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NONZERO SYMMETRY CLASSES OF SMALLEST DIMENSION

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1. Introduction. Let U be a k-dimensional vector space over the complex numbers. Let $\otimes^m U$ denote the mth tensor power of U where $m \geqq 2$. For each permutation σ in the symmetric group S_m, there exists a linear mapping $P(\sigma)$ on $\otimes^m U$ such that

$$P(\sigma)(x_1 \otimes \ldots \otimes x_m) = x_{\sigma^{-1}(1)} \otimes \ldots \otimes x_{\sigma^{-1}(m)}$$

for all $x_1, \ldots, x_m$ in U.

Let G be a subgroup of S_m and λ an irreducible (complex) character on G. The symmetrizer

$$T(G, \lambda) = \frac{\lambda(1)}{|G|} \sum_{\sigma \in G} \lambda(\sigma) P(\sigma)$$

is a projection of $\otimes^m U$. Its range is denoted by $U_\lambda{}^m(G)$ or simply $U_\lambda(G)$ and is called the symmetry class of tensors corresponding to G and λ.

The problem of characterizing all groups G and irreducible characters λ and G for which $U_\lambda(G) = 0$ was considered in [**10**], [**27**] and [**7**, **8**]. The main result of this paper characterizes those $U_\lambda(G)$ with dimension equal to $\lambda(1)$ when $m = 2k$ (Theorem 13). Its proof relies on the results concerning (k)-groups studied by the first author [**4**, **5**, **6**]. It was proved in [**9**] that for $m \leqq 2k - 2$, dim $U_\lambda(G) = 1$ if and only if $m = k$, $G = S_k$ and λ is the sign character ϵ.

2. Some preliminaries. Let $\Gamma_{m,k}$ be the set of all functions from $M = \{1, 2, \ldots, m\}$ into $K = \{1, 2, \ldots, k\}$. Let $e_1, \ldots, e_k$ be a basis of U. Then

$$\{e_\alpha{}^\otimes = e_{\alpha(1)} \otimes \ldots \otimes e_{\alpha(m)} : \alpha \in \Gamma_{m,k}\}$$

is a basis of $\otimes^m U$. It follows that $\{e_\alpha{}^* = T(G, \lambda)e_\alpha{}^\otimes : \alpha \in \Gamma_{m,k}\}$ spans $U_\lambda(G)$.

We define an equivalence relation on $\Gamma_{m,k}$ as follows: For $\alpha, \beta \in \Gamma_{m,k}$, $\alpha \equiv \beta \pmod G$ if and only if there exists a $\sigma \in G$ such that $\alpha\sigma = \beta$. Let Δ be the system of distinct representatives for the equivalence relation formed by taking the element in each equivalence class which is first in lexi-

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cographic order. For each $\alpha \in \Gamma_{m,k}$ let G_α be the stabilizer subgroup of α, i.e., $G_\alpha = \{\sigma \in G: \alpha\sigma = \alpha\}$. Then it is well-known that $e_\alpha{}^* = 0$ if and only if

$$\sum_{\sigma \in G_\alpha} \lambda(\sigma) = 0.$$

Let

$$\bar{\Delta} = \{\alpha \in \Delta : e_\alpha{}^* \neq 0\}.$$

Then it was proved in [**24**] that

$$(1) \quad U_\lambda(G) = \sum_{\alpha \in \bar{\Delta}} \langle e_{\alpha\sigma}{}^* : \sigma \in G\rangle,$$

the sum being direct.

For each $\alpha \in \bar{\Delta}$, the subspace $\langle e_{\alpha\sigma}{}^* : \sigma \in G\rangle$ is called the *orbital subspace of $U_\lambda(G)$ corresponding to α.* In [**13**], Freese proved that

$$(2) \quad \dim \langle e_{\alpha\sigma}{}^* : \sigma \in G\rangle = \frac{\lambda(1)}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \lambda(\sigma).$$

Thus if $U_\lambda(G) \neq 0$ then $\dim U_\lambda(G) \geqq \lambda(1)$. It is known [**17**, p. 79] that

$$(3) \quad \dim U_\lambda(G) = \frac{\lambda(1)}{|G|} \sum_{\sigma \in G} \lambda(\sigma) k^{c(\sigma)}$$

where $\dim U = k$ and $c(\sigma)$ denotes the number of cycles in the disjoint cycle decomposition of σ (including cycles of length 1).

