while T is infinite. Set

$$\begin{aligned} b_i &= a_{0,i} + \ldots + a_{p-1,i}, \\ h(x) &= q(p, x) = \textstyle\sum_{i<x} b_i. \end{aligned} \tag{2}$$

Note that $h(x)$ is the partial sum-function of b_n (**2**, p. 86). Also, a_{xy} is a recursive and combinatorial function of x and y; hence b_x is a recursive and combinatorial function of x. In addition, since $q(x, y)$ is a recursive and combinatorial function, it follows that $h(x)$ is also a recursive and combinatorial function. Let $H(X)$ denote the canonical extension of $h(x)$. Since, for all x, $h(x) = q(p, x)$ we have by (**9**, Theorem 7.3) that $H(X) = Q(p, X)$ for $X \in \Omega$, and, in particular,

$$H(T) = Q(p, T). \tag{3}$$

Hence

$$Q(S, T) = Q(p, T) = H(T) \qquad \text{by (3)},$$
$$= \sum\nolimits_T b_i \qquad \text{by Theorem B},$$
$$= \sum\nolimits_T (a_{0,i} + \ldots + a_{p-1,i}) \qquad \text{by (2)},$$
$$= \sum\nolimits_{(S,T)} a_{ij}.$$

Assume now that both S and T are infinite regressive isols. Let s_n and t_n be regressive functions ranging over the sets $\sigma \in S$ and $\tau \in T$, respectively. Let s'_n and t'_n be the regressive functions corresponding to s_n and t_n; cf. Definition 4. They range over the sets 2^σ and 2^τ respectively. In addition, let $\{c_{ij}\}$ and $\{d_{ij}\}$ denote the double sequences of combinatorial coefficients of the functions a_{xy} and $q(x, y)$, respectively. Then

(4)
$$\sum_{j(k,l)=0}^{\infty} j_3[s_k, t_l, \nu(a_{kl})] \in \sum_{(S,T)} a_{ij},$$
$$\sum_{j(k,l)=0}^{\infty} j_3[s'_k, t'_l, \nu(d_{\tau(k),\tau(l)})] \in \sum\nolimits_{(2^S,2^T)} d_{\tau(i),\tau(j)}.$$

By Theorem 1,

$$Q(S, T) = \sum\nolimits_{(2^S,2^T)} d_{\tau(i),\tau(j)},$$

and therefore

(5)
$$\sum_{j(k,l)=0}^{\infty} j_3[s'_k, t'_l, \nu(d_{\tau(k),\tau(l)})] \in Q(S, T).$$

Put

$$\psi = \sum_{j(k,l)=0}^{\infty} j_3[s_k, t_l, \nu(a_{kl})],$$
$$\zeta = \sum_{j(k,l)=0}^{\infty} j_3[s'_k, t'_l, \nu(d_{\tau(k),\tau(l)})].$$

To prove (1), it suffices to establish

(6)
$$\psi = \zeta.$$

Put

$$\bar{j} = \begin{cases} 0, & \text{if } j = 0, \\ \max(\rho_j), & \text{if } j > 0, \end{cases}$$

$$\psi_{kl} = j_3[s_k, t_l, \nu(a_{kl})],$$
$$\zeta_{kl} = \{j_3(s'_k, t'_l, z) | \bar{x} = k,\ \bar{y} = l \text{ and } z < d_{\tau(x),\tau(y)}\}.$$

We claim that

(a)
$$\psi = \sum_{j(k,l)=0}^{\infty} \psi_{kl}, \quad \text{and} \quad \zeta = \sum_{j(k,l)=0}^{\infty} \zeta_{kl},$$

where $(\psi_{kl})_{k,l\in\epsilon}$ and $(\zeta_{kl})_{k,l\in\epsilon}$ are classes of mutually disjoint sets;

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(b) card ψ_{kl} = card ζ_{kl} for $k, l \in \epsilon$;

(c) given any element $j_3(s, t, z) \in \psi$, we can compute the numbers k, l such that $j_3(s, t, z) \in \psi_{kl}$ and also the members and cardinality of each of the two sets ψ_{kl} and ζ_{kl};

(d) given any element $j_3(s', t', z) \in \zeta$, we can compute the numbers k, l such that $j_3(s', t', z) \in \zeta_{kl}$ and also the members and cardinality of each of the two sets ζ_{kl} and ψ_{lk}.

