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RESEARCH ARTICLE

Tits Alternative for 2-dimensional CAT(0) complexes

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Abstract
We prove the Tits Alternative for groups acting on 2-dimensional CAT(0) complexes with a bound on the order of the cell stabilisers.

1. Introduction

A *triangle complex* X is a 2-dimensional simplicial complex, possibly not locally compact, with a following *piecewise smooth Riemannian metric*. Namely, we have a family of smooth Riemannian metrics σ_T, σ_e on the triangles and edges, such that the restriction of σ_T to e is σ_e for each $e \subset T$. The Riemannian metrics σ_T, σ_e induce metrics (i.e. distance functions) on triangles and edges. We then equip X with the *quotient pseudometric* d (see [BH99, Chapter I.5.19]). We assume that for each metric ball B, the simplices of X intersecting B have only finitely many isometry types (note that the only time we will apply it to B of radius nonzero is in the proof of Remark 2.5). Then (X, d) is a complete length space, which can be deduced from [BH99, Chapters I.5.20 and I.7.13] using a bilipschitz map from each B to a piecewise Euclidean complex. Note that we study triangle complexes, as opposed to piecewise Euclidean 2-dimensional simplicial complexes, for applications to groups such as $\mathrm{Tame}(\mathbf{k}^3)$ (Corollary D). All group actions on X will be by simplicial isometries.

We say that a group acts on a cell complex X *almost freely* if there is a bound on the order of the cell stabilisers. Note that an almost free action on a triangle complex is proper in the sense of [BH99, Chapter I.8.2]. Furthermore, any subgroup of a group acting properly and cocompactly acts almost freely. A group is *virtually cyclic* (respectively, *virtually* $\mathbb{Z}^2$, *virtually abelian, virtually solvable*), if it has a finite index subgroup that is cyclic (respectively, $\mathbb{Z}^2$, abelian, solvable).

Theorem A. *Let G be a finitely generated group acting almost freely on a* CAT(0) *triangle complex X. Then G is virtually cyclic, or virtually $\mathbb{Z}^2$, or it contains a nonabelian free group.*

By [BH99, Chapters II.7.5 and 7.7(2)] and Remarks 2.4 and 2.5, if G acts almost freely on a CAT(0) triangle complex with finitely many isometry types of simplices, then every sequence $G_1 < G_2 < \cdots$ of virtually abelian subgroups of G stabilises. Consequently:

Corollary B. *If X has finitely many isometry types of simplices, then Theorem A holds also for G infinitely generated.*

As explained in [OP21, page 3], one cannot omit in Theorem A the assumption on almost freeness.

Here are some examples of applications of Theorem A to particular groups. The first result, which is a consequence of Corollary B, was studied independently by Paul Tee. We are assuming that G below acts freely instead of almost freely, since A is torsion free [CD95, Theorem B].

Corollary C. *Let G be a subgroup of a 2-dimensional Artin group A acting freely on the modified Deligne complex of A (see [CD95]). Then G is cyclic, $\mathbb{Z}^2$, the fundamental group of the Klein bottle, or it contains a nonabelian free group.*

The second result concerns the tame automorphism group $\mathrm{Tame}(\mathbf{k}^3)$, which is a subgroup of $\mathrm{Aut}(\mathbf{k}^3)$, where $\mathbf{k}$ is a field (see [LP21]). In [LP21, Sections 2 and 5], we introduced a cell complex $\mathbf{X}$ with an action of $\mathrm{Tame}(\mathbf{k}^3)$. We proved that $\mathbf{X}$ is CAT(0) for $\mathbf{k}$ of characteristic 0 [LP21, Theorem A]. Some cells of $\mathbf{X}$ are polygons instead of triangles, but we can easily transform $\mathbf{X}$ into a triangle complex by subdividing.

Corollary D. *Let G be a finitely generated subgroup of* $\mathrm{Tame}(\mathbf{k}^3)$*, with* $\mathbf{k}$ *of characteristic* 0*. Suppose that G acts almost freely on the cell complex* $\mathbf{X}$*. Then G is virtually cyclic, or virtually $\mathbb{Z}^2$, or it contains a nonabelian free group.*

An ingredient in the proof of Theorem A is the following characterisation of CAT(0) triangle complexes using a link condition. In [BB96, Theorem 7.1], this was proved only for locally compact triangle complexes, and in [BH99, II.5.2], only for piecewise Euclidean and piecewise hyperbolic triangle complexes.

Theorem E. *A triangle complex X is locally* CAT(0) *if and only if*

(i) *the Gaussian curvature of σ_T at any interior point of a triangle T of X is ≤ 0, and*
(ii) *the sum of geodesic curvatures in any two distinct triangles of X at any interior point of a common edge is ≤ 0 and*
(iii) *for each vertex v of X, the girth of the link lk_v^X is $\geq 2\pi$.*

Motivation and relation to other results.

The term *Tits Alternative* usually refers to the property that all finitely generated subgroups either are virtually solvable or contain a nonabelian free group. The name comes from the theorem of Tits [Tit72], who proved that every finitely generated linear group either is virtually solvable or contains a nonabelian free group. It is widely believed (see, e.g. [Bes00, Question 2.8], [Bri06], [Bri07, Question 7.1], [FHT11, Problem 12], [Cap14, Section 5]) that all CAT(0) *groups* (groups acting *geometrically*, that is, properly and cocompactly, on CAT(0) spaces) satisfy the Tits Alternative. This was proved only in a limited number of cases (see [NV02, SW05, CS11, MP20, MP22, OP21] and references therein). Groups acting geometrically on 2-dimensional CAT(0) complexes were studied thoroughly by Ballmann and Brin in [BB95], where they proved the Rank Rigidity Conjecture for such groups. They also proved that such groups either are virtually abelian, or they contain a nonabelian free group (statements of this type are sometimes called the *Weak Tits Alternative* [SW05]). However, the Tits Alternative for such groups has been open till our current work (e.g. in [FHT11, Problem 12] the question on the Tits Alternative is asked specifically in dimension 2). Just recently, together with Norin, we were able to show in [NOP22], among other results, that the groups in question do not contain infinite torsion subgroups. This property might be seen as the first step towards the Tits Alternative. In [OP21], we proved the Tits Alternative for the class of 2-dimensional recurrent complexes. This class contains all 2-dimensional Euclidean buildings, 2-dimensional systolic complexes, as well as some complexes outside the CAT(0) setting.

Regarding Corollary C, for right-angled Artin groups, the Tits Alternative follows from the work of Baudisch [Bau81]. In our previous work [OP21], we showed the Tits Alternative for a subclass of 2-dimensional Artin groups, containing all large-type Artin groups. Recently, in [MP22], we proved the Tits Alternative for 2-dimensional Artin groups of hyperbolic type, and in [MP20], we proved

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it for FC-type Artin groups. An approach to the Tits Alternative for subgroups of 2-dimensional Artin groups acting not almost freely on the modified Deligne complex has been developed by Martin [Mar22]. As for Corollary D, Cantat proved that the group of birational transformations $\mathrm{Bir}(\mathbb{C}^2)$ satisfies the Tits Alternative [Can11]. Earlier, Lamy proved the Tits Alternative for the group of polynomial automorphisms $\mathrm{Aut}(\mathbb{C}^2)$ [Lam01]. These proofs extend to any field $\mathbf{k}$ of characteristic 0 [Lam22]. The same statement for $\mathrm{Aut}(\mathbf{k}^3)$ seems at the moment out of reach, since there is no analogue for $\mathrm{Aut}(\mathbf{k}^3)$ of the CAT(0) complex $\mathbf{X}$ from [LP21]. However, we believe that for $\mathrm{Tame}(\mathbf{k}^3) \lneq \mathrm{Aut}(\mathbf{k}^3)$, with $\mathbf{k}$ of characteristic 0, one could study the subgroups acting not almost freely on $\mathbf{X}$ by generalising the methods of the current paper.[1]

One of the main obstacles to generalising Theorem A to higher-dimensional complexes lies in the proof of Proposition 3.1, where it is used that the fundamental groups of graphs are free, whereas the fundamental groups of higher-dimensional complexes can be arbitrary.