A character χ of G is called a (k)-character of G if, for each $\alpha \in \Gamma_{m,k}$, we have

$$\sum_{\sigma \in G_\alpha} \chi(\sigma) = 0.$$

If χ is a (k)-character of G then $G_\alpha \neq \{1\}$ for all $\alpha \in \Gamma_{m,k}$ and this is precisely another version of the definition of (k)-group [**6**]. Clearly $U_\lambda(G) = 0$ if and only if λ is a (k)-character where $\dim U = k$.

Let G_1 and G_2 be permutation groups on $M = \{1, \ldots, m\}$ and $N = \{1, \ldots, n\}$ respectively. Each permutation of the wreath product $G = G_1 \wr G_2$ can be uniquely expressed in the form $(g_1, \ldots, g_n : h)$ where $g_i \in G_1$, $h \in G_2$. If λ_i is a linear character on G_i, $i = 1, 2$, the mapping λ defined by

$$\lambda(g_1, \ldots, g_n; h) = \lambda_1(g_1) \ldots \lambda_1(g_n)\lambda_2(h)$$

is a linear character on $G_1 \wr G_2$ and is denoted by $\lambda_1 \wr \lambda_2$. In [**28**, Theorem 2], Williamson proved that

$$(4) \quad \dim U_\lambda(G_1 \wr G_2) = \dim [(U_{\lambda_1}(G_1))_{\lambda_2}(G_2)].$$

Let H be a normal subgroup of G of index 2. Let λ be an irreducible character of G. Then the irreducible character λ' on G such that

$$\lambda'(\sigma) = \lambda(\sigma), \quad \sigma \in H$$
$$\lambda'(\sigma) = -\lambda(\sigma), \quad \sigma \notin H$$

is called the associated character of λ. If $\lambda = \lambda'$, we say that λ is self-associated.

For each irreducible character χ of H and a $\pi \in G\backslash H$, we can define an irreducible character $\bar{\chi}$ of H as follows:

$$\bar{\chi}(\sigma) = \chi(\pi^{-1}\sigma\pi)$$

for all $\sigma \in H$. Note that $\bar{\chi}$ is independent of the choice of π and is called the conjugate character of χ. χ is called self-conjugate if $\bar{\chi} = \chi$. The relation between associated characters of G and the conjugate characters of H is given in the following theorem [**1**].

THEOREM 1. (a) *If χ and χ' are associated irreducible characters of G and $\chi \neq \chi'$, then $\chi|_H = \chi'|_H$ is a self-conjugate irreducible character of H. Conversely, every self-conjugate irreducible of H is the restriction of a pair of associated irreducible characters of G.*

(b) *If χ is a self-associated irreducible character of G then $\chi|_H = \lambda + \bar{\lambda}$ where λ and $\bar{\lambda}$ are irreducible conjugate characters of H. Conversely, the sum of a pair of distinct conjugate irreducible characters of H is the restriction of a self-associated irreducible character of G.*

The last line of the following theorem follows from Lemma 5 in [**24**], while the rest of the theorem is a special case of Theorems 3 and 4 in [**20**]. We remark that the theorem could also be proved easily by using formula (3).

THEOREM 2. *Let G be a subgroup of S_m and H a normal subgroup of G of index 2.*

(a) *If λ is a self-conjugate irreducible character of H induced by the associated irreducible characters χ and χ' of G, then*

$$U_\lambda(H) = U_\chi(G) \oplus U_{\chi'}(G).$$

(b) *If λ and $\bar{\lambda}$ is a pair of conjugate irreducible characters of H such that $\chi|_H = \lambda + \bar{\lambda}$ where χ is a self-associated irreducible character of G, then*

$$U_\chi(G) = U_\lambda(H) \oplus U_{\bar{\lambda}}(H)$$

and

$$\dim U_\lambda(H) = \dim U_{\bar{\lambda}}(H).$$

We now describe irreducible characters on the wreath product $G = S_n \wr S_2$. Consider G as a permutation group on $\{1, 2, \ldots, 2n\}$ with the complete block system $N_1 = \{1, \ldots, n\}$, $N_2 = \{n + 1, \ldots, 2n\}$. We

