A CONTRIBUTION TO THE THEORY OF CHROMATIC POLYNOMIALS

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SUMMARY

Two polynomials $\theta(G, n)$ and $\phi(G, n)$ connected with the colourings of a graph G or of associated maps are discussed. A result believed to be new is proved for the lesser-known polynomial $\phi(G, n)$. Attention is called to some unsolved problems concerning $\phi(G, n)$ which are natural generalizations of the Four Colour Problem from planar graphs to general graphs. A polynomial $\chi(G, x, y)$ in two variables x and y, which can be regarded as generalizing both $\theta(G, n)$ and $\phi(G, n)$ is studied. For a connected graph $\chi(G, x, y)$ is defined in terms of the "spanning" trees of G (which include every vertex) and in terms of a fixed enumeration of the edges. The invariance of $\chi(G, x, y)$ under a change of this enumeration is apparently a new result about spanning trees. It is observed that the theory of spanning trees now links the theory of graph-colourings to that of electrical networks.

1. Introduction. A *graph* G consists of a set $V(G)$ of elements called *vertices* together with a set $E(G)$ of elements called *edges*, the two sets having no common element. With each edge there are associated either one or two vertices called its *ends*.

An edge of G is a *loop* or *link* according as the number of its ends is 1 or 2. For convenience we sometimes say that a link has two distinct ends and a loop two equal ends.

We restrict ourselves to *finite* graphs, that is graphs for which $V(G)$ and $E(G)$ are both finite.

If $V(G) = 0$ we must have $E(G) = 0$ also.

A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and each edge of H has the same ends in H as in G. The subgraph H of G is a *spanning* subgraph of G if $V(H) = V(G)$. The subgraph of G for which $V(H)$ is a given subset W of $V(G)$ and $E(H)$ is the set of all edges of G having no end outside W, will be denoted by $G[W]$.

A sequence $(a_0, A_1, a_1, A_2, a_2, \ldots, A_n, a_n)$, in which the terms are alternately vertices a_i and edges A_j of G is a *path* from a_0 to a_n in G if it satisfies the following conditions.

(i) If $1 \leqslant i \leqslant n$ the ends of A_i are a_{i-1} and a_i.

(ii) If $1 \leqslant i \leqslant n$ then $a_{i-1} = a_i$ if and only if A_i is a loop.

It is not required that the terms of the sequence shall be all distinct. If they are distinct the path is *simple*. If the sequence has more than one term and its terms are distinct except that $a_0 = a_n$ then the path is *circular*.

If x and y are elements of $V(G)$ we say x and y are *connected* in G if there is a path from x to y in G. The relation of connection in G is clearly an equivalence

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relation. Hence if $V(G)$ is non-null it can be partitioned into disjoint non-null subsets $V_1, \ldots, V_k$ such that two vertices of G are connected in G if and only if they belong to the same set V_i. The subgraphs $G[V_i]$ of G are the *components* of G. Together they include all the edges and vertices of G, and no two of them have an edge or vertex in common. We denote the number of components of G by $p_0(G)$. The graph G is *connected* if $p_0(G) = 0$ or 1. The first case arises only when $V(G) = 0$ and $E(G) = 0$. Clearly each component of a graph is connected.

A connected graph in which there is no circular path is a *tree*.

We write $\alpha_0(G)$ and $\alpha_1(G)$ for the numbers of elements of $V(G)$ and $E(G)$ respectively.

Let Q_n be a finite set of $n > 0$ elements. Let f be a mapping of $V(G)$ into Q_n. We call f an *n-colouring* of G if each edge of G has two ends x and y such that $f(x) \neq f(y)$. We denote the number of n-colourings of G, defined in terms of Q_n, by $P(G, n)$. If $V(G) = 0$ we take this number to be 1. We observe that $P(G, n) = 0$ if G has a loop.

$P(G, n)$ is not altered by replacing Q_n by another set of n elements. We find it convenient to take Q_n as the ring of residue classes mod n.

The function $P(G, n)$ was studied by Hassler Whitney **(6; 7)**. He showed that when G is loopless, $P(G, n)$ is a polynomial in n of degree $\alpha_0(G)$. For planar graphs G the polynomial has been studied in great detail by Birkhoff and Lewis **(1)**, who associated it with the dual map of G. Following them we call $P(G, n)$ the *chromatic polynomial* of G.