Finally, note that assuming in Theorem A that X is locally compact would allow us to avoid proving Theorem E, but otherwise it would not simplify the proof.

Organisation

In Section 2, we prove Theorem E. In Section 3, we recall the method of invariant cocompact subcomplexes from [OP21], which allows us to reduce Theorem A to Proposition 3.2 that assumes the existence of edges of degree ≥ 3 in our complex X. Under this assumption, we can exclude the cases of virtually cyclic or $\mathbb{Z}^2$ groups in Section 4. In technical Section 5, which we recommend to skip at the first reading, we arrange our complex X to have no 'unfoldable' links.[2] We give criteria for finding 'rank 1' elements, and consequently free subgroups, in Section 6. In the absence of 'rank 1' elements, we obtain a particular rationality property of the complex X in Section 7. In Section 8, we give new criteria for distinguishing the endpoints of certain piecewise geodesics. Together with a Poincaré recurrence argument, this allows us to prove Proposition 3.2 in Section 9.

2. Characterisation of CAT(0) triangle complexes

In this section, we prove Theorem E, which characterises CAT(0) triangle complexes. The following result is known under the name of the Cartan–Hadamard theorem.

Theorem 2.1 [BH99, II.4.1(2)]. *Let X be a complete connected metric space. If X is simply connected and locally* CAT(0)*, then it is* CAT(0).

We also have the following consequence of [BH99, II.1.7(4) and II.4.14(2)].

Theorem 2.2. *Let X be a complete* CAT(0) *space. A piecewise local geodesic in X with Alexandrov angles π at the breakpoints is a geodesic.*

Let x be a point of a triangle complex X. Let lk_x^X be the metric graph that is the *link* of x, as defined in [BB95, page 176]. Namely, if x is a vertex of X, then the vertices of lk_x^X correspond to the edges of X containing x, and the edges of lk_x^X correspond to the triangles of X containing x. The length of each edge is the angle in the corresponding triangle of X. Since we assumed that triangles containing x belong to only finitely many isometry classes of σ_T, there are only finitely many possible edge lengths in a given lk_x^X. If x lies in the interior of an edge e of X, then lk_x^X has two vertices corresponding to the components of $e \setminus x$, and edges of length π corresponding to the triangles of X containing e. If x lies in the interior of a triangle, then lk_x^X is a circle of length 2π. We denote by d_x^X (or, shortly, d_x) the length metric on lk_x^X. By [NOP22, Lemma 2.1], we can identify lk_x^X with the completion of the space of directions at x

[1] We have recently extended Corollary D to such subgroups using methods tailored to $\mathrm{Tame}(\mathbf{k}^3)$ [LP22].

[2] This solves an issue that seems to have been overlooked in the proof of [BB95, Theorem C], page 197, line 9. Namely, not all angles 2π are excluded there, since in [BB95, Lemma 7.6], one cannot remove the assumption $\xi \neq \eta$ for ω of length 2π, for example, for S_v a wedge of two circles of length 2π.

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(see [BH99, II.3.18]). Thus, a local geodesic in X starting at x determines a point in lk^X_x. The *angle* at x between two such local geodesics is defined to be the distance between the two corresponding points in lk^X_x with respect to the metric d_x. By [NOP22, Lemma 2.1], if this angle is $< \pi$, then it coincides with the Alexandrov angle, and if it is $\geq \pi$, then the Alexandrov angle equals π.

Proof of Theorem E. In the 'only if' part, condition (i) follows from [BH99, II.1A.6]. The proof of condition (ii) is identical to that in [BB96, Theorem 7.1], and the proof of condition (iii) was given in [NOP22, Section 2]. For the proof of the 'if' part, suppose that a triangle complex X satisfies conditions (i)–(iii). By condition (i) and [BH99, II.1A.6], we have that X is locally CAT(0) at any interior point of a triangle.

Consider now an edge e of X. Let $\mathrm{St}(e)$ be the union of all the closed triangles containing e. We will show that $\mathrm{St}(e)$ is CAT(0), which implies that X is locally CAT(0) at any interior point x of e, since the metrics on $\mathrm{St}(e)$ and on X coincide on a sufficiently small neighbourhood of x. Let $Y \subset \mathrm{St}(e)$ be the union of the triangles T for which there exists a point on e with positive geodesic curvature in T. By condition (ii), there is at most one such triangle of given isometry type T_0 and given embedding $e \subset T_0$, so Y has finitely many triangles. We denote this number by $m(e)$ for future reference. For each triangle T of $\mathrm{St}(e)$ outside Y, denote $Y_T = Y \cup T$. By conditions (i) and (ii), and by [BB96, Theorem 7.1], we have that each Y_T is locally CAT(0), hence, CAT(0) by Theorem 2.1 (note that a geodesic in Y_T might enter and exit a given triangle infinitely many times). Furthermore, the inclusion $Y \subset Y_T$ is an isometric embedding, since points of e have nonpositive geodesic curvature in T. By [BH99, II.11.3], the union $\mathrm{St}(e)$ of Y_T is CAT(0), as desired.

Consider now a vertex v of X. After possibly subdividing X, we can assume that in each triangle T containing v, the local geodesic γ_T starting at v and bisecting the angle of T at v ends at the opposite side of T. We will prove that the union $\mathrm{St}(v)$ of all the closed triangles and edges containing v is CAT(0). This will imply that X is locally CAT(0) at v, since the metrics on $\mathrm{St}(v)$ and on X coincide on a sufficiently small neighbourhood of v.

Claim. $\mathrm{St}(v)$ *is geodesic, and there is* $M > 0$*, such that each geodesic in* $\mathrm{St}(v)$ *intersects the interiors of at most* M *triangles.*

To justify the Claim, we employ the idea of a taut string [BH99, Chapter I.7.20]. Let θ be the minimum of π and the minimum length of an edge in lk^X_v, and set $N = 2 + \frac{\pi}{\theta}$. Let $M = 1 + N(1 + \max_e m(e))$, where $m(e)$ is defined as above and the maximum is taken over all the edges e of $\mathrm{St}(v)$ containing v. For each such edge e, let $\mathrm{St}'(e) \subset \mathrm{St}(e)$ be the closure of the component containing e of $\mathrm{St}(e) \setminus \bigcup_{T \subset \mathrm{St}(e)} \gamma_T$, for γ_T as above. Since $\mathrm{St}'(e)$ is convex in $\mathrm{St}(e)$, it is CAT(0) [BH99, II.1.15(1)].