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shall write every permutation in $S_n \wr S_2$ as $\sigma_1\sigma_2\pi$ where $\sigma_1 \in S_{N_1}$, $\sigma_2 \in S_{N_2}$ and $\pi \in S_2$. Let λ and ρ be two irreducible characters corresponding to irreducible representations D_λ and D_ρ of S_{N_1} and S_{N_2} respectively. Then the character $\lambda \# \rho$ corresponding to the outer tensor product $D_\lambda \# D_\rho$ defined by

$$(D_\lambda \# D_\rho)(\sigma_1\sigma_2) = D_\lambda(\sigma_1) \otimes D_\rho(\sigma_2)$$

where $\sigma_1 \in S_{N_1}$, $\sigma_2 \in S_{N_2}$ is an irreducible character of the product $S_{N_1} \cdot S_{N_2}$ (see [**11**], [**15**]). In fact

$$(\lambda \# \rho)(\sigma_1\sigma_2) = \lambda(\sigma_1)\rho(\sigma_2).$$

If $\lambda \neq \rho$, then the induced character $(\lambda \# \rho)^G$ is an irreducible character of $G = S_n \wr S_2$. If $\lambda = \rho$, then we first extend $D_\lambda \# D_\lambda$ to an irreducible representation $\widetilde{D_\lambda \# D_\lambda}$ of $S_n \wr S_2$ as follows: for each $\sigma_1\sigma_2\pi \in S_n \wr S_2$, if

$$D_\lambda(\sigma_1) = (a_{i_1 j_1})$$
$$D_\lambda(\sigma_2) = (b_{i_2 j_2})$$
$$D_\lambda \# D_\lambda(\sigma_1\sigma_2) = (a_{i_1 j_1} b_{i_2 j_2})$$

we put

$$\widetilde{D_\lambda \# D_\lambda}(\sigma_1\sigma_2\pi) = (a_{i_1 j_{\pi(1)}} b_{i_2 j_{\pi(2)}}).$$

Now for each irreducible character χ of S_2 corresponding to irreducible representation D_χ of S_2 we can define a representation $\tilde{D}_\chi$ of $S_n \wr S_2$ as follows:

$$\tilde{D}_\chi(\sigma_1\sigma_2\pi) = D_\chi(\pi).$$

Then the inner tensor product $D_\lambda \# D_\lambda \otimes \tilde{D}_\chi$ is an irreducible representation of $S_n \wr S_n$ and its corresponding character is the character $\lambda \wr \chi$. We shall need the following result (see [**15**]) concerning the irreducible characters of wreath product $S_n \wr S_2$ in the next section.

THEOREM 3. *Every irreducible character of the wreath product* $G = S_n \wr S_2$ *is either equal to* $(\lambda \# \rho)^G$ *or* $\lambda \wr \chi$ *where* λ, ρ *are distinct irreducible characters of* S_n *and* χ *is an irreducible character of* S_2.

THEOREM 4. *Let* G_1 *and* G_2 *be permutation groups on* $\{1, \ldots, m\}$ *and* $\{m + 1, \ldots, m + t\}$ *respectively. Let* λ_1 *and* λ_2 *be irreducible characters of* G_1 *and* G_2 *respectively. If* $\lambda = \lambda_1 \# \lambda_2$ *is the irreducible character on the product* $G_1 \cdot G_2$ *corresponding to the outer tensor product representation, then*

$$\dim U_\lambda^{m+t}(G_1 \cdot G_2) = \dim U_{\lambda_1}(G_1) \cdot \dim U_{\lambda_2}(G_2).$$

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Proof. In view of (3),

$$\dim U_\lambda(G_1 \cdot G_2) = \frac{\lambda(1)}{|G_1 \cdot G_2|} \sum_{\sigma_i \in G_i} (\lambda_1 \# \lambda_2)(\sigma_1\sigma_2) k^{c(\sigma_1\sigma_2)}$$
$$= \frac{\lambda_1(1)}{|G_1|} \frac{\lambda_2(1)}{|G_2|} \sum_{\sigma_i \in G_i} \lambda_1(\sigma_1)\lambda_2(\sigma_2) k^{c(\sigma_1)+c(\sigma_2)}$$
$$= \left(\frac{\lambda_1(1)}{|G_1|} \sum_{\sigma_1 \in G_1} \lambda_1(\sigma_1) k^{c(\sigma_1)}\right) \left(\frac{\lambda_2(1)}{|G_2|} \sum_{\sigma_2 \in G_2} \lambda_2(\sigma_2) k^{c(\sigma_2)}\right)$$
$$= \dim U_{\lambda_1}(G_1) \cdot \dim U_{\lambda_2}(G_2).$$

When λ_1, λ_2 are linear, Theorem 4 was proved in [**26**] by a different method.