Re (a). It is clear that $(\psi_{kl})_{k,l\in\epsilon}$ and $(\zeta_{kl})_{k,l\in\epsilon}$ are classes of mutually disjoint sets and that

$$\psi = \sum_{j(k,l)=0}^{\infty} \psi_{kl} .$$

Moreover, it is readily seen that

$$\sum_{j(k,l)=0}^{\infty} \zeta_{kl} \subset \zeta . \tag{7}$$

Now assume that $j_3(s'_x, t'_y, z) \in \zeta$. Taking into account that $d_{0i} = d_{i0} = 0$ and $z < d_{r(x),r(y)}$, we see that $r(x), r(y) \neq 0$; hence $\rho_x, \rho_y \neq 0$ and $x, y > 0$. Thus

$$j_3(s'_x, t'_y, z) \in \zeta_{kl}, \qquad \text{for } k = \bar{x},\ l = \bar{y}.$$

We have now proved that

$$\zeta \subset \sum_{j(k,l)=0}^{\infty} \zeta_{kl}. \tag{8}$$

In view of (7) and (8), the proof of (a) is complete.

Re (b). Clearly,

$$\text{card } \psi_{kl} = \text{card } \nu(a_{kl}) = a_{kl}. \tag{9}$$

We now determine the cardinality of ζ_{kl}. Any set with n as its maximum must be non-empty and have cardinality $\leqslant n + 1$. Thus

$$j_3(s'_x, t'_y, z) \in \zeta_{kl} \Rightarrow 1 \leqslant r(x) \leqslant k + 1 \quad\text{and}\quad 1 \leqslant r(y) \leqslant l + 1,$$

and therefore

$$\zeta_{kl} = \sum_{u=1}^{k+1} \sum_{v=1}^{l+1} \{j_3(s'_x, t'_y, z) | \bar{x} = k,\ \bar{y} = l,\ r(x) = u,\ r(y) = v, \text{ and } z < d_{uv}\}.$$

Let (u, v) be any ordered pair such that $1 \leqslant u \leqslant k + 1$, $1 \leqslant v \leqslant l + 1$. We wish to find out for how many ordered triples (x, y, z) the statement

$$j_3(s'_x, t'_y, z) \in \zeta_{kl}, \quad f(x) = u, \quad r(y) = l, \quad \text{and} \quad z < d_{uv} \tag{10}$$

is true. Every set with u as cardinality and k as maximum is of the form

$$(k) + \text{some } (u - 1)\text{-element subset of } \nu_k.$$

Thus the number of such sets is $\binom{k}{j-1}$ and the x in (10) can be chosen in $\binom{k}{u-1}$ ways. Similarly, the y in (10) can be chosen in $\binom{l}{v-1}$ ways. Finally, the z in (10) can be chosen in d_{uv} ways. Hence

$$\begin{aligned} \text{card}\, \zeta_{kl} &= \sum_{u=1}^{k+1} \sum_{v=1}^{l+1} \binom{k}{u-1}\binom{l}{v-1} d_{uv} \\ &= \sum_{u=0}^{k} \sum_{v=0}^{l} \binom{k}{u}\binom{l}{v} d_{u+1,v+1} \\ &= \sum_{u=0}^{k} \sum_{v=0}^{l} \binom{k}{u}\binom{l}{v} c_{uv} \qquad \text{by Proposition 9, Corollary,} \\ &= a_{kl} \qquad \text{by the definition of } c_{uv}. \end{aligned}$$

Combining this last result with (9), we complete the proof of (b).

Re (c). Let $j_3(s, t, z) \in \psi$. The functions s_n and t_n are regressive; hence we can compute the numbers k and l such that $s = s_k$ and $t = t_l$. Moreover, a_{xy} is a recursive function; thus we can compute the elements and cardinality of the set $\psi_{kl} = j_3[s_k, t_l, \nu(a_{kl})]$. From the numbers k, l, s_k, t_l, we can compute

(i) the 2^k numbers i such that $\bar{\imath} = k$,
(ii) the 2^l numbers j such that $\bar{\jmath} = l$,
(iii) the $k + 1$ elements $s_0, \ldots, s_k$,
(iv) the $l + 1$ elements $t_0, \ldots, t_l$.

By (i) and (iii), we can effectively find the 2^k sets $s(\rho_i)$ for which $\bar{\imath} = k$, hence also the 2^k numbers s'_i for which $\bar{\imath} = k$. Similarly, we can, by (ii) and (iv), effectively find the 2^l numbers t'_i for which $\bar{\jmath} = l$. The functions $r(x)$ and d_{ij} are recursive; hence so is the function $d_{r(i),r(j)}$. We can therefore compute the elements and cardinality of the set

$$\zeta_{kl} = \{j_3(s'_i, t'_j, z) | \bar{\imath} = k,\ \bar{\jmath} = l, \text{ and } z < d_{r(i),r(j)}\}.$$

Re (d). Let $j_3(s', t', z) \in \zeta$. The functions s'_n, t'_n, being regressive, we can compute the numbers x and y such that $s' = s'_x$ and $t' = t'_y$. We already noted in the proof of (a) that $x, y > 0$. We can now compute the numbers $k = \bar{x}$ and $l = \bar{y}$. Hence

$$j_3(s', t', z) = j_3(s'_x, t'_y, z) \in \zeta_{kl}.$$

From the relations $\rho_{s'(x)} = s(\rho_x)$ and $k = \bar{x}$, we conclude that $s_k \in \rho_{s'(x)}$. The number s_k can therefore be computed from $s'(x)$ and k. In view of (b) and $j_3(s'_x, t'_y, z) \in \zeta_{kl}$, we see that $\psi_{kl} \neq 0$. Thus

$$(11) \qquad j_3(s_k, t_l, 0) \in \psi_{kl},$$

where $j_3(s_k, t_l, 0)$ can be computed. Using (c) it follows that the elements and cardinality of each of the two sets ψ_{kl} and ζ_{kl} can be computed. This completes the proof of (d).