For $x, y \in \mathrm{St}(v)$, a *string* between x and y is a sequence of edges $e_1, \ldots, e_n$ of $\mathrm{St}(v)$ containing v and points $x_0 = x, x_1, \ldots, x_n = y$ with $\mathrm{St}'(e_i)$ containing both x_{i-1} and x_i. The *length* of the string is the sum $\Sigma_{i=1}^n d_i(x_{i-1}, x_i)$, where d_i is the metric on $\mathrm{St}'(e_i)$. The distance between x and y in $\mathrm{St}(v)$ is the infimum of the lengths of strings between x and y. A string is *taut*, if $n \leq 2$, or

- for each $0 \leq i \leq n$, the point x_i is distinct from v, and
- for each $0 < i < n$, the point x_i belongs to one of the γ_T defined above and
- for each $0 < i < n$, the concatenation of geodesics $x_{i-1}x_i$ in $\mathrm{St}'(e_i)$ and $x_i x_{i+1}$ in $\mathrm{St}'(e_{i+1})$ is a geodesic in $\mathrm{St}'(e_i) \cup \mathrm{St}'(e_{i+1})$.

We now justify that for each string $(e_i), (x_i)$ between x and y, we can find a taut string between x and y whose length does not exceed the length of $(e_i), (x_i)$. Indeed, by discarding some x_i, we can first assume that consecutive e_i are distinct, and so for each $0 < i < n$, the point x_i belongs to one of the γ_T. Since γ_T are compact, there is a choice of x'_i in the same γ_T as x_i, minimising the length of the string $(e_i), (x'_i)$. Then the concatenation of geodesics $x'_{i-1}x'_i$ in $\mathrm{St}'(e_i)$ and $x'_i x'_{i+1}$ in $\mathrm{St}'(e_{i+1})$ is a geodesic in $\mathrm{St}'(e_i) \cup \mathrm{St}'(e_{i+1})$. Finally, if an x'_{i-1} equals v, and $x'_i \neq x_n$, then for $x'_i \in \gamma_T$, the subpath of γ_T from x'_{i-1} to x'_i is a geodesic in both $\mathrm{St}'(e_i)$ and $\mathrm{St}'(e_{i+1})$, and consequently, we can discard x'_i and e_i from the string. Repeating the argument, we arrive at $i - 1 = n - 1$ or $i - 1 = n$. Analogously, we obtain $i - 1 = 1$ or $i - 1 = 0$, and so $n \leq 2$.

We will now show that a taut string satisfies $n \le N$. We can assume $n > 2$, and so none of x_i equals v. For $0 < i < n$, let θ_i be the Alexandrov angle at x_i in $\mathrm{St}'(e_{i+1})$ between the geodesics $x_i x_{i+1}$ and $x_i v$. The concatenation of geodesics $x_{i-1}x_i$ in $\mathrm{St}'(e_i)$ and $x_i x_{i+1}$ in $\mathrm{St}'(e_{i+1})$ is a geodesic in $\mathrm{St}'(e_i) \cup \mathrm{St}'(e_{i+1})$, which is CAT(0) by [BH99, II.11.3]. Consequently, by the definition of θ, we have $\theta_i \ge \theta_{i-1} + \theta$. Thus, $\pi \ge \theta_{n-1} \ge (n-2)\theta + \theta_1$, and so, $n - 2 \le \frac{\pi}{\theta} = N - 2$.

Since there are finitely many isometry types of simplices in $\mathrm{St}(v)$, and each taut string satisfies $n \le N$, the distance between x and y in $\mathrm{St}(v)$ is realised by the length of some taut string $(e_i), (x_i)$. Then the concatenation of all the geodesics $x_{i-1}x_i$ in $\mathrm{St}'(e_i)$ is a geodesic between x and y, proving that $\mathrm{St}(v)$ is geodesic. Furthermore, for any geodesic γ from x to y in $\mathrm{St}(v)$, one can choose points on γ forming a taut string. Since any taut string satisfies $n \le N$, and we have that γ intersects the interiors of at most $1 + n(1 + \max_e m(e))$ triangles, the Claim follows.

Returning to the proof of Theorem E, we follow the scheme in [BB96, Theorem 7.1] to find a sequence of CAT(0) spaces Gromov–Hausdorff converging to $\mathrm{St}(v)$. Namely, realise each (isometry type of a) triangle T of $\mathrm{St}(v)$ as $T \subset \mathbb{R}^2$, with metric induced from some smooth Riemannian metric of nonpositive Gaussian curvature on $\mathbb{R}^2$ defined in a neighbourhood U of T. Denote by e, f the edges of T containing v and by g the remaining edge of T. Let l denote the length of e. For each $n > 0$, we decompose e into paths $a^1 \cdot a^2 \cdots a^n$ of length $\frac{l}{n}$, and we define κ^k to be the integral of the geodesic curvature along a^k. Let e_n be the piecewise geodesic in U that starts at v tangent to e, has n locally geodesic pieces of length $\frac{l}{n}$ and exterior angle at the k-th breakpoint that equals κ^k. For n sufficiently large the path e_n exists, and they C^1-converge to e as n tends to ∞. We define paths f_n analogously. We define g_n to be any piecewise geodesics joining the endpoints of e_n and f_n and C^1-converging to g. This gives us a triangle $T_n \subset U$ bounded by e_n, f_n and g_n, whose boundary is piecewise geodesic (one can pass to a union of triangles with locally geodesic boundary by subdividing). Furthermore, we have a map $T_n \to T$, whose restriction to e_n, f_n preserves length and which is bilipschitz with the bilipschitz constant converging to 1 as n tends to ∞. Glueing various T_n along the sides corresponding to the ones of T that we glued to form $\mathrm{St}(v)$ yields a triangle complex that we call $\mathrm{St}(v)_n$. Then $\mathrm{St}(v)$ is a Gromov–Hausdorff limit of $\mathrm{St}(v)_n$. Note that since $\mathrm{St}(v)$ satisfied conditions (i)–(iii), we have that $\mathrm{St}(v)_n$ satisfies conditions (i)–(iii). Since $\mathrm{St}(v)_n$ has locally geodesic edges, and is geodesic by the Claim (applied to $\mathrm{St}(v)_n$ instead of $\mathrm{St}(v)$), the same proof as for [BB96, Theorem 7.1, case 1] shows that $\mathrm{St}(v)_n$ is locally CAT(0), hence, CAT(0) by Theorem 2.1. Consequently, by [BH99, II.3.10], we have that $\mathrm{St}(v)$ is CAT(0). □

We have the following immediate consequence of Theorem E.

Corollary 2.3. *If X is a triangle complex that is locally* CAT(0)*, then all of its subcomplexes are locally* CAT(0).

Note that while we did not apply the Claim to $\mathrm{St}(v)$ in the proof of Theorem E, it will be used in the following remarks.

Remark 2.4. Suppose that X is a CAT(0) triangle complex with finitely many isometry types of simplices. Then the constant M in the Claim does not depend on v. As in the '$\Psi_n \Rightarrow \Phi_n$' part of the proof of [BH99, Chapter I.7.28], we obtain that for each l there is $M' > 0$, such that each geodesic in X of length $\le l$ intersects the interiors of at most M' simplices. Using this in the place of [Bri99, Lemma 1] in the proof of [Bri99, Lemma 2 and Theorem A], we obtain that every simplicial isometry g of X is *semisimple*: it fixes a point, or is *loxodromic*, meaning that there is a geodesic line ω in X (called an *axis*), such that g preserves ω and acts on it as a nontrivial translation.