THEOREM 5. *Let* λ *and* ρ *be distinct irreducible characters on* S_n. *Let* $G = S_n \wr S_2$ *and* $\chi = (\lambda \# \rho)^G$. *Then*

$$\dim U_\chi(G) = 2 \dim U_\lambda(S_{N_1}) \dim U_\rho(S_{N_2})$$

where $N_1 = \{1, \ldots, n\}$ *and* $N_2 = \{n + 1, \ldots, 2n\}$.

Proof. Since $(\lambda \# \rho)^G$ is self-associated with respect to $S_{N_1} \cdot S_{N_2}$ and

$$\chi|_{S_{N_1} \cdot S_{N_2}} = \lambda \# \rho + \overline{\lambda \# \rho},$$

it follows from Theorem 2 and Theorem 4 that

$$\dim U_\chi(G) = 2 \dim U_{\lambda\#\rho}(S_{N_1} \cdot S_{N_2})$$
$$= 2 \dim U_\lambda(S_{N_1}) \dim U_\rho(S_{N_2}).$$

COROLLARY 1 [**8**]. *Let* λ *and* ρ *be distinct irreducible characters of* S_n. *Then* $(\lambda \# \rho)^G$ *is a* (k)*-character of* $G = S_n \wr S_2$ *if and only if either* λ *or* ρ *is a* (k)*-character of* S_n.

Proof. This follows immediately from Theorem 5.

3. Nonzero symmetry classes of smallest dimension. In this section we shall determine those subgroups G of S_m and those irreducible characters λ on G such that $\dim U_\lambda(G) = \lambda(1)$ when $m = 2k$ where $k = \dim U$.

Throughout the rest of the paper we assume that $\dim U = k$, $M = \{1, 2, \ldots, m\}$ and $K = \{1, 2, \ldots, k\}$.

THEOREM 6. *Let* $0 \neq \dim U_\lambda(G) < k\lambda(1)$. *If* $\alpha \in \bar{\Delta}$ *then* $|\alpha(M)| = k$ *and* $|\alpha^{-1}(i)| = m/k$ *for* $i = 1, 2, \ldots, k$.

Proof. If $k > m$, let Q be the set of all mappings β in $\Gamma_{m,k}$ such that

$$\alpha(1) < \alpha(2) < \ldots < \alpha(m).$$

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Then $Q \subseteq \bar{\Delta}$ and hence from (1) and (2) we have

$$\dim U_\lambda(G) \geqq |Q|\lambda(1) = {}_mC_k\lambda(1) \geqq k\lambda(1)$$

a contradiction. Hence $k \leqq m$.

Suppose now $|\alpha(M)| = s \neq k$. Then for each $i \in \alpha(M)$ and $j \notin \alpha(M)$, $1 \leqq j \leqq k$, let $\sigma_{ij} = (ij)$ be the transposition in S_k. Then

$$\{e^*_{\sigma_{ij}\alpha} : i \in \alpha(M), j \notin \alpha(M), 1 \leqq j \leqq k\} \cup \{e_\alpha^*\}$$

is a set with $s(k - s) + 1$ elements and different elements of the set belong to different orbital subspaces of $U_\lambda(G)$. Hence

$$\dim U_\lambda(G) \geqq [s(k - s) + 1]\lambda(1) \geqq k\lambda(1),$$

a contradiction. Hence $|\alpha(M)| = k$.

Let $D = \{j : |\alpha^{-1}(j)| = |\alpha^{-1}(1)|\}$. Suppose that $|D| = t \neq k$. Then for each $i \in D$ and $j \in K\backslash D$, let τ_{ij} be the transposition (ij) in S_k. Then

$$\{e^*_{\tau_{ij}\alpha} : i \in D, j \in K\backslash D\} \cup \{e_\alpha^*\}$$

is a set with $t(k - t) + 1$ elements and different elements of the set belong to different orbital subspaces of $U_\lambda(G)$. Hence

$$\dim U_\lambda(G) \geqq [t(k - t) + 1]\lambda(1) \geqq k\lambda(1),$$

a contradiction. Hence $|\alpha^{-1}(1)| = |\alpha^{-1}(i)|$ for $i = 1, \ldots, k$. This completes the proof.