We now establish that $\psi = \zeta$. Let, for each $(k, l) \in \epsilon^2$, $p_{kl}(z)$ be the 1–1 function that maps ψ_{kl} in the order-preserving manner onto ζ_{kl}. Let

$$g(z) = p_{kl}(z), \qquad \text{for } z \in \psi,$$

where (k, l) is the ordered pair such that $z \in \psi_{kl}$. Thus g maps ψ 1–1 onto ζ. Using (c) and (d), we can prove that both g and g^{-1} have partial recursive extensions. This shows that $\psi \simeq \zeta$, that i.e. (6) is true, and completes the proof of the theorem.

To illustrate Theorem 2, we evaluate $\sum_{(S,T)} a_{ij}$ for one simple recursive combinatorial function a_{ij}, namely $i \cdot j$. Clearly,

$$\sum_{i<m} \sum_{j<m} i \cdot j = \frac{(m-1)m(n-1)n}{4} \qquad (= 0 \text{ if } m = 0 \text{ or } n = 0).$$

Thus, for $S, T \in \Lambda_R$,

$$\sum_{(S,T)} i \cdot j = \begin{cases} 0 & \text{if } S = 0 \text{ or } T = 0, \\ \dfrac{(S-1)S(T-1)T}{4} & \text{if } S, T \neq 0. \end{cases}$$

THEOREM 3. *Let a_{xy} be a recursive function. Let $q(x, y)$ be the partial sum-function of a_{xy} and $Q(X, Y)$ its canonical extension. Let $S, T \in \Lambda_R$. Then*

$$\sum_{(S,T)} a_{ij} = Q(S, T).$$

Proof. Assume the statement of the theorem. Let a_{xy}^+ and a_{xy}^- be the *positive* and *negative* parts, respectively, associated with the function a_{xy}; cf. **(8)**. Then a_{xy}^+ and a_{xy}^- are recursive and combinatorial functions such that

$$a_{ij} = a_{ij}^+ - a_{ij}^-.$$

Since a_{ij} is always non-negative, we have $a_{ij}^+ \geqslant a_{ij}^-$. Combining this fact with the corollary to Proposition 5, we obtain

$$(1) \qquad \sum_{(S,T)} a_{ij} = \sum_{(S,T)} a_{ij}^+ - \sum_{(S,T)} a_{ij}^-.$$

Let $q^+(x, y)$ and $q^-(x, y)$ be the partial sum-functions of a_{xy}^+ and a_{xy}^-, respectively, having the respective canonical extensions $Q^+(X, Y)$ and $Q^-(X, Y)$. It easily follows that for all $x, y \in \epsilon$,

$$q(x, y) = q^+(x, y) - q^-(x, y);$$

in fact, $q^+(x, y)$ and $q^-(x, y)$ are the positive and negative parts, respectively, associated with the function $q(x, y)$. Hence

$$(2) \qquad Q(X, Y) = Q^+(X, Y) - Q^-(X, Y) \qquad \text{for } X, Y \in \Lambda.$$

In addition, by Theorem 3, we have

$$(3) \qquad \sum_{(S,T)} a_{ij}^+ = Q^+(S, T) \qquad \text{and} \qquad \sum_{(S,T)} a_{ij}^- = Q^-(S, T).$$

The desired conclusion now follows from (1), (2), and (3).

Remark. We wish to observe here one consequence of Theorem 3, the fact that the minimum function for pairs of regressive isols introduced in **(3)** and the canonical extension, restricted to Λ_R^2, of the well-known minimum function for pairs of numbers are equivalent. For this purpose, let us first recall the definition of the function d_{xy} given by

$$d_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j; \end{cases}$$

and also the fact (Proposition 3(b)) that for $S, T \in \Lambda_R$,

$$(*) \qquad \sum_{(S,T)} d_{ij} = \min (S, T).$$

Clearly the function d_{xy} is recursive, and in addition, it easily follows that its partial sum-function is the function minimum $(x, y) : \epsilon^2 \rightarrow \epsilon$. Combining this fact with Theorem 3, we see that the left side of $(*)$, and hence also the right side, represents the canonical extension of the function minimum (x, y) evaluated at the ordered pair (S, T). Since this is true for each ordered pair (S, T) of regressive isols, the desired equivalence follows.

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