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Some remarks on stable graphs

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We introduce some methods of constructing stable graphs and characterize a few classes of stable graphs. We also give a counter example to disprove Holton's conjecture.

1. Introduction

A graph G is a strict graph in the sense of Tutte. Let $G_{v_1v_2\ldots v_k}$ be the graph obtained by removing the vertices $v_1, v_2, \ldots, v_k$ and all the edges incident with these vertices, from G. Let $A(G)$ be the automorphism group of G and $A(G)_{v_1v_2\ldots v_k}$ be the stabilizer of $\{v_1, v_2, \ldots, v_k\}$ such that each element in $A(G)_{v_1v_2\ldots v_k}$ fixes v_i individually for all $i = 1, 2, \ldots, k$. Let $|V(G)|$ be the cardinality of the vertex set $V(G)$ of G. If there exists a sequence $S = \{v_1, v_2, \ldots, v_n\}$, $n = |V(G)|$, of distinct vertices of G such that $A\left(G_{v_1v_2\ldots v_k}\right) = A(G)_{v_1v_2\ldots v_k}$ for each $k = 1, 2, \ldots, n$, then G is said to be stable (otherwise unstable - perhaps an alternative term is recommended as some other writers have used this term in a different sense) and S is called a stabilising sequence of G.

In [1], Holton proves, among other results, that

(1) if G_v is stable for some $v \in A(G)$, and $A\left(G_v\right) = A(G)_v$, then G is stable;

(2) the union of m graphs G_i is stable if and only if each

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G_i is stable;

(3) if $A(G)$ is a subgroup of the dihedral group D_n, $n = |V(G)| \geq 5$, then G is unstable.

We now extend some of Holton's results.

2. Construction problems

It is well known that the complement $\overline{G}$ of a disconnected graph G is connected. Also, it is clear that G is stable if and only if $\overline{G}$ is stable. Hence, we can restrict ourselves to the construction of connected stable graphs.

Let G be a stable graph (connected or disconnected) with stabilising sequence $\{v_1, v_2, \ldots, v_n\}$. Let $V_1, V_2, \ldots, V_r$ be the orbits of $V(G)$ under $A(G)$. We define G^* to be the graph obtained from G by adding a new vertex v_0 to G and adding edges joining v_0 to all the vertices in V_i for some i (one or more) or all $i = 1, 2, \ldots, r$.

THEOREM 1. *G^* is stable.*

Proof. We shall prove that $\{v_0, v_1, \ldots, v_n\}$ is a stabilising sequence of G^*.

We know from the definition that $G^*_{v_0} = G$. Hence $A\left(G^*_{v_0}\right) = A(G)$.

It is clear that $A(G^*)_{v_0} \leq A(G)$.

We now prove that $A(G) \leq A(G^*)_{v_0}$.

Suppose $\phi \in A(G)$. We define a mapping ϕ^* of $V(G^*)$ onto $V(G^*)$ as follows:

$$v_0\phi^* = v_0 , \quad v_i\phi^* = v_i\phi \text{ for every } i = 1, 2, \ldots, n .$$

Then $(v_i, v_j) \in E(G^*)$ ((v_i, v_j) is an edge of G^*), $i, j \neq 0$, implies that $(v_i, v_j)\phi^* = (v_i\phi, v_j\phi) \in E(G^*)$ and $(v_i, v_0) \in E(G^*)$ implies that $(v_i, v_0)\phi^* = (v_i\phi, v_0) \in E(G^*)$ because $v_i, v_i\phi$ belong to the same orbit

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V_j for some j. Hence $\phi^* \in A(G^*)_{v_0}$. If we identify ϕ with ϕ^*, then $A(G) \leq A(G^*)_{v_0}$ and so $A\left(G^*_{v_0}\right) = A(G^*)_{v_0}$. The rest of the proof is immediate.

As a special case of Theorem 1, we have

COROLLARY. *If G is a stable graph and G^* is the graph obtained from G by adding a new vertex v_0 to G and adding all the edges joining v_0 to each vertex of G, then G^* is stable.*

Let H be an induced subgraph of G. Let $v_i \in V(H)$; we define

$$D_1(v_i, H) = \{v_j \in V(H);\ (v_i, v_j) \in E(H)\} .$$

Let H and K be two connected stable graphs. Let $\{u_1, u_2, \ldots, u_m\}$ be a stabilising sequence of H, $\{v_1, v_2, \ldots, v_n\}$ be a stabilising sequence of K, and $m \leq n$. We define $G = H \dot{+} K$ to be the graph obtained from H and K by identifying u_1 with v_1 and putting the two graphs H and K side by side. In other words, G is obtained from the union of H and K by identifying u_1 with v_1.

THEOREM 2. *Let $G = H \dot{+} K$. Suppose K_{v_1} is connected. If H_{u_1} is not isomorphic with K_{v_1} or if H_{u_1} is isomorphic with K_{v_1} such that*

$$D_1(u_1, H)\phi = D_1(v_1, K)$$

for every isomorphism ϕ of H_{u_1} to K_{v_1}, then $\{u_1, u_2, \ldots, u_m, v_2, v_3, \ldots, v_n\}$ is a stabilising sequence of G.

Proof. $G_{u_1} = H_{u_1} \cup K_{v_1}$, union of the two disjoint induced subgraphs H_{u_1} and K_{v_1} of G.

If H_{u_1} is not isomorphic with K_{v_1} then, since K_{v_1} is connected,

$$A\left(G_{u_1}\right) = A\left(H_{u_1}\right) \times A\left(K_{v_1}\right) ,$$

the direct product of $A\left(H_{u_1}\right)$ and $A\left(K_{v_1}\right)$. Hence

$$A\left(G_{u_1}\right) = A(G)_{u_1} .$$

If H_{u_1} is isomorphic with K_{v_1} such that

$$D_1\left(u_1, H\right)\phi = \left\{u_i\phi;\ u_i \in D_1\left(u_1, H\right)\right\} = D_1\left(v_1, K\right)$$

for every isomorphism ϕ of H_{u_1} to K_{v_1} , then

$$A\left(G_{u_1}\right) = A\left(H_{u_1}\right) \sim S_2$$

the wreath product of $A\left(H_{u_1}\right)$ and S_2 , symmetric group of $\{1, 2\}$.

Hence $A\left(G_{u_1}\right) = A(G)_{u_1}$.

The rest of the proof is clear.

REMARKS. If $m > n$, but none of $A\left(H_{u_1u_2\ldots u_k}\right)$, $k = 1, 2, \ldots, m$, is isomorphic with $A\left(K_{v_1}\right)$, then by similar methods, we can show that $G = H \dotplus K$ is stable.

It is not difficult to see that the complete bipartite graphs $K_{m,n}$ and in particular, the star graphs $K_{1,t}$ are stable. Hence, we have, with appropriate order of composition and a few restrictions, the following corollaries to Theorem 2.

COROLLARY 1. *$K_{m,n} \dotplus K_{r,s}$ is stable.*

COROLLARY 2. *$K_{m,n} \dotplus K_r$ is stable.*

COROLLARY 3. *$K_m \dotplus K_n$ is stable.*

Let G be a connected, stable graph with stabilising sequence

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$\{u_1, u_2, \ldots, u_m\}$. We define $G' = G + K_2$ to be the graph obtained from G and K_2 by putting G and K_2 side by side and adding a new edge E joining u_1 to a vertex v_1 of K_2 .

THEOREM 3. *If* $u_1\phi = u_1$ *for each* $\phi \in A(G)$ *or there are no monovalent vertices in* G *then* G' *is stable.*

Proof. Suppose $V(K_2) = \{v_1, v_2\}$.

If $u_1\phi = u_1$ for each $\phi \in A(G)$, we can verify that $\{v_1, v_2, u_1, u_2, \ldots, u_m\}$ is a stabilising sequence of G' .

If there are no monovalent vertices in G , we can verify that $\{v_2, u_1, v_1, u_2, \ldots, u_m\}$ is a stabilising sequence of G' .

Applying Theorems 1, 2, and 3, together with Holton's result (2), we can construct all stable graphs with 3, 4, 5 and 6 vertices from the basic graph K_2 . It is unknown to the author whether we may or may not be able to obtain all the stable graphs G with $|V(G)| \geq 7$ by applying only these methods and accepting that $K_{1,t}$ is stable.

3. Characterization problems

Let G be a stable graph with stabilising sequence $\{v_1, v_2, \ldots, v_n\}$. Suppose $\phi \in A(G)$. Then $\{v_1\phi, v_2\phi, \ldots, v_n\phi\}$ is also a stabilising sequence of G . It would be interesting to investigate the role that stabilising sequences will play in the characterization problems of stable graphs. For instance, it is not difficult to show that if G is connected and stable, then any sequence of the vertex set of G is a stabilising sequence of G if and only if G is the complete graph. We now use this fact to prove

THEOREM 4. *Let* G *be connected and stable. If* $\{v_1, v_{2\alpha}, v_{3\alpha}, \ldots, v_{n\alpha}\}$ *is a stabilising sequence for each permutation* α *of* $\{2, 3, \ldots, n\}$ *, then* G *is either* K_n *or* $K_{1,n-1}$.

Proof. If G_{v_1} is connected, then by the previous remark,

$G_{v_1} = K_{n-1}$ and so $G = K_n$.

Suppose G_{v_1} is disconnected. Let

$$G_{v_1} = H_1 \cup H_2 \cup \ldots \cup H_r ,$$

where each H_i is a connected component of G_{v_1} . We can verify without much difficulty that $|V(H_j)| \geq 2$ for some j is impossible. Hence G_{v_1} is the trivial graph with $n - 1$ vertices and 0 edges. Hence $A\left(G_{v_1}\right) = S_{n-1}$ and this implies that $G = K_{1,n-1}$.

The following is another characterization with respect to the automorphism group.

THEOREM 5. *Let F be a group such that for every nontrivial subgroup F_1 of F, any graph whose automorphism group is isomorphic with F_1 has vertex number greater than the order of F_1, then any graph G with $A(G) = F$ is unstable.*

Proof. The smallest order of F with the above property is 3 . The only group whose order is 3 is the cyclic group C_3 . Any graph G with $A(G) = C_3$ has vertex number greater than 3 and G is easily seen to be unstable.

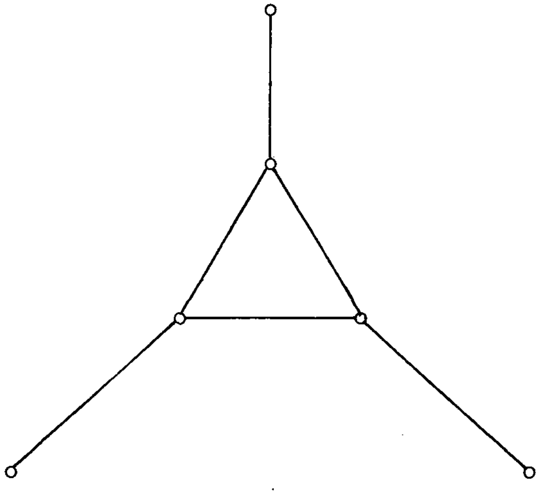
Let F be any group with order greater than 3 and G be any graph with $A(G) = F$. Then for any vertex v of G , $A\left(G_v\right) = F_1$ is a subgroup of F . If F_1 is the identity group, then by Holton's result (3), G_v is unstable. If F_1 is nontrivial, then by induction hypothesis, G_v is unstable and so G is unstable.

COROLLARY. *If $A(G) = C_n$, the cyclic group of order n , and n is* *odd, then* G *is unstable.*

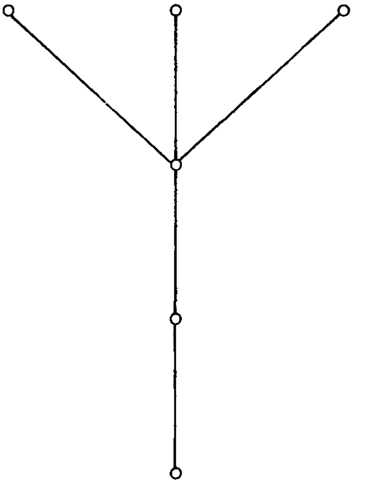
4. A counter example to Holton's conjecture

In [1], Holton conjectured that if two graphs G_1 and G_2 are such that $A(G_1) = A(G_2)$ where all G_1, $\overline{G}_1$, G_2, $\overline{G}_2$ are connected, then G_1 is stable if and only if G_2 is stable.

We now give a counter example to show that Holton's conjecture is not true. The two graphs G_1, G_2 given below satisfy all the conditions in Holton's conjecture. But G_1 is unstable whereas G_2 is stable.



G_1



G_2

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[1] D.A. Holton, "A report on stable graphs", *J. Austral. Math. Soc.* 15 (1973), 163-171.

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