COROLLARY 2. *If* $\dim U_\lambda(G) = \lambda(1)$, *then k is a divisor of m.*

The following result was proved in [**2**, Corollary 1].

THEOREM 7. *Let λ be the irreducible character of S_m corresponding to a Young diagram $(\lambda_1, \ldots, \lambda_t)$. Then* $\dim U_\lambda(S_m) = 0$ *if and only if $t > k$.*

The following result follows from the Proposition in [**12**, p. 20] and Theorem 1 in [**25**].

THEOREM 8. *Let λ be the irreducible character of S_m corresponding to a Young diagram $(\lambda_1, \ldots, \lambda_t)$. Then* $\dim U_\lambda(S_m) = \lambda(1)$ *if and only if $t = k$ and $\lambda_1 = \lambda_2 = \ldots = \lambda_k$.*

We remark that the necessity of the above theorem also follows easily from the Theorem in [**21**] and Theorem 6.

Let A_m denote the alternating group of degree m.

THEOREM 9. *Let λ be an irreducible character of A_m. Let $m = ks$ and $k \geqq s$. Then* $\dim U_\lambda(A_m) = \lambda(1)$ *if and only if $s = k$ and λ is the restriction of the self-associated irreducible character of S_m corresponding to the Young diagram $(\lambda_1, \ldots, \lambda_s)$ where $\lambda_1 = \ldots = \lambda_s = k$.*

Proof. If λ is the self-conjugate irreducible character induced by the associated characters χ and χ' on S_m, then by Theorem 2,

$$U_\lambda(A_m) = U_\chi(S_m) \oplus U_{\chi'}(S_m).$$

If $\dim U_\lambda(A_m) = \lambda(1)$ then we may assume without loss of generality that $\dim U_\chi(S_m) = \lambda(1)$ and $\dim U_{\chi'}(S_m) = 0$. Hence by Theorem 8, χ corresponds to the Young diagram $(\lambda_1, \ldots, \lambda_t)$ where $t = k$, $\lambda_1 = \lambda_2 = \ldots = \lambda_t = s$. Hence χ' corresponds to a Young diagram with s rows. However Theorem 7 implies that $s > k$, a contradiction.

If λ is not self-conjugate then, by Theorem 1, $\lambda + \bar{\lambda} = \chi|_{A_m}$ for some self-associated irreducible character χ of S_m where $\bar{\lambda}$ is the conjugate of λ. In view of Theorem 2,

$$\begin{aligned} \dim U_\lambda(A_m) = \lambda(1) &\Leftrightarrow \dim U_{\bar{\lambda}}(A_m) = \lambda(1) \\ &\Leftrightarrow \dim U_\chi(S_m) = \chi(1) \end{aligned}$$

$\Leftrightarrow \chi$ corresponds to the Young diagram $(\lambda_1, \ldots, \lambda_t)$ with $t = k$ and $\lambda_1 = \ldots = \lambda_t = s$.

Since χ is self-associated, we must have $s = k$. This completes the proof.

Two permutation groups H_1 and H_2 on N_1 and N_2 respectively are said to be of the same *type* if there exists an injection $\phi: N_1 \to N_2$ and an isomorphism $f: H_1 \to H_2$ such that

$$\phi(\sigma(i)) = f(\sigma)(\phi(i)) \text{ for all } i \in N_1, \sigma \in H_1.$$

The following result is useful in the sequel.

THEOREM 10 [**8**]. *Suppose $m \leqq 2k = 2 \dim U$. Then $U_\lambda(G)$ is trivial if and only if one of the following holds:*

1. G contains a subgroup of type S_n with $n > k$ and $\lambda|_{S_n}$ is a multiple of an irreducible character of S_n corresponding to a Young diagram $(\lambda_1, \ldots, \lambda_t)$ where $t > k$.

2. G contains a subgroup of type $S_k \wr S_2$ and

$$\lambda|_{S_k \wr S_2} = \lambda(1)\rho \wr \chi$$

where ρ is the sign character of S_k and χ is the sign character of S_2.