Remark 2.5. Suppose that X is a CAT(0) triangle complex with finitely many isometry types of simplices. Then the set of translation lengths of simplicial isometries of X is a discrete subset of $[0, \infty)$, which is proved using the Claim exactly as [Bri99, Proposition]. Similarly, we obtain the following: Let X be a CAT(0) triangle complex with a subcomplex Y on which some group of simplicial isometries of X acts coboundedly. Since any metric ball in X intersects finitely many isometry types of simplices, we have that each bounded neighbourhood of Y intersects finitely many isometry types of simplices. Then

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for each simplicial isometry g of X, the set $\inf_{y\in Y} d(y, gy)$ attains its infimum, which we denote $|g|_Y$. Moreover, the set of $|g|_Y$ over all simplicial isometries g of X is a discrete subset of $[0, \infty)$.

3. *G*-cocompact subcomplexes

Let X be a simplicial complex with an action of a group G. We say that a subcomplex $Z \subset X$ is an *invariant cocompact subcomplex with respect to* G (shortly, G*-c.s.*) if Z is G-invariant, and the quotient Z/G is compact. Note that a G-c.s. is not required to be connected.

A 2-dimensional simplicial complex is *essential* if every edge has degree at least 2 and none of connected components is a single vertex. An essential simplicial complex is *thick* if it has an edge of degree at least 3.

A *disc diagram* D is a compact contractible simplicial complex with a fixed embedding in $\mathbb{R}^2$. Its *boundary path* is the attaching map of the cell at ∞. If X is a simplicial complex, *a disc diagram in* X is a nondegenerate simplicial map $\varphi\colon D \to X$, and its *boundary path* is the composition of the boundary path of D and φ. We say that φ is *reduced* if it maps triangles sharing an edge to two distinct triangles. By [OP21, Remark 3.6], for each contractible closed edge-path α in a simplicial complex X, there is a reduced disc diagram in X with boundary path α.

A group G acts on a simplicial complex X *without inversions* if for any $g \in G$ stabilising a simplex σ of X, we have that g fixes σ pointwise. More generally, we say that G acts *without weak inversions* if for each vertex v of X, there is no $g \in G$ sending v to a distinct vertex in a common edge.

The first ingredient in our proof of Theorem A is the following earlier result.

Proposition 3.1 [OP21, Proposition 3.7]. *Let G be a finitely generated group acting almost freely and without inversions on a simply connected 2-dimensional simplicial complex X that contains no simplicial 2-spheres. If X contains no thick G-c.s., then G is virtually cyclic, or virtually $\mathbb{Z}^2$, or it contains a nonabelian free group.*

The second ingredient in the proof of Theorem A is the following, the proof of which will occupy the present article.

Proposition 3.2. *Let G be a group acting almost freely and without weak inversions on a* CAT(0) *triangle complex X that is an increasing union of connected essential G-c.s. If X contains an edge of degree ≥ 3, then G contains a nonabelian free group.*

We now show how Theorem A follows from these two ingredients.

Proof of Theorem A. By passing to a subdivision (see [NOP22, Lemma 2.1]), we can assume that G acts without weak inversions. By Proposition 3.1, we can assume that X contains a thick G-c.s. Z_1. We will prove that G contains a nonabelian free group. By passing to a connected component of Z_1 and its stabiliser G' in G (which is finitely generated, since it acts properly and cocompactly on a connected complex), we can assume that Z_1 is connected. If Z_1 contains a closed edge-path that is not contractible in Z_1, repeatedly attaching to Z_1 the images of reduced disc diagrams and their G-translates, we obtain an increasing sequence $Z_1 \subset Z_2 \subset \cdots$ of connected essential G-c.s., such that their union X' is simply connected. By Corollary 2.3, we have that X' is locally CAT(0), and so X' is CAT(0) by Theorem 2.1. It remains to apply Proposition 3.2. □

4. Not virtually cyclic or $\mathbb{Z}^2$

The first step of the proof of Proposition 3.2 is the following.

Lemma 4.1. *Let G be a group acting almost freely on a* CAT(0) *triangle complex X. If X contains a subcomplex Z that is a connected thick G-c.s., then*

(i) *G is not virtually cyclic and*
(ii) *G is not virtually $\mathbb{Z}^2$.*

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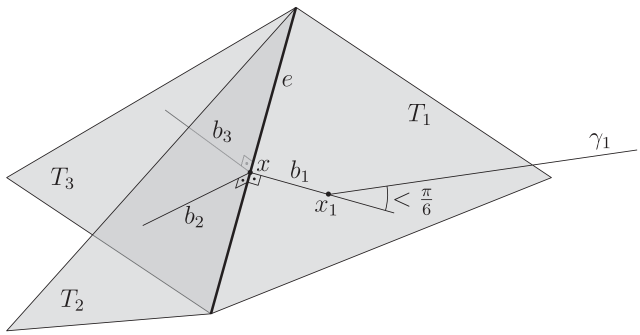

Figure 1. *The geodesic ray γ_1.*

In the proof of Lemma 4.1, we will need the following vocabulary. Let X be a triangle complex. We say that a ray $\gamma\colon [0,\infty) \to X$ or a path $\gamma\colon [0,1] \to X$ *starts* (respectively, *ends*) in a simplex σ, if for some $\varepsilon > 0$ the points $\gamma(0,\varepsilon)$ (respectively, $\gamma(1-\varepsilon,1)$) all lie in the interior of σ. If $\gamma(0)$ (respectively, $\gamma(1)$) lies in the interior of an edge e, then γ *starts* (respectively, *ends*) *perpendicularly to e* if the angle at $\gamma(0)$ (respectively, $\gamma(1)$) between γ and e is $\frac{\pi}{2}$.

We will also need the following, which generalises an argument in the proof of [MP21, Theorem A].

Lemma 4.2. *Let A be a group isomorphic to $\mathbb{Z}^2$ acting freely on a* CAT(0) *triangle complex W with finitely many isometry types of simplices. Then there is an isometrically embedded A-cocompact subcomplex in W isometric to the Euclidean plane.*

Proof. Since W has finitely many isometry types of simplices, by Remark 2.4, all elements of A act as loxodromic isometries on W. By [BH99, II.7.20(1)], we have $\mathrm{Min}(A) = Y \times \mathbb{R}^n$ with A preserving the product structure and acting trivially on Y. By [BH99, II.7.20(2)], we have $n \leq 2$, but since A acts freely by simplicial isometries, we have $n = 2$, and so Y is a point, as desired. □

Proof of Lemma 4.1. Let e be an edge of Z of degree ≥ 3, and let x be a point in the interior of e. Let b_1, b_2, b_3, be geodesics starting at x perpendicularly to e contained in distinct triangles T_1, T_2, T_3. For each $i = 1, 2, 3$, the set of starting directions at points in ∂T_i of geodesics intersecting b_i at angle $< \frac{\pi}{6}$ has positive Liouville measure (see [BB95, Section 3]). Let S denote the union of all the open edges in the links lk^Z_y for all $y \in Z^1 \setminus Z^0$, with the (infinite) Liouville measure. We say that $(\xi_j)_j \in S^{\mathbb{Z}}$ with $\xi_j \in \mathrm{lk}^Z_{y_j}$ *determines* a locally geodesic oriented line γ in $Z \setminus Z^0$ transverse to Z^1 if γ intersects Z^1 exactly at points y_j in directions ξ_j, in that order. Since G acts on Z properly and cocompactly, the set of $(\xi_j)_j \in S^{\mathbb{Z}}$ that determine locally geodesic oriented lines projects to a full measure subset in each coordinate S (see [BB95, Section 3], which relies on [CFS82, Chapter 6]). Consequently, for $i = 1, 2, 3$, there exists a locally geodesic ray γ_i in Z starting at an interior point x_i of b_i at angle $< \frac{\pi}{6}$ from b_i, disjoint from Z^0 and transverse to Z^1. Let $a_i = xx_i \subset b_i$. See Figure 1.

