A NOTE ON REGULAR LOCAL NOETHER LATTICES II

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Let (R, M) be a local ring and let R^* be the M-adic ring completion of R. It is well known that R is a regular local ring if and only if R^* is a regular local ring. The purpose of the note is to show that this result is essentially a consequence of a more general theory concerning local Noether lattices which was developed in [6].

By a *multiplicative lattice* we will mean a complete lattice on which there is defined a commutative, associative, totally join distributive multiplication for which the unit element of the lattice is an identity for multiplication (written juxtaposition). Let $\mathscr{L}$ be a multiplicative lattice. An element P of $\mathscr{L}$ is said to be *meet principal* if $AP \wedge B = A \wedge (B : P)$, for all A and B in $\mathscr{L}$; P is said to be *join principal* if $(A \vee BP) : P = B \vee (A : P)$, for all A and B in $\mathscr{L}$; and P is said to be *principal* if P is both meet and join principal. $\mathscr{L}$ will be called *principally generated* if each element of $\mathscr{L}$ is a join (finite or infinite) of principal elements of $\mathscr{L}$. $\mathscr{L}$ is called a *Noether lattice* in case $\mathscr{L}$ is modular, principally generated, and satisfies the ascending chain condition on elements. A Noether lattice $\mathscr{L}$ is said to be *local* if it has a unique maximal (proper) prime M. In this case we shall write $(\mathscr{L}, M)$. In general we adopt the lattice terminology of [2] and [6].

Let $(\mathscr{L}, M)$ be a local Noether lattice. As in section 2 of [3] we let $\mathscr{L}^*$ be the collection of all formal sums $\sum_{n=1}^{\infty} A_n$ of elements of $\mathscr{L}$ such that $A_n = A_{n+1} \vee M^n$, for $n = 1, 2, \ldots$. On $\mathscr{L}^*$ define

$$\sum_{n=1}^{\infty} A_n \leqq \sum_{n=1}^{\infty} B_n \quad \text{if and only if} \quad A_n \leqq B_n,\ n = 1, 2, \ldots$$

$$\sum_{n=1}^{\infty} A_n \cdot \sum_{n=1}^{\infty} B_n = \sum_{n=1}^{\infty} (A_n B_n \vee M^n)$$

so that $\mathscr{L}^*$ becomes a multiplicative lattice satisfying the ascending chain condition [3, Theorem 2.1]. For each element A in $\mathscr{L}$, set $A^* = \sum_{n=1}^{n} A_n$ where $A_n = A \vee M^n$ and note that $A^* \in \mathscr{L}^*$.

Let $\mathscr{L}$ be a Noether lattice, $D \in \mathscr{L}$, and set $\mathscr{L}/D = \{A \in \mathscr{L} \mid A \geqq D\}$. If we define $A \circ B = AB \vee D$, for all $A, B \in \mathscr{L}/D$, then $\mathscr{L}/D$ becomes a Noether lattice [2, Lemma 4.1]. If $(\mathscr{L}, M)$ is a local Noether lattice, $D \in \mathscr{L}$, $D \leqq M$, then $\mathscr{L}/D$ is a local Noether lattice with maximal element M [2, Corollary 4.1]. A local Noether lattice $(\mathscr{L}, M)$ is called *M-complete* if, given any decreasing sequence $\langle A_i \rangle$ of elements of $\mathscr{L}$ and any $n \geqq 1$, it follows that $A_j \leqq \bigwedge_i A_i \vee M^n$, for all large integers j.

REMARK 1. We will require the following known properties of $\mathscr{L}^*$. We refer the reader to [3, p. 331] for their proof.

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(i) $\mathscr{L}^*$ is a local Noether lattice with maximal element $M^* = \sum_{n=1}^{\infty} M$.

(ii) For each natural number n, the map $A \mapsto A^*$ from $\mathscr{L}/M^n \to \mathscr{L}^*/M^{*n}$ is a multiplicative lattice isomorphsm.

(iii) $\mathscr{L}^*$ is M^*-complete.

The following result will be needed in the proof of Theorem 3 [3, Corollary 1.3].

LEMMA 2. *Let $(\mathscr{L}_1, M_1)$ and $(\mathscr{L}_2, M_2)$ be local Noether lattices and $\{\varphi_i : \mathscr{L}_1/M_1^i \to \mathscr{L}_2/M_2^i\}$ a sequence of multiplicative lattice homomorphisms of $\mathscr{L}_1/M_1^i$ onto $\mathscr{L}_2/M_2^i$ such that φ_{i+1} extends φ_i for all i. If $\mathscr{L}_2$ is M_2-complete, then $\mathscr{L}_1$ is embeddable in $\mathscr{L}_2$. If also $\mathscr{L}_1$ is M_1-complete, then $\mathscr{L}_1$ and $\mathscr{L}_2$ are isomorphic as multiplicative lattices.*

We shall in general adopt the ring terminology of [7]. In particular, a local ring is commutative, Noetherian, and has an identity. If R is a ring, we denote the multiplicative lattice of ideals of R by $\mathscr{L}(R)$. If R is a local ring, $\mathscr{L}(R)$ is a local Noether lattice [2, p. 486].