THEOREM 11. *If G has t orbits $O_1, O_2, \ldots O_t$ such that $|O_1| = \ldots = |O_t| = k$, then* $\dim U_\lambda(G) = \lambda(1)$ *if and only if $G = S_{O_1} \ldots S_{O_t}$ and $\lambda = \epsilon$.*

Proof. The sufficiency follow from Theorem 4. To prove the necessity, let $1 \leqq i \leqq t$. Given distinct elements $s, j \in O_i$, let $\alpha \in \Gamma_{m,k}$ such that

$$\begin{aligned} |\alpha(O_n)| &= k \quad &&\text{for } n \neq i, \\ |\alpha(O_i)| &= k - 1 \quad &&\text{and } \alpha(s) = \alpha(j). \end{aligned}$$

By Theorem 6, $e_\alpha^* = 0$. Hence $G_\alpha \neq \{1\}$ and therefore $(sj) \in G$. Hence $S_{0_i} \subseteq G$. This shows that $G = S_{0_1} \ldots S_{0_t}$. Hence $\lambda = \lambda_1 \# \ldots \# \lambda_t$ for some irreducible characters λ_i of S_i, $i = 1, 2, \ldots, t$. By Theorem 4,

$$\dim U_\lambda(G) = \prod_{i=1}^{t} \dim U_{\lambda_i}(S_{0_i}) = \lambda(1).$$

Hence $\dim U_{\lambda_i}(S_{0_i}) = \lambda_i(1)$ for all $i = 1, \ldots, t$. Since $|O_i| = k$ and $\dim U = k$ it follows from Theorem 8 that $\lambda_i = \epsilon$. This completes the proof.

Lemma 1. *Let G be a subgroup of S_6 containing neither 2-cycles nor 3-cycles. If* $\dim U = k = 3$, *then* $\dim U_\lambda(G) > \lambda(1)$ *for any irreducible character λ of G.*

Proof. Suppose that $\dim U_\lambda(G) \leqq \lambda(1)$. Let $\alpha \in \Gamma_{6,3}$ such that

$$\alpha^{-1}(1) = \{1, 2\}, \alpha^{-1}(2) = \{3, 4, 5\}, \alpha^{-1}(3) = \{6\}.$$

By Theorem 6, $e_\alpha^* = 0$ and hence $G_\alpha \neq \{1\}$. Suppose $|G_\alpha| > 2$. Then G contains a 2-cycle or a 3-cycle, a contradiction. Hence $|G_\alpha| = 2$. We may assume that $(12)(34) \in G_\alpha$. Then

$$\sum_{\sigma \in G_\alpha} \lambda(\sigma) = \lambda(1) + \lambda((12)(34)) = 0$$

and hence $\lambda((12)(34)) = -\lambda(1)$. Similarly, we can show that for $\beta_1, \beta_2 \in \Gamma_{6,3}$ defined by

$$\begin{aligned}
&\beta_1^{-1}(1) = \{3, 4\}, \beta_1^{-1}(2) = \{1, 5, 6\}, \beta_1^{-1}(3) = \{2\}\\
&\beta_2^{-1}(1) = \{1, 2\}, \beta_2^{-1}(2) = \{4, 5, 6\}, \beta_2^{-1}(3) = \{3\},\\
&G_{\beta_1} = \{1, (34)(56)\}, G_{\beta_2} = \{1, (12)(56)\} \text{ and}\\
&\lambda((34)(56)) = \lambda((12)(56)) = -\lambda(1).
\end{aligned}$$

Now for $\gamma \in \Gamma_{6,3}$ defined by

$$\gamma^{-1}(1) = \{1, 2\}, \gamma^{-1}(2) = \{3, 4\}, \gamma^{-1}(3) = \{5, 6\},$$

we have $G_\gamma = \{1, (12)(34), (12)(56), (34)(56)\}$. It follows that

$$\sum_{\sigma \in G_\alpha} \lambda(\sigma) = \lambda(1) - 3\lambda(1) = -2\lambda(1),$$

which contradicts the fact that $|G_\gamma|^{-1} \sum_{\sigma \in G_\gamma} \lambda(\sigma)$ is a non-negative integer. Hence $\dim U_\lambda(G) > \lambda(1)$.

Lemma 2. *If* $\dim U_\lambda(G) = \lambda(1)$ *then for any $(k - 1)$-dimensional subspace W of U, $W_\lambda(G) = 0$.*

Proof. This follows immediately from Theorem 6.

Theorem 12. *If* $\dim U_\lambda(G) = \lambda(1)$ *and λ is not linear, then G is a (k)-group.*

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Proof. Since $\dim U_\lambda(G) = \lambda(1)$, $\bar{\Delta} = \{\alpha\}$ for some $\alpha \in \Delta$ and by (2) we have

$$1 = \frac{1}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \lambda(\sigma).$$

If $G_\alpha = \{1\}$ then $1 = \lambda(1)$, a contradiction. Hence $G_\alpha \neq \{1\}$. This implies that G is a (k)-group.