Since X is CAT(0), by Theorem 2.2, we have that γ_i are geodesic rays in X and $a_i^{-1} \cdot a_j$ are geodesics in X. Since each $\gamma_i^{-1} \cdot a_i^{-1} \cdot a_j \cdot \gamma_j$ is a piecewise geodesic with angles $> \frac{5\pi}{6}$ at the two breakpoints, by [BH99, II.9.3], the rays γ_i, γ_j are not asymptotic and they determine points at distance $> \frac{2\pi}{3}$ in the Tits boundary of X. In particular, Z cannot be quasi-isometric to $\mathbb{R}$ since it contains three pairwise nonasymptotic geodesic rays. This proves (i).

For (ii), assume for contradiction, that G is virtually $\mathbb{Z}^2$ generated by elements g, h. Let α, β be edge-paths in Z connecting a basepoint $y \in Z^0$ to gy, hy, respectively. Then the concatenation $\alpha \cdot g\beta \cdot h\alpha^{-1} \cdot \beta^{-1}$ is a closed edge-path, and since X is simply connected, there is a reduced disc diagram $D \to X$ with that boundary path. Let $Z' \subset X$ be the connected thick G-c.s. obtained from Z by adding the translates under G of the image of D. The complex Z' is locally CAT(0) by Corollary 2.3. Let $\widetilde{Z}' \to Z'$ be the universal cover of Z', which is CAT(0) by Theorem 2.1. The action of G on Z' lifts to an almost free

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action of a group $\widetilde{G}$ on $\widetilde{Z}'$ fitting into the short exact sequence $\pi_1 Z' \to \widetilde{G} \to G$. Since $D \to Z'$ lifts to $D \to \widetilde{Z}'$, we have commuting $\tilde{g}, \tilde{h} \in \widetilde{G}$ mapping to $g, h \in G$, and, hence, generating a subgroup $A < \widetilde{G}$ isomorphic to $\mathbb{Z}^2$.

Since A acts almost freely and is torsion free, we have that it acts freely on $\widetilde{Z}'$. By Lemma 4.2 applied with $W = \widetilde{Z}'$, there is an isometrically embedded A-cocompact subcomplex $E \subset \widetilde{Z}'$ isometric to the Euclidean plane. We now justify that the composition $\phi\colon E \subset \widetilde{Z}' \to Z' \subset X$ is an isometric embedding.

Indeed, for two triangles T, T' of E containing a common edge e', the sum of the geodesic curvatures in $\phi(T)$ and $\phi(T')$ at any point of $\phi(e')$ equals 0, and so the geodesic curvature at $\phi(e')$ in any triangle of X distinct from $\phi(T), \phi(T')$ is nonpositive. Consequently, ϕ is a local isometric embedding at e'. Furthermore, since E is isometric to the Euclidean plane, for any geodesic γ in E passing through a vertex v, the angle (see Section 2) between the incoming and outgoing directions of γ at v equals π. Since the map that ϕ induces between lk^E_v and $\mathrm{lk}^X_{\phi(v)}$ is locally injective, the angle between the incoming and outgoing directions of $\phi(\gamma)$ at $\phi(v)$ equals π as well. By Theorem 2.2, we have that $\phi(\gamma)$ is a geodesic. Thus, ϕ is an isometric embedding, as desired.

Since the image of A in G is of finite index, it acts cocompactly on Z'. Consequently, the geodesic rays γ_i from the proof of part (i) are at bounded distance from $\phi(E)$ in Z'. Since X is CAT(0), we obtain that each γ_i is asymptotic to a geodesic ray in $\phi(E)$ and these three rays are pairwise at angle $> \frac{2\pi}{3}$ in $\phi(E)$, which is a contradiction. □

5. Folding

This section is devoted to a technical reduction of Proposition 3.2 to the case where the vertex links of X are not ‘unfoldable’.

By a *graph*, we mean a (possibly infinite) metric graph with finitely many possible edge lengths. A closed edge-path embedded in a graph Λ is a *cycle of* Λ. An edge-path I in Λ that is embedded, except possibly at the endpoints, is a *segment of* Λ if the endpoints of I have degree ≥ 3 in Λ, but every internal vertex of I has degree 2.

Definition 5.1. Let S be a set with an equivalence relation $\sim$, each of whose equivalence classes has size ≥ 2. A graph is a $\sim$-*clover* if it is obtained from the disjoint union of intervals $S \times [0, \pi]$ by identifying all the points in $S \times 0$ to one point called the *basepoint* and identifying each $s \times \pi$ with $s' \times \pi$ for $s \sim s'$. A graph is a *clover* if it is a $\sim$-clover for some $S, \sim$. A graph Γ is *unfoldable* at a vertex y if Γ is a wedge $\Gamma_1 \vee \Gamma_2$ at y of a cycle Γ_1 of length 2π and a clover Γ_2 with basepoint y (in particular, Γ is also a clover). See Figure 2.

Suppose that we have a triangle complex X' and a vertex w' contained in distinct edges $e_1 = w'v_1$, $e_2 = w'v_2$ of the same length. Suppose that $\mathrm{lk}^{X'}_{v_1}$ is a circle of length 2π and $\mathrm{lk}^{X'}_{v_2}$ is a clover with basepoint corresponding to e_2. Then the quotient map $p'\colon X' \to X$ with X obtained from X' by identifying v_1 with v_2 and e_1 with e_2 is called a *folding* (note that X might not be a simplicial complex,

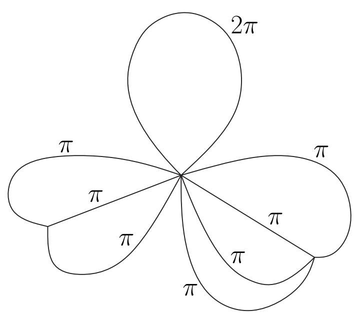

Figure 2. *An unfoldable graph.*

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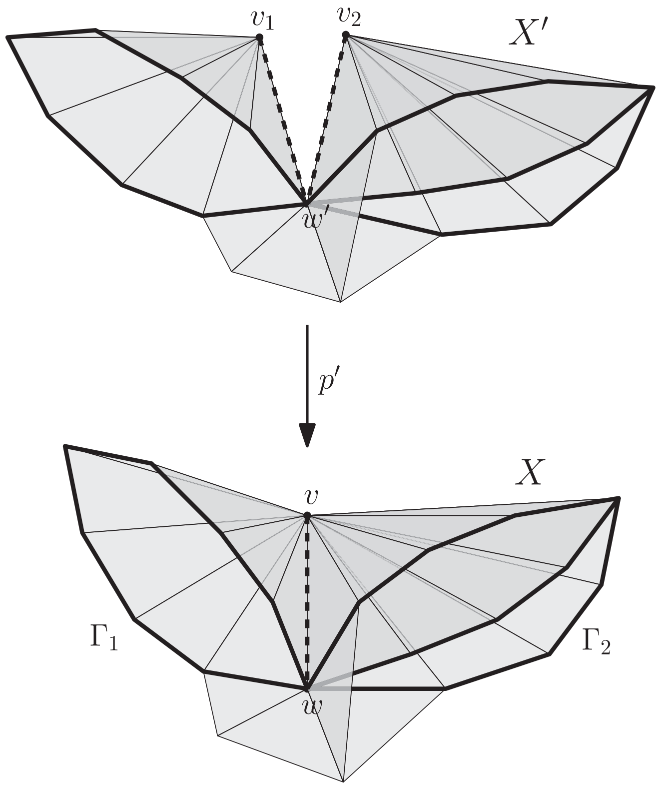

Figure 3. *A folding $p' : X' \to X$ over Γ_1. Subgraphs corresponding to the links Γ_1 and Γ_2 and their preimages are thickened, the edge vw and its preimage are dashed.*

but in this article, we will be using only the inverse operation to folding which does result in a simplicial complex).