THEOREM 3. *Let (R, M) be a local ring with M-adic completion (R^*, MR^*). Then $\mathscr{L}(R^*)$ and $\mathscr{L}(R)^*$ are isomorphic as multiplicative lattices.*

Proof. For each i, $i = 1, 2, \ldots$, define

$$\lambda_i : \mathscr{L}(R)/M^i \to \mathscr{L}(R^*)/(MR^*)^i \quad \text{by} \quad \lambda_i : A \mapsto AR^*$$

so that λ_i is the canonical multiplicative ideal lattice isomorphism. For each i, define

$$\alpha_i : \mathscr{L}(R)^*/M^{*i} \to \mathscr{L}(R)/M^i \quad \text{by} \quad \alpha_i : \sum_{n=1}^{\infty} A_n \mapsto \bigcap_{n=1}^{\infty} A_n$$

$$\psi_i : \mathscr{L}(R)/M^i \to \mathscr{L}(R)^*/M^{*i} \quad \text{by} \quad \psi_i : A \mapsto A^*.$$

For each i, ψ_i is a multiplicative lattice isomorphism (Remark 1) and by [6, p. 160, Remark 1] α_i is the inverse of ψ_i, thus α_i is a multiplicative lattice isomorphism. For each i, set $\varphi_i = \lambda_i \alpha_i$ so that

$$\varphi_i : \mathscr{L}(R)^*/M^{*i} \to \mathscr{L}(R^*)/(MR^*)^i \quad \text{and} \quad \varphi_i : \sum_{n=1}^{\infty} A_n \mapsto \bigcap_{n=1}^{\infty} A_n R^*.$$

Thus each φ_i is a multiplicative lattice isomorphism and φ_{i+1} extends φ_i. Since $\mathscr{L}(R)^*$ is M^*-complete (Remark 1) and $\mathscr{L}(R^*)$ is MR^*-complete [8, Theorem 1] it follows that $\mathscr{L}(R)^*$ and $\mathscr{L}(R^*)$ are isomorphic as multiplicative lattices by Lemma 2.

The *height* of a prime element P of a Noether lattice $\mathscr{L}$ is defined to be the supremum of all integers n for which there exists a prime chain $P_0 < P_1 < \ldots < P_n = P$ in $\mathscr{L}$, and the *altitude* of $\mathscr{L}$ is defined to be the supremum of the heights of the prime elements of $\mathscr{L}$. A local Noether lattice $(\mathscr{L}, M)$ of altitude k is said to be *regular* in case M is the join of k principal elements.

LEMMA 4. *Let (R, M) be a local ring. Then R is a regular local ring if and only if $\mathscr{L}(R)$ is a regular local Noether lattice.*

Proof. Clearly the altitudes of R and $\mathscr{L}(R)$ are the same. Let d be their common altitude. If R is regular, there exist d elements $a_1, a_2, \ldots, a_d$ in R such that $M = a_1R + \ldots + a_dR$. Since each a_iR is principal in $\mathscr{L}(R)$ [2, p. 482), $\mathscr{L}(R)$ is regular. Conversely, if $\mathscr{L}(R)$ is regular, there exist principal elements $A_1, A_2, \ldots, A_d$ in $\mathscr{L}(R)$ such that $M = A_1 \vee \ldots \vee A_d$. Since $\mathscr{L}(R)$ is local, for each i, $1 \leqq i \leqq d$, there exists a_i in R such that $A_i = a_iR$ [5, Corollary 6] and so R is regular.

The proof of the following theorem may be found in [6, Theorem 3].

THEOREM 5. *Let $(\mathscr{L}, M)$ be a local Noether lattice. Then $\mathscr{L}$ is a regular local Noether lattice if and only if $\mathscr{L}^*$ is a regular local Noether lattice.*

By Lemma 4, R is regular if and only if $\mathscr{L}(R)$ is regular and similarly for R^* and $\mathscr{L}(R^*)$. By Theorem 5, $\mathscr{L}(R)$ is regular if and only if $\mathscr{L}(R)^*$ is regular. These results in conjunction with Theorem 3 immediately yield the following theorem.

THEOREM 6. *Let (R, M) be a local ring and let R^* be the M-adic completion of R. Then R is a regular local ring if and only if R^* is a regular local ring.*

As we have seen (Lemma 4) the ideal lattice $\mathscr{L}(R)$ of a regular local ring R is contained in class of regular local Noether lattices. The following example (due to Bogart [1]) shows the existence of regular local Noether lattices which are not the ideal lattice of any regular local ring. Let F be a field, let $x_1, x_2, \ldots, x_n$ be indeterminates, and let RL_n be the collect of elements of $\mathscr{L}(F[x_1, x_2, \ldots, x_n])$ which are joins of products of the principal ideals (x_1), $(x_2), \ldots, (x_n)$. RL_n is a sublattice of $\mathscr{L}(F[x_1, x_2, \ldots, x_n])$ and is a regular local Noether lattice of altitude n. For $n \geqq 2$, it can be shown RL_n is not isomorphic to the ideal lattice of any ring. We refer the reader to [1, p. 169] for the details.

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