THEOREM 13. *For $m = 2k = 2 \dim U$, $\dim U_\chi(G) = \chi(1)$ if and only if one of the following holds:*

(a) $G = S_{0_1} \cdot S_{0_2}$ *where* $|O_1| = |O_2| = k$, $\chi = \epsilon$.

(b) $G = S_m$ *and χ corresponds to the Young diagram* $(\chi_1, \ldots, \chi_k)$ *where* $\chi_1 = \ldots = \chi_k$.

(c) *G is of type* $S_k \wr S_2$, $\chi = \epsilon \wr 1$.

(d) *G is of type* $S_2 \wr S_3$, $\chi = \epsilon \wr \epsilon$, $k = 3$.

(e) $G = A_4$, $\chi \not\equiv 1$, *χ is linear*, $k = 2$.

Proof. The sufficiency follows from Theorems 11, 8 and 9 and formula (4). The proof of the necessity is divided into three cases:

Case 1. G is intransitive. Suppose G has an orbit O such that $|O| < k$. Let $\alpha \in \bar{\Delta}$ and $\pi = (12 \ldots k) \in S_k$. Then $\alpha \not\equiv \pi\alpha \pmod{G}$, $e_{\pi\alpha}^* \neq 0$. Hence $\dim U_\chi(G) \geqq 2\chi(1)$, a contradiction. Hence G has only two orbits O_1 and O_2 with $|O_1| = |O_2| = k$. By Theorem 11, we obtain (a).

Case 2. G is primitive.

(1) If $k = 1$, then $G = S_2$, $\chi \equiv 1$ and we obtain (b).

(2) If $k = 2$, then $G = A_4$ or S_4. In the first case, by Theorem 9, we have (e). In the second case, by Theorem 8, we have (b).

(3) If $k = 3$, then G is of the type A_6, S_6, $\langle (126)(354), (12345), (2345) \rangle$ or $\langle (126)(354), (12345), (25)(34) \rangle$ (see [**3**]). The first case cannot occur by Theorem 9. The second case implies (b) by Theorem 8. The third and fourth cases cannot happen by Lemma 1.

(4) $k = 4$. If χ is not linear then by Theorem 12 G is a (4)-group. Since G is primitive, by Theorem 3.6 in [**6**], $G \supseteq A_8$. If χ is linear then by Lemma 2, $W_\chi(G) = 0$ for some 3-dimensional subspace W of U. Hence by the theorem in [**7**], $G \supseteq A_8$ since G is primitive. Thus by Theorems 8 and 9 we obtain (b).

(5) $k > 4$. Since $\dim U_\chi(G) = \chi(1)$, by Lemma 2, G is $(k-1)$-group. Hence by Theorem 6.3 in [**6**], G contains A_m. Appealing to Theorems 8 and 9 we obtain (b).

Case 3. G is imprimitive transitive. Let $\{N_1, \ldots, N_t\}$ be a complete block system of G. Suppose $t = 2$. For each function β_1 from N_1 to $\{1, 2, \ldots, k-1\}$, let $\beta \in \Gamma_{m,k}$ be defined by

$\beta(i) = \beta_1(i)$ for $i \in N_1$,

$|\beta^{-1}(j) \cap N_2| = 1$ for $j = 1, \ldots, k$.

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Since $|\beta^{-1}(j)| \neq 2$ for some j, by Theorem 6, $e_\beta{}^* = 0$ and hence

$$\sum_{\sigma \in G_\beta} \chi(\sigma) = 0.$$

Hence $G_{N_2} = \{g \in G : g(i) = i, i \in N_2\}$ is a $(k-1)$-group and $\chi|_{G_{N_2}}$ is a $(k-1)$-character. By Theorem 10,

$$G_{N_2} = S_{N_1} \text{ and } \chi|_{G_{N_2}} = \chi(1)\epsilon.$$

Similarly, we can show that

$$G_{N_1} = S_{N_2} \text{ and } \chi|_{G_{N_2}} = \chi(1)\epsilon.$$

Hence $G = S_k \wr S_2$. By Theorem 3, χ is of the form $(\lambda \# \rho)^G$, $\lambda \wr 1$ or $\lambda \wr \epsilon$ where λ and ρ are distinct irreducible characters of S_k.