Conversely, suppose that X is a triangle complex with a vertex v whose link $\Gamma_1 \vee \Gamma_2$ is unfoldable at a point y corresponding to an edge vw. Then, up to an isometry, there exists a unique triangle complex X' and a folding $p' : X' \to X$ identifying edges $w'v_1, w'v_2$ to wv and such that the links of v_i in X' map isometrically to the graphs Γ_i in the link of v. See Figure 3. We call p' the *folding over* Γ_1 (since it is uniquely determined by Γ_1).

Suppose now that $p' : X' \to X, \hat{p}' : \hat{X}' \to X$ are foldings over $\Gamma_1 \neq \hat{\Gamma}_1$. Suppose that $v \neq \hat{w}$ and $\hat{v} \neq w$. We have that $\hat{\Gamma}_1$ lifts to a link $\mathrm{lk}^{X'}_{v'}$, which is again, unfoldable, except when $\mathrm{lk}^{X'}_{v'} = \hat{\Gamma}_1$. In that exceptional case, we have $\Gamma_2 = \hat{\Gamma}_1$ and $p' = \hat{p}'$, and we set $p'' = \mathrm{id}$. Otherwise, let $p'' : X'' \to X'$ be the folding over $\hat{\Gamma}_1$. We call $p'' \circ p'$ the *folding over* $\Gamma_1, \hat{\Gamma}_1$. Note that the folding over $\Gamma_1, \hat{\Gamma}_1$ coincides with the folding over $\hat{\Gamma}_1, \Gamma_1$.

Analogously, given a finite family of foldings $p'^{\lambda} : X'^{\lambda} \to X$ over Γ_1^{λ} (where λ is an index) with all v^{λ} distinct from all w^{λ}, the *folding over* $\{\Gamma_1^{\lambda}\}$ is the composition of foldings over the lifts of Γ_1^{λ}, which does not depend on the order. For a countable family of such foldings $p'^{\lambda} : X'^{\lambda} \to X$, the *folding over* $\{\Gamma_1^{\lambda}\}$ is the inverse limit of the foldings over the finite subsets of $\{\Gamma_1^{\lambda}\}$.

Lemma 5.2. *Let $p_{\mathcal{F}} : X_{\mathcal{F}} \to X$ be the folding over a (finite or countable) family $\mathcal{F} = \{\Gamma_1^{\lambda}\}$.*

(i) *The map $p_{\mathcal{F}}$ is a homotopy equivalence.*
(ii) *If X is locally* $\mathrm{CAT}(0)$*, then $X_{\mathcal{F}}$ is locally* $\mathrm{CAT}(0)$.
(iii) *If X is essential, then $X_{\mathcal{F}}$ is essential.*

Proof. For part (i), note that if for some indices λ, μ, we have $v^{\lambda} = v^{\mu}$, then $w^{\lambda} = w^{\mu}$, since the point

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y^λ in $\mathrm{lk}^X_{v^\lambda}$ does not depend on Γ_1^λ. Consequently, distinct edges $v^\lambda w^\lambda$ might intersect only along w^λ. Thus, their union $V \subset X$ is a forest, and so the quotient map $q\colon X \to X^*$ collapsing each component of V into a point is a homotopy equivalence. Similarly, the subcomplex $V_\mathcal{F} = p_\mathcal{F}^{-1}(V) \subset X_\mathcal{F}$ is a forest, and so the quotient map $q_\mathcal{F}\colon X_\mathcal{F} \to X^*$ collapsing each component of $V_\mathcal{F}$ into a point is also a homotopy equivalence. Thus, the identity $q_\mathcal{F} = q \circ p_\mathcal{F}$ implies part (i).

Part (ii) follows from the fact that the maps that $p_\mathcal{F}$ induces between the links of $X_\mathcal{F}$ and X are locally injective and from Theorem E.

The map $p_\mathcal{F}$ is a local isometry at the open edges outside $V_\mathcal{F}$. Thus, for part (iii), it suffices to justify that the links of the vertices in each $p_\mathcal{F}^{-1}(v^\lambda)$ have no leaves. But each such link is a cycle Γ_1^μ (for $v^\mu = v^\lambda$) or a clover, as desired. □

Proposition 5.3. *Let G be a group acting almost freely and without weak inversions on a* CAT(0) *triangle complex X that is an increasing union of connected essential G-c.s. Then G acts almost freely and without weak inversions on a* CAT(0) *triangle complex X' that is an increasing union of connected essential G-c.s., and none of whose links are unfoldable.*

Furthermore, if X contains an edge of degree ≥ 3, then X' contains an edge of degree ≥ 3 or G contains a nonabelian free group.

Proof. We fix an increasing sequence $Z_k \subset X$ of connected essential G-c.s. exhausting X. A *multifolding* $(X', (Z'_k), p')$ is a:

(i) CAT(0) triangle complex X' with an action of G,
(ii) a sequence (Z'_k) of essential G-c.s. exhausting X' and
(iii) a G-equivariant simplicial map $p'\colon X' \to X$ that
- maps bijectively the set of triangles of each Z'_k to the set of triangles of Z_k and
- whose restriction $Z'_k \to Z_k$ is a homotopy equivalence.

We introduce a partial order $\leq$ on the set of multifoldings, writing $(X', (Z'_k), p') \leq (X'', (Z''_k), p'')$ (or, shortly, $X' \leq X''$) if there is a G-equivariant simplicial map $r\colon X'' \to X'$ satisfying $p'' = p' \circ r$. Multifoldings X', X'' are *equivalent* if $X' \leq X''$ and $X'' \leq X'$. Let $\mathcal{X}$ be the set of equivalence classes of multifoldings.

We claim that every chain of elements X'^λ in $\mathcal{X}$ (where λ is an index) has an upper bound. Indeed, denote by p'_k the restriction of p' to Z'_k and write $(Z'_k, p'_k) \leq (Z''_k, p''_k)$ whenever there is a G-equivariant simplicial map $r\colon Z''_k \to Z'_k$ satisfying $p''_k = p'_k \circ r$. For each k, since G acts properly and cocompactly on Z'^λ_k, by the first bullet point, we have that $(Z'^\lambda_k, p'^\lambda_k)$ can take on only finitely values up to the appropriate equivalence. Thus, there exists a largest element among the $(Z'^\lambda_k, p'^\lambda_k)$, which we call Z'^∞_k. Furthermore, since $(Z'_{k+1}, p'_{k+1}) \leq (Z''_{k+1}, p''_{k+1})$ implies $(Z'_k, p'_k) \leq (Z''_k, p''_k)$, we have natural injective maps $Z'^\infty_k \to Z'^\infty_{k+1}$. Let X'^∞ be their direct limit, equipped with the limit map p'^∞ to X. Since each Z'^∞_k is locally CAT(0) (Corollary 2.3), we have that X'^∞ is locally CAT(0) (Theorem E).