The following explicit formula for $P(G, n)$ is due to Hassler Whitney.

$$P(G, n) = \sum_S (-1)^{\alpha_1(S)} n^{p_0(S)}. \tag{1}$$

The summation is over all spanning subgraphs S of G. We shall find another explicit formula in terms of the *spanning trees* of G valid when G is connected. A spanning tree is a spanning subgraph which is a tree.

At this stage it is convenient to apply some of the concepts of elementary combinatorial topology. We *orient* G by distinguishing one end of each edge A as the *positive* end $p(A)$ and one as the *negative* end $q(A)$. The positive and negative ends coincide if A is a loop but not if A is a link. If $a \in V(G)$ and $A \in E(G)$ we write $\eta(A, a) = 0$ if A is a loop or if a is not an end of A. Otherwise we write $\eta(A, a) = 1$ or -1 according as a is the positive or the negative end of A. A mapping f of $V(G)$ or $E(G)$ into Q_n is a *0-chain* or *1-chain* respectively *on G over Q_n*.

If $V(G)$ is null we consider that there is just one 0-chain on G over Q_n. Similarly if $E(G)$ is null there is just one 1-chain on G over Q_n.

If h is a 0-chain on G over Q_n its *coboundary* δh is the 1-chain on G over Q_n satisfying

$$(\delta h)(A) = \sum_a \eta(A, a)\, h(a) \tag{2}$$

for each $A \in E(G)$. This may be rewritten as

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(2a) $$(\delta h)(A) = h(p(A)) - h(q(A)).$$

If g is a 1-chain on G over Q_n its *boundary* ∂g is the 0-chain on G over Q_n satisfying

(3) $$(\partial g)(a) = \sum_A \eta(A, a)g(A)$$

for each $a \in V(G)$. We call g a *1-cycle* on G over Q_n if $\partial g = 0$, that is $(\partial g)(a) = 0$ for each a.

2. Colour-coboundaries and colour-cycles. A *colour-coboundary* or *colour-cycle* on G over Q_n is a 1-chain g on G over Q_n which is a coboundary or a 1-cycle respectively and which satisfies $g(A) \neq 0$ for each $A \in E(G)$.

We denote the numbers of colour-coboundaries and colour-cycles on G over Q_n by $\theta(G, n)$ and $\phi(G, n)$ respectively. These numbers are independent of the orientation of G, by (2a) and (3). We consider that $\theta(G, n) = \phi(G, n) = 1$ if G has no edge.

The colour-coboundaries on G over Q_n are the coboundaries of the n-colourings of G, by (2a). Another consequence of (2a) is that $\delta h_1 = \delta h_2$ for 0-chains h_1 and h_2 on G over Q_n if and only if $h_1(a) - h_2(a)$ is constant in each component of G. Accordingly

(4) $$\theta(G, n) = n^{-p_0(G)}P(G, n).$$

It follows that $\theta(G, n) = 0$ if G has a loop. The function $\phi(G, n)$ need not vanish if G has a loop. Indeed if g is a 1-chain on G over Q_n and A is a loop of G then the 0-chain ∂g is independent of $g(A)$, by (3). Hence if G_0 is the graph obtained from G by suppressing the loops, say $l(G)$ in number, we have

(5) $$\phi(G_0, n) = (n - 1)^{-l(G)}\phi(G, n).$$

However $\phi(G, n)$ does vanish if G has an *isthmus*. An edge A of G with ends x and y is called an isthmus of G if each path from x to y has A as a term. Thus an isthmus is necessarily a link. If G_A' is the graph obtained from G by suppressing A we may say that A is an isthmus of G provided x and y belong to different components of G_A'. An equivalent definition is that A is an isthmus of G provided that it is a term of no circular path in G. For if A is a term of such a circular path then x and y are clearly connected in G_A'. And if a path from x to y exists in G_A' the path of this kind with fewest terms is simple and can be extended to form a circular path in G having A as a term.[1]

We observe that a tree may be defined as a connected graph in which each edge is an isthmus.

The proof that $\phi(G, n)$ vanishes when G has an isthmus A is as follows. Let H be the component of G_A' having the end x of A in G as a vertex. Let g be any 1-cycle on G over Q_n. Then

$$\sum_B \eta(B, b)g(B) = 0$$

[1]Our term "isthmus" applies to each of the two kinds of edge for which König uses the terms "Brücke" and "Endkante" (**3**, pp. 3, 179).

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for each $b \in V(H)$, where B runs through $E(G)$, by (3). Summing this over all the vertices of H we obtain $\eta(A, x)g(A) = 0$. Hence $g(A) = 0$. Accordingly no 1-cycle on G over Q_n is a colour-cycle.

The connection between the function $\phi(G, n)$ and the ordinary theory of map-colourings is best seen by considering two dual graphs G and G^* on the sphere. It may be shown—though we do not prove it here—that $\phi(G^*, n) = \theta(G, n)$. Accordingly each of the following unproved propositions is equivalent to the famous Four Colour Conjecture.

(i) $\theta(G, 4) > 0$ if G is a planar graph without a loop.

(ii) $\phi(G, 4) > 0$ if G is a planar graph without an isthmus.

I wish to draw attention to some unsolved problems related to (ii) but having to do with general graphs. They are the problems of proving or disproving the following conjectures.

CONJECTURE I: *There exists a positive integer m such that $\phi(G, n) > 0$ whenever $n \geqslant m$ and G has no isthmus.*

CONJECTURE II: $\phi(G, n) > 0$ *whenever $n \geqslant 5$ and G has no isthmus.*

Conjecture II is a stronger version of Conjecture I. We cannot replace 5 by a smaller integer because it can be shown that the Petersen graph (**3**, p. 194) satisfies $\phi(G, 4) = 0$.

We prove $\phi(G, 4) = 0$ for the Petersen graph as follows. If $\phi(G, 4) > 0$ then for any orientation of G we can find a colour-cycle g on G over Q_4. Let $[m]$ denote the residue class of an integer m modulo 4. If a is any vertex of G the three residue classes $\eta(A, a)g(A)$ corresponding to the edges A having a as an end are non-zero and sum to zero. Their values must be either [1], [1], and [2] or [− 1], [− 1] and [2]. In the first case we call a a *positive* vertex, in the second a *negative* vertex. The edges A such that $g(A) = [1]$ or $[-1]$ are therefore the edges of some disjoint circular paths no two of which have a common term and which together involve all the vertices. Each of these paths has an even number of edges since positive and negative vertices must alternate in it. It follows that the edges of G can be arranged in three disjoint classes so that each vertex is an end of one member of each class. But it is well known that this is not true for the Petersen graph (**4**).

We may perhaps regard the following theorem as a very short first step towards a verification of Conjecture I.

THEOREM: *If* $\phi(G, n) > 0$ *then* $\phi(G, n + 1) > 0$.

Proof. In the preceding combinatorial definitions we may replace Q_n by the ring of ordinary integers, obtaining *integral* 0-chains, 1-chains, 1-cycles, etc.

If $\phi(G, n) > 0$ there exists a colour-cycle g on G over Q_n. It follows that there is an integral 1-cycle g' on G such that $g'(A) \in g(A)$ and $|g'(A)| < n$ for each $A \in E(G)$. This is a consequence of Theorem IV of (**5**). It is true that that theorem is stated only for the case in which G is a simplicial 1-complex, that is a graph without loops and in which no two links have the same two ends, but its proof is valid with only trivial modifications in the general case. Now for each $A \in E(G)$ we have $g'(A) \not\equiv 0 \bmod (n+1)$. Replacing each integer $g'(A)$ by its residue class mod $(n+1)$ we obtain a colour-cycle on G over Q_{n+1}. The theorem follows.

The methods now available for the computation of $\theta(G, n)$ and $\phi(G, n)$ are laborious. They depend on some recursion formulae which we exhibit below.

If A is an edge of G not a loop we define G_A'' as the graph obtained from G by suppressing A and then identifying the ends of A in G to form a single vertex t.

By examining the relationships between the colour-coboundaries, and between the colour-cycles, of the three graphs G, G_A' and G_A'', where A is any edge of G not a loop or isthmus, we obtain the identities

(6) $$\theta(G, n) = \theta(G_A', n) - \theta(G_A'', n),$$

(7) $$\phi(G, n) = \phi(G_A'', n) - \phi(G_A', n).$$

For a disconnected graph G with components $G_1, \ldots, G_k$ we evidently have

(8) $$\theta(G, n) = \prod_{i=1}^{k} \theta(G_i, n),$$

(9) $$\phi(G, n) = \prod_{i=1}^{k} \phi(G_i, n).$$

LEMMA: *If J is a graph in which every edge is an isthmus then every 1-chain on J over Q_n is a coboundary on J.*

Proof. If $\alpha_1(J) = 0$ this is trivial. Suppose it is true whenever $\alpha_1(J)$ is less than some positive integer q. Consider the case $\alpha_1(J) = q$.

Let h be any 1-chain on J over Q_n. Let h_A be the 1-chain on J_A' over Q_n such that $h_A(B) = h(B)$ for each $B \in E(J) - \{A\}$. By the inductive hypothesis h_A is the coboundary of a 0-chain f on G_A'. Let x be the positive and y the negative end of A in J. Let J_0 be the component of J_A' of which x (but not y) is a vertex. If we replace $f(a)$ by $f(a) + s$ for each $a \in V(J_0)$, where s is an element of Q_n, we shall not alter the coboundary of f in J_A'. We may therefore suppose that $f(x) - f(y) = h(A)$. Then h is the coboundary of f in J.

The Lemma follows by induction.

Now consider a graph for which each edge is either a loop or an isthmus. Suppose such a graph H has $l(H)$ loops and $i(H)$ isthmuses. We have, as a consequence of the Lemma,

(10) $$\theta(H, n) \begin{cases} = 0 & \text{if } l(H) > 0, \\ = (n-1)^{i(H)} & \text{if } l(H) = 0. \end{cases}$$

As a consequence of (5) we have also

(11) $$\phi(H, n) \begin{cases} = 0 & \text{if } i(H) > 0, \\ = (n-1)^{l(H)} & \text{if } i(H) = 0. \end{cases}$$

3. The dichromate of a graph. We now define a function $\chi(G, x, y)$ of two variables x and y, which may be regarded as generalizing both $\theta(G, n)$ and $\phi(G, n)$. We call it the *dichromate* of G.

If G has no edge we write

$$\chi(G, x, y) = 1. \tag{12}$$

If G has an edge and is connected we proceed as follows. First we enumerate the edges of G as $A_1, \ldots, A_m$.

Consider any spanning tree T of G. Suppose A_j is an edge of T. Then T_{A_j}' has two components, C and D say. Each has one end of A_j as a vertex. We say A_j is *internally active* in T if each edge A_k of G other than A_j which has one end a vertex of C and one end a vertex of D satisfies $k < j$.

Now suppose A_j is not an edge of T. Denote its ends by a and b. (They may not be distinct.) There is a simple path P in T from a to b. There is only one such path. For suppose there are two distinct simple paths P_1 and P_2 in T from a to b. Then we may suppose some edge A_k of T appears in P_1 but not in P_2. Let its ends be c and d, c preceding A_k in P_1. Then in T_{A_k}' there are paths from d to b, from a to b and from a to c. Hence c and d are vertices of the same component of T_{A_k}'. This is impossible since A_k is an isthmus of T. We say A_j is *externally active* in T if each A_k which is a term of P satisfies $k < j$.

If A_j is an edge of G we write $\lambda(T, A_j) = 1$ or 0 according as A_j is or is not internally active in T. We write also $\mu(T, A_j) = 1$ or 0 according as A_j is or is not externally active in T. We call $\lambda(T, A_j)$ and $\mu(T, A_j)$ the *internal* and *external activities* respectively of A_j in T. We denote by $r(T)$ and $s(T)$ the numbers of edges of G which are internally and externally active respectively in T.

We define the dichromate of G by the formula

$$\chi(G, x, y) = \sum_T x^{r(T)} y^{s(T)}, \tag{13}$$

the summation being over all the spanning trees of G.

We note that at least one spanning tree of G exists. This is proved by König (**3**, p. 60). (Our "spanning tree" is König's Gerüst). Hence the polynomial on the right of (13) is not identically zero.

To make the above definition significant we must show that $\chi(G, x, y)$, as defined by (13), is independent of the particular enumeration of the edges of G which is used.

To prove this we study the effect of interchanging the symbols A_i and A_{i+1} between the two corresponding edges. With respect to the new enumeration let $\lambda'(T, A_j)$ and $\mu'(T, A_j)$ denote the internal and external activities respectively in the spanning tree T of G, of the edge initially denoted by A_j. For each T let the interchange of the two symbols replace $r(T)$ and $s(T)$ by $r'(T)$ and $s'(T)$ respectively.

The following argument is stated in terms of the initial enumeration.

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First we observe that the interchange leaves $\lambda(T, A_j)$ and $\mu(T, A_j)$ unaltered if A_j is not A_i or A_{i+1}. Hence

$$(14) \qquad r'(T) = r(T) - \lambda(T, A_i) - \lambda(T, A_{i+1}) + \lambda'(T, A_i) + \lambda'(T, A_{i+1}),$$

$$(15) \qquad s'(T) = s(T) - \mu(T, A_i) - \mu(T, A_{i+1}) + \mu'(T, A_i) + \mu'(T, A_{i+1}).$$

We partition the set of all spanning trees of G into three disjoint classes X, Y, and Z as follows. $T \in X$ if A_i and A_{i+1} are both edges of T, $T \in Y$ if neither A_i nor A_{i+1} is an edge of T, and $T \in Z$ if one but not both of A_i and A_{i+1} is an edge of T.

If $T \in X$ or $T \in Y$ it is clear that the internal and external activities in T of A_i and A_{i+1} are not altered by the interchange. Hence $r'(T) = r(T)$ and $s'(T) = s(T)$ in these cases, by (14) and (15).

If $T \in Z$ let A_j be the member of the pair $\{A_i, A_{i+1}\}$ which is an edge of T and let A_k be the other member. Let the ends of A_j be a and b. Let C and D be the two components of T_{A_j}', having a and b respectively as a vertex. Let c and d be the ends, not necessarily distinct, of A_k. We partition the set Z into two disjoint subclasses Z_1 and Z_2 by the following rule: $T \in Z_1$ if c and d are vertices of the same component of T_{A_j}', and $T \in Z_2$ otherwise.

If $T \in Z_1$ the simple path P from c to d in T does not have A_j as a term. Accordingly the internal and external activities of A_j and A_k in T are not affected by the interchange. So $r'(T) = r(T)$ and $s'(T) = s(T)$ in this case also.

Suppose $T \in Z_2$. Then we may suppose that c is a vertex of C and d is a vertex of D. Let $\sigma(T)$ be the spanning subgraph of G obtained by suppressing the edge A_j and adjoining the edge A_k. Clearly $\sigma(T)$ is connected. We show that it is a spanning tree of G. For otherwise there is a circular path P in $\sigma(T)$. This has A_k as a term since it is not a path in T. This implies that there is a simple path from c to d in $(\sigma(T))_{A_k}'$, that is in T_{A_j}', which is false. Now clearly $\sigma(T) \in Z_2$ and $\sigma(\sigma(T)) = T$. We note that A_j and A_k must be redefined in terms of $\sigma(T)$ before the operation σ is repeated.

We deduce that Z_2 can be partitioned into disjoint pairs of the form $\{T, \sigma(T)\}$ such that A_i is an edge of T. In what follows we take T to be the first member of such a pair.

Suppose first that some edge A_w of G distinct from A_i and A_{i+1} is internally active in T but not in $\sigma(T)$.

Without loss of generality we may suppose A_w is an edge of the tree C. Let C_1 and C_2 be the two components of C_{A_w}', a being a vertex of C_2. Let A_w have ends $\alpha \in V(C_1)$ and $\beta \in V(C_2)$. If $c \in V(C_2)$ then since $\lambda(\sigma(T), A_w) = 0$ there is an edge A_v of G such that $v > w$ and which has one end in $V(C_1)$ and one end in $V(C_2)$ or $V(D)$. But then A_w cannot be internally active in T, contrary to its definition. We deduce that $c \in V(C_1)$. Now since $\lambda(\sigma(T), A_w) = 0$ it follows that there is an edge A_v of G having one end γ in $V(C_2)$ and one end δ in $V(D)$ or $V(C_1)$, and which satisfies $v > w$. Actually $\delta \in V(D)$ since $\lambda(T, A_w) = 1$. This state of affairs is represented in Figure 1.

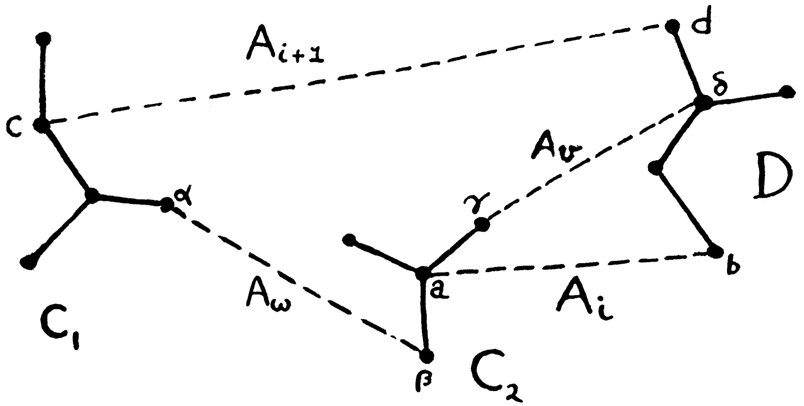

Figure 1

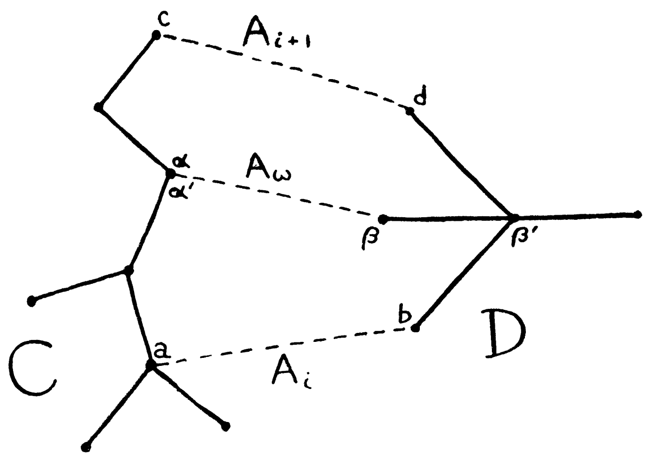

Figure 2

Since $\lambda(T, A_w) = 1$ we have $v > w > i + 1$. Hence

(16) $\lambda(T, A_i) = \lambda'(T, A_i) = 0, \ \lambda(\sigma(T), A_{i+1}) = \lambda'(\sigma(T), A_{i+1}) = 0.$

There is a circular path J from a to a in G which has A_i, A_{i+1} and A_w as terms. Apart from these terms it is made up of three simple paths, one from b to d in D, one from c to α in C_1 and one from β to a in C_2. It follows that the simple paths from c to d in T and from a to b in $\sigma(T)$ each have A_w as a term. Hence

(17) $\mu(T, A_{i+1}) = \mu'(T, A_{i+1}) = 0, \ \mu(\sigma(T), A_i) = \mu'(\sigma(T), A_i) = 0.$

Suppose next that some edge A_w of G distinct from A_i and A_{i+1} is externally active in T but not in $\sigma(T)$. Let its ends be α and β. They are not vertices of the same tree C or D; otherwise the simple paths from α to β in T and $\sigma(T)$ would be identical and this would imply $\mu(T, A_w) = \mu(\sigma(T), A_w)$. Hence we may suppose $\alpha \in V(C)$ and $\beta \in V(D)$. (See Fig. 2.)

Let P_1 and P_2 be the simple paths in T and $\sigma(T)$ respectively from α to β. Then A_i is a term of P_1 and A_{i+1} is a term of P_2. Since $\mu(T, A_w) = 1$ we have $w > i$ and therefore $w > i + 1$. Hence formula (16) holds in this case also.

In P_1 let α' be the last vertex of G preceding A_i which is a term of P_2, and let β' be the first vertex of G succeeding A_i which is a term of P_2. Clearly $\alpha' \in V(C)$ and $\beta' \in V(D)$. Let R_1 and R_2 be the subsequences of P_1 and P_2 respectively extending from α' to β'. There is a circular path J in G formed by taking first the terms of R_1 and then the terms of R_2 in reverse order. It has A_i and A_{i+1} as terms, for they are terms of R_1 and R_2 respectively. The subsequences of P_1 and P_2 extending from α to α' are identical, since C is a tree. Similarly the subsequences of P_1 and P_2 extending from β' to β are identical.

Since $\mu(T, A_w) = 1$ and $\mu(\sigma(T), A_w) = 0$ there must be an edge A_v of G which is a term of J and satisfies $v > w$. Then the simple paths from c to d in T and from a to b in $\sigma(T)$ each have A_v as a term. Hence formula (17) still holds.

Next we consider the case in which some edge of G is internally or externally active in $\sigma(T)$ but not in T. We first go over to the new enumeration by interchanging the symbols A_i and A_{i+1}. This interchanges T and $\sigma(T)$. The foregoing argument shows that (16) and (17) are true in the new enumeration. They are relations between the two enumerations. To state them in terms of the old enumeration we have merely to interchange the symbols λ and λ', μ and μ', A_i and A_{i+1} and finally T and $\sigma(T)$. But the sets of equations are invariant under this operation.

In all these three cases (16) and (17) are true. Hence by (14) and (15) we have $r'(T) = r(T)$, $r'(\sigma(T)) = r(\sigma(T))$, $s'(T) = s(T)$ and $s'(\sigma(T)) = s(\sigma(T))$.

We now consider the remaining case, in which $\lambda(T, A_w) = \lambda(\sigma(T), A_w)$ and $\mu(T, A_w) = \mu(\sigma(T), A_w)$ for each edge A_w of G other than A_i and A_{i+1}. If there is an edge A_v of G satisfying $v > i + 1$ and having ends in both $V(C)$ and $V(D)$, then $\lambda(T, A_i) = \lambda'(T, A_i) = \lambda(\sigma(T), A_{i+1}) = \lambda'(\sigma(T), A_{i+1}) = 0$. If there is no such edge A_v we have instead $\lambda(T, A_i) = \lambda'(\sigma(T), A_{i+1}) = 0$ and $\lambda(\sigma(T), A_{i+1}) = \lambda'(T, A_i) = 1$.

There is a circular path J from a to a in G having A_i and A_{i+1} as terms and otherwise consisting of a simple path from a to c in C and another from d to b in D. If J has a term A_v such that $v > i + 1$, then $\mu(T, A_{i+1}) = \mu'(T, A_{i+1}) = \mu(\sigma(T), A_i) = \mu'(\sigma(T), A_i) = 0$. If J has no such term we have instead $\mu(T, A_{i+1}) = \mu'(\sigma(T), A_i) = 1$ and $\mu(\sigma(T), A_i) = \mu'(T, A_{i+1}) = 0$.

It follows that $r'(T) = r(\sigma(T))$, $r'(\sigma(T)) = r(T)$, $s'(T) = s(\sigma(T))$ and $s'(\sigma(\mathrm{T})) = s(T)$.

The foregoing analysis shows that the sum on the right of (13) is not affected by the interchange of the symbols A_i and A_{i+1}. All that happens is that the contributions to the sum of certain pairs of trees are interchanged. But any permutation of the symbols $A_1, \ldots, A_m$ can be effected by a finite number of interchanges of consecutive symbols. Hence the function $\chi(G, x, y)$ defined by (13) is independent of the particular enumeration of edges employed.

We extend the definition of the dichromate to graphs which are not connected as follows. If the components of G are $G_1, \ldots, G_k$, then

$$(18) \qquad \chi(G, x, y) = \prod_{i=1}^{k} \chi(G_i, x, y).$$

This is consistent with (12) in the case of an edgeless graph.

We note some general properties of the dichromate.

(i) *$\chi(G, x, y)$ is a polynomial of degree $\alpha_0(G) - p_0(G)$ in x and of degree $\alpha_1(G) - \alpha_0(G) + p_0(G)$ in y.*

Proof. By a simple induction we find that a graph S in which each edge is an isthmus satisfies $\alpha_1(S) = \alpha_0(S) - p_0(S)$. A connected graph G has at least one spanning tree, and each such tree T satisfies $\alpha_1(T) = \alpha_0(G) - p_0(G)$.

If G is connected and has an edge the theorem follows from (13). For the contribution to the sum on the right of (13) of any spanning tree T is of degree at most $\alpha_1(T)$ in x and at most $\alpha_1(G) - \alpha_1(T)$ in y. By choosing a suitable enumeration of the edges of G we can arrange that either of these values is attained. The proposition follows in this case.

We extend it to all G by applying (12) and (18).

(ii) *If A is an edge of G not a loop or an isthmus, then*

$$(19) \qquad \chi(G, x, y) = \chi(G_A', x, y) + \chi(G_A'', x, y).$$

Proof. This proof depends on the observation that for a connected G the spanning trees of G_A' are those spanning trees of G which do not have A as an edge, while the spanning trees of G_A'' are the graphs T_A'' such that T is a spanning tree of G having A as an edge. We enumerate the edges of G so that $A = A_1$. We obtain corresponding enumerations for G_A' and G_A'' by rejecting A_1 and then reducing each suffix by 1. With these enumerations each tree not having A as an edge makes the same contribution to $\chi(G_A', x, y)$ as to $\chi(G, x, y)$, and any other tree T makes the same contribution to $\chi(G, x, y)$ as does T_A'' to $\chi(G_A'', x, y)$.

The proposition follows for a connected G.

If G is not connected let G_1 be its component having A as an edge. Then G_A' and G_A'' have the same components as G except that G_1 is replaced by $(G_1)_A'$ and $(G_1)_A''$ respectively. Since (19) is true for G_1 it follows from (18) that it is true also for G.

(iii) *Let H be a graph having $l(H)$ loops, $i(H)$ isthmuses, and no other edge. Then*

$$(20) \qquad \chi(H, x, y) = x^{i(H)} y^{l(H)}.$$

Proof. If H is connected form H_0 from it by suppressing the loops. Clearly H_0 is the only spanning tree of H. So (20) follows from (12) and (13). Using (18) we readily extend the formula to the general case.

Formulae (19) and (20) provide a method for computing the dichromate of a given graph G. If G has an edge A which is not a loop or an isthmus then (19) expresses the dichromate in terms of the dichromates of simpler graphs. Otherwise (20) gives the dichromate directly.

Such computations may sometimes be shortened by using the following theorem.

(iv) *If G consists of two connected graphs H_1 and H_2 having just one vertex b in common, then*

$$\chi(G, x, y) = \chi(H_1, x, y)\ \chi(H_2, x, y).$$

To prove this we observe that a subgraph of G is a spanning tree of G if and only if it is the union of a spanning tree of H_1 and a spanning tree of H_2. We then apply (13).

Comparing (19) and (20) with (6) and (10), or with (7) and (11), we arrive at inductive proofs of the following formulae.

$$\theta(G, n) = (-1)^{\alpha_0(G) - p_0(G)} \chi(G, 1 - n, 0), \tag{21}$$

$$\phi(G, n) = (-1)^{\alpha_1(G) - \alpha_0(G) + p_0(G)} \chi(G, 0, 1 - n). \tag{22}$$

These formulae justify our description of $\chi(G, x, y)$ as generalizing both $\theta(G, n)$ and $\phi(G, n)$.

The result that for a connected graph G the sum on the right of (13) is invariant under a change of enumeration of the edges is an interesting theorem about the spanning trees of G. As one of its corollaries we have:

For each enumeration of the edges of G there exist spanning trees T_1 and T_2 of G such that each edge of T_1 is internally active in T_1 and each edge of G not an edge of T_2 is externally active in T_2.

The number $C(G)$ of spanning trees of a graph G is important in the theory of electrical networks in which the conductance of each wire is unity. A summary of this theory is given in **(2)**. So the theory of spanning trees provides a link between the theory of graph-colourings and the theory of electrical networks. The dichromate can be regarded as a generalization of $C(G)$, for we have

$$C(G) = \chi(G, 1, 1).$$

$C(G)$ has a simple expression as a determinant, and its properties are well known. Perhaps some of them will suggest new properties of the dichromate and hence of the chromatic polynomials.

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