If $\chi = (\lambda \# \rho)^G$, then Theorem 5 implies that

$$\dim U_\lambda(S_{N_1}) = \lambda(1), \dim U_\rho(S_{N_2}) = \rho(1).$$

By Theorem 4, $\lambda = \epsilon$ and $\rho = \epsilon$, a contradiction. If $\chi = \lambda \wr 1$ or $\lambda \wr \epsilon$, we have

$$\chi|_{S_{N_i}} = \lambda(1)\lambda = \chi(1)\epsilon, \ i = 1, 2.$$

Hence $\lambda = \lambda(1)\epsilon$. By the irreducibility of λ we have $\lambda(1) = 1$. Hence $\chi(1) = 1$. Using formula (4), we have $\chi = \epsilon \wr 1$. This gives (c).

We now consider individual values of k.

For $k = 2$, we have $t = 2$ and this implies that we have (c).

For $k = 3$, we have $t = 2$ or 3. We need only to consider $t = 3$. Let $N_1 = \{x_1, x_2\}$, $N_2 = \{y_1, y_2\}$ and $N_3 = \{z_1, z_2\}$. Let $\alpha \in \Gamma_{6,3}$ be defined by

$$\alpha^{-1}(1) = \{x_1, y_1\}, \alpha^{-1}(2) = \{y_2, z_2\}, \alpha^{-1}(3) = \{z_1, x_2\}.$$

Then $G_\alpha = \{1\}$ and hence G is not a (3)-group. By Theorem 12, χ is linear. Now let $\beta \in \Gamma_{6,3}$ be defined as follows:

$$\beta^{-1}(1) = \{x_1, x_2, y_1\}, \beta^{-1}(2) = \{y_2, z_2\}, \beta^{-1}(3) = \{z_1\}.$$

Since $\dim U_\chi(G) = \chi(1)$, by Theorem 6 we have $e_\beta{}^* = 0$. Hence $(x_1x_2) \in G$ and

$$0 = \sum_{\sigma \in G_\beta} \chi(\sigma) = 1 + \chi((x_1x_2)).$$

Hence $\chi((x_1x_2)) = -1$. Similarly we can show that (y_1y_2), $(z_1z_2) \in G$ and

$$\chi((y_1y_2)) = \chi((z_1z_2)) = -1.$$

It follows that $S_{N_1} \cdot S_{N_2} \cdot S_{N_3} \subseteq G$ and $\chi|_{S_{N_i}} = \epsilon$, $i = 1, 2, 3$. Next, let $\gamma \in \Gamma_{6,3}$ be defined by

$$\gamma^{-1}(1) = \{x_1, y_1, z_1\}, \gamma^{-1}(2) = \{x_2, y_2\}, \gamma^{-1}(3) = \{z_2\}.$$

In view of Theorem 6, $e_\gamma^* = 0$. Hence

$$(x_1y_1)(x_2y_2) \in G \text{ and } \chi((x_1y_1)(x_2y_2)) = -1.$$

Similarly we can show that

$$(x_1z_1)(x_2z_2) \in G \text{ and } \chi((x_1z_1)(x_2z_2)) = -1;$$
$$(y_1z_1)(y_2z_2) \in G \text{ and } \chi((y_1z_1)(y_2z_2)) = -1.$$

Hence $G = S_2 \wr S_3$ and $\chi = \epsilon \wr \epsilon$.

For $k \geqq 4$ we have $2k < 3(k - 1)$. Since G is a $(k - 1)$-group, G is of type $S_2 \wr S_4$ where $k = 4$ or $t = 2$ (see Lemma 8.7 and Corollary 6.2 in [**6**]). The second case implies (c). Suppose that G is of the type $S_2 \wr S_4$. Let $\delta \in \Gamma_{8,4}$ be defined by

$$|\delta^{-1}(1) \cap N_i| = 1, \quad i = 1, 2, 3.$$
$$|\delta^{-1}(2) \cap N_i| = 1, \quad i = 1, 4.$$
$$|\delta^{-1}(3) \cap N_i| = 1, \quad i = 2, 4.$$
$$|\delta^{-1}(4) \cap N_i| = 1, \quad i = 3.$$

Then $G_\delta = \{1\}$ and hence $e_\delta^* \neq 0$. By Theorem 6, we obtain a contradiction. This completes the proof.

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