To prove that X'^∞ is an upper bound for our chain in $\mathcal{X}$, by Theorem 2.1, it remains to prove that X'^∞ is simply connected. Let α be a closed edge-path in the 1-skeleton of X'^∞, and fix k, such that α lies in Z'^∞_k. Fix λ with $Z'^\lambda_k = Z'^\infty_k$, and keep the notation α for its copy in Z'^λ_k. Since X'^λ is simply connected, there is a disc diagram $D \to X'^\lambda$ with boundary path α. Fix l, such that the image of D is contained in Z'^λ_l. Since, by the second bullet point, the induced map $\pi_1 Z'^\infty_l \to \pi_1 Z'^\lambda_l$ is an isomorphism, we have that α is trivial in $\pi_1 Z'^\infty_l$ and, hence, in $\pi_1 X'^\infty$. Consequently, by the Kuratowski–Zorn lemma, there is a maximal element $X' \in \mathcal{X}$.

We now prove that none of the links of X' are unfoldable. Otherwise, suppose that X' has a vertex v whose link $\Gamma = \Gamma_1 \vee \Gamma_2$ is unfoldable at a point corresponding to an edge vw of X'. Let $\mathcal{F} = \{g\Gamma_1\}$, for $g \in G$. Since G acts without weak inversions, we have $gv \neq hw$, for all $g, h \in G$. Thus, we can define the folding over $\mathcal{F}$, which we denote by $X_\mathcal{F} \to X'$. The action of G on X' lifts to an action of G on $X_\mathcal{F}$. For each essential G-c.s. $Z' \subset X'$, let $Z_\mathcal{F} \subset X_\mathcal{F}$ be the closure of the union of all the open triangles of $X_\mathcal{F}$ mapping into Z'. Note that if Z' does not contain v, or if Z' contains v, but $\mathrm{lk}^{Z'}_v$ is contained in Γ_1 or Γ_2 (in the latter case, $\mathrm{lk}^{Z'}_v \subset \bigcap_{g \in \mathrm{Stab}(v)} g\Gamma_2$), then $Z_\mathcal{F} \to Z'$ is an isometry. Otherwise, $\mathrm{lk}^{Z'}_v$

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contains edges lying on both the cycle Γ_1 and the clover Γ_2. Since Z' is essential, the link $\mathrm{lk}_v^{Z'}$ is the wedge of Γ_1 and a clover, so it is unfoldable. Then $Z_{\mathcal{F}} \to Z'$ is the folding over the family $\{g\Gamma_1\}$, for $g \in G$. By Lemma 5.2(i,iii), we have that $Z_{\mathcal{F}}$ is essential and $Z_{\mathcal{F}} \to Z'$ is a homotopy equivalence. By Lemma 5.2(i,ii), we have that $X_{\mathcal{F}}$ is simply connected and locally CAT(0), and, hence, CAT(0) by Theorem 2.1. Consequently, we have $X_{\mathcal{F}} \in \mathcal{X}$ and $X_{\mathcal{F}} > X'$, contradicting the maximality of X'. Thus, none of the links of X' are unfoldable.

For the last assertion, note that, by Lemma 4.1 applied to X, we have that G is neither virtually cyclic, nor virtually $\mathbb{Z}^2$. Moreover, G is finitely generated, since it acts properly and cocompactly on Z_1, which is connected. Consequently, if X' does not have edges of degree 3, then by Proposition 3.1, we have that G contains a nonabelian free group, as desired. □

6. Criteria for rank 1 elements

In this section, we give criteria for finding 'rank 1' elements, and, consequently, free subgroups in G.

Definition 6.1. Let γ be a geodesic line in a CAT(0) triangle complex X. We say that γ is *curved* if γ passes through a vertex v and its incoming and outgoing directions at v are at angle $> \pi$.

Lemma 6.2. *Let g be a loxodromic isometry of a* CAT(0) *triangle complex X with a curved axis γ. Then there exists M, such that the projection to γ of each closed metric ball in X disjoint from γ has diameter $\leq M$.*

Proof. Suppose that γ passes through a vertex v with incoming and outgoing directions at angle $> \pi+\kappa$, for some $\frac{\pi}{2} > \kappa > 0$. Let R be the translation length of g. We will prove that $M = R\lceil\frac{2\pi}{\kappa}\rceil$ satisfies the lemma. Otherwise, let x, y be points in a closed metric ball disjoint from γ, such that the projections x', y' of x, y to γ are at distance $> M$.

There are at least $n = \lceil\frac{2\pi}{\kappa}\rceil$ translates of v under $\langle g\rangle$ on $x'y'$ distinct from x', y'. We denote these translates by $v'_1, \ldots, v'_n$, in the order in which they appear on $x'y'$. By the continuity of the projection map, there are points $v_1, \ldots, v_n$ lying on the geodesic xy in that order, such that each v'_i is the projection of v_i to γ. We additionally denote $v_0 = x, v'_0 = x', v_{n+1} = y, v'_{n+1} = y'$. For $0 \leq i \leq n$, let β_i denote the geodesic quadrilateral $v_iv_{i+1}v'_{i+1}v'_iv_i$. The sum of the four Alexandrov angles of each β_i is $\leq 2\pi$ [BH99, II.2.11], so the sum of all the Alexandrov angles of all β_i is $\leq (n+1)2\pi$.

On the other hand, for $0 \leq i < n$, the sum of the Alexandrov angles of β_i and β_{i+1} at v_{i+1} is $\geq \pi$. We will now prove that the sum of the Alexandrov angles of β_i and β_{i+1} at v'_{i+1} is $> \pi + \kappa$. Indeed, if one of them is not equal to the angle in the usual sense (see Section 2), then it equals π. However, since v'_{i+1} is the projection of v_{i+1}, the second Alexandrov angle is $\geq \frac{\pi}{2}$, so their sum is $\geq \frac{3\pi}{2}$, as desired. Consequently, we have $n(\pi+\pi+\kappa) < (n+1)2\pi$, and so $n\kappa < 2\pi$, which is a contradiction. □

Lemma 6.3. *Let G be a group acting almost freely on a* CAT(0) *triangle complex X with a fixed point ξ in the visual boundary of X. Suppose that there is a curved axis γ for some $g \in G$ with one of the limit points ξ. Then G is virtually cyclic.*

Proof. Consider the space of geodesic rays $\rho\colon [0,\infty) \to X$ representing ξ, with the pseudometric $d(\rho_1,\rho_2) = \inf_{t_1,t_2} d(\rho_1(t_1),\rho_2(t_2))$. Identifying ρ_1 with ρ_2 for $d(\rho_1,\rho_2) = 0$, we obtain a metric space whose metric completion X_ξ is CAT(0) [Lee00, Proposition 2.8]. Since X has geometric dimension ≤ 2 (see [Kle99]), by [Cap09, Remark after Corollary 4.4], we have that X_ξ has geometric dimension ≤ 1 (more precisely, as we learned from Pierre-Emmanuel Caprace, for any ρ representing ξ, the space $X_\xi \times \mathbb{R}$ embeds isometrically in the pointed ultralimit of $(X,\rho(n))_n$, which has geometric dimension ≤ 2 by [Lyt05, Lemma 11.1]). Since G fixes ξ, the action of G on X induces an action of G on X_ξ.

We will now justify that a complete CAT(0) space X_ξ of geometric dimension ≤ 1 is an $\mathbb{R}$-tree, which we learned also from Pierre-Emmanuel Caprace. For a geodesic triangle xyz in X_ξ, let x' be the projection of x to the geodesic yz. If $x' \neq x, y$, then the direction of the geodesic $x'x$ in $\mathrm{lk}_{x'}^{X_\xi}$ is distinct from that of the geodesic $x'y$. Thus, the geodesic xy must pass through x', since otherwise mapping it

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to $\mathrm{lk}_{x'}^{X_\xi}$ would give a path between distinct points of a discrete set. Analogously, the geodesic xz passes through x', and so xyz is 0-thin, justifying that X_ξ is an $\mathbb{R}$-tree.

Suppose, first, that there is $h \in G$ acting loxodromically on X_ξ. Let M be the constant given by Lemma 6.2 for γ. Let ρ be a ray in γ representing ξ. Since h acts loxodromically on X_ξ, after possibly replacing h by its power, we can assume $d(\rho, h\rho) > M$. Assume, without loss of generality, that the Busemann function (see [BH99, II.8.17 and II.8.20]) satisfies $B_\xi(\rho(0)) \leq B_\xi(h\rho(0))$. Then for each $t \geq 0$, the projection $p(t)$ of $\rho(t)$ to $h\gamma$ is contained in $h\rho$, and for $D = d(\rho(0), h\rho(0))$, we have $d(\rho(t), p(t)) \leq D$. Pick $k \in \mathbb{N}$ with $kM > 2D$. Then, by the triangle inequality, we have $d(p(0), p(2kM)) \geq 2kM - 2D > kM$. Consequently, there is $0 \leq n < k$, such that $d(p(2nM), p(2(n+1)M)) > M$. Thus, the closed ball of radius M centred at $\rho((2n+1)M)$ is disjoint from $h\gamma$ and contains points $\rho(2nM), \rho(2(n+1)M)$ whose projections to $h\gamma$ are at distance $> M$. This contradicts the choice of M.

Consequently, G has a global fixed point in X_ξ (which might not be represented by a geodesic ray but be a point added in the completion). Thus, there is $D > 0$, such that for any $\varepsilon > 0$, there is a geodesic ray ρ' representing ξ at distance $\leq D$ from ρ and satisfying $d(\rho', g\rho') < \varepsilon$ for each $g \in G$. Consider the homomorphism $\psi\colon G \to \mathbb{R}$ defined by $\psi(g) = B_\xi(gx) - B_\xi(x)$ for any $x \in X$. We will now justify that ψ has discrete image. Otherwise, for any $\varepsilon > 0$ and ρ' as above, there is $t > 0$, such that $d(\rho'(t), g\rho'(t)) < 2\varepsilon$, but g does not fix a point of X. This contradicts Remark 2.5 applied with Y containing the D-neighbourhood of γ.

Let K be the kernel of ψ. Arguing as in the previous paragraph, we obtain that every $g \in K$ fixes a point of X. By [NOP22, Theorem 1.1(i)], every finitely generated subgroup of K fixes a point of X. Since K acts almost freely, we have that K is finite and so G is virtually cyclic. □

Proposition 6.4. *Let G be a group acting almost freely on a* CAT(0) *triangle complex X. Assume that G contains a loxodromic element g with a curved axis γ. Then G is virtually cyclic or contains a nonabelian free group.*

Proof. By Lemma 6.2, g is rank 1 in the sense of [BF09, Definition 5.1]. By Lemma 6.3, we can assume that G does not have a finite index subgroup fixing a limit point of γ. Then there is $f \in G$ with γ and $f\gamma$ having disjoint limit point pairs. Consequently, by [BF09, Proposition 5.9], for some n, the elements g^n and fg^nf^{-1} generate a nonabelian free group. □

7. Extrationality

The main result of this section will be Proposition 7.4, where we will show that in the absence of unfoldable vertices and curved axes, the complex X enjoys a particularly strong rationality property of angles, which we call extrationality.

Definition 7.1. The *branching locus E* of a triangle complex X is the subcomplex of X that is the union of all the closed edges of degree ≥ 3. A *patch* of X is a maximal connected subspace P of $X \setminus E$, such that $P \setminus X^0$ is connected (see Figure 4). If X is simply connected, then by Van Kampen's theorem, P is a planar surface, and so we can choose an orientation on P.

We equip $P \setminus X^0$ with the length metric induced from $X \setminus X^0$ (see [BH99, Chapter I.3.24]). Let $\overline{P}$ denote the completion of $P \setminus X^0$, which admits an obvious embedding of P. Furthermore, $\overline{P}$ admits an obvious triangle complex structure and a simplicial map $\overline{P} \to X$ fitting the following commutative diagram.

$$\begin{array}{ccc} & & P \\ & \swarrow & \downarrow \\ \overline{P} & \longrightarrow & X \end{array}$$

Note that $\overline{P}$ is a connected surface with boundary, which we denote ∂P.

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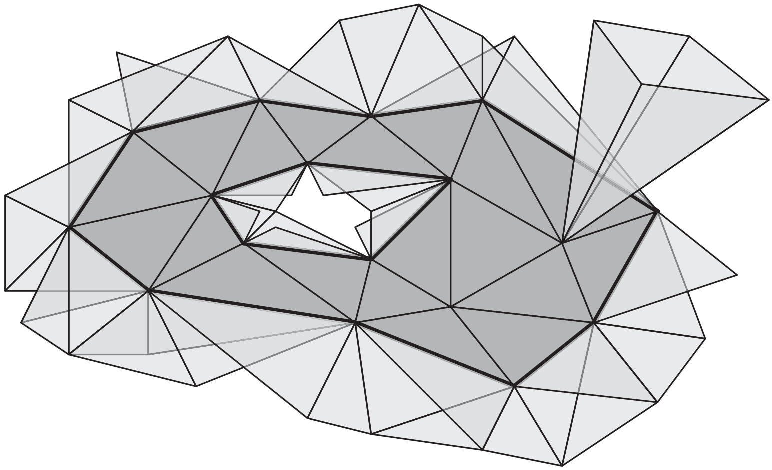

Figure 4. *A patch (in dark grey).*

Definition 7.2. A triangle complex X is *piecewise Euclidean* if all its triangles are geodesic Euclidean triangles. A piecewise Euclidean triangle complex X is *rational* if for any vertex v of X, all cycles and segments (see Section 5) in the link of v have lengths commensurable with π. In particular, the angle at v between any edges of the branching locus E is then commensurable with π. A rational triangle complex X is *extrational* if

- for any vertex v of X with a component C of lk_v^X a circle, we have that the length of C is 2π and
- each homomorphism ψ defined below is trivial.

We define $\psi = \psi(P)$ for each patch P of X. Consider the chain complex $C_*(\overline{P}, \partial P)$ consisting of those singular chains that are affine with respect to the affine structure on $\overline{P}$ induced by the piecewise Euclidean metric. Note that the affine structure on $\overline{P}$ has singularities at the points x of ∂P with $\mathrm{lk}_x^{\overline{P}}$ of length $\neq \pi$, and so we require our affine chains to be disjoint from such x except possibly at the vertices. For each $x \in \overline{P}$, choose (not necessarily continuously) a direction $\xi_x \in \mathrm{lk}_x^{\overline{P}}$ at x, with the only restriction that for $x \in \partial P$, the direction ξ_x corresponds to one of the edges in ∂P containing x. For an affine singular 1-simplex $\sigma \to \overline{P}$ with endpoints x and y, let $\psi(\sigma) \in \mathbb{R}/\pi\mathbb{Q}$ be the oriented angle between ξ_x and σ at x minus the oriented angle between ξ_y and σ at y. Note that since X was rational, this equals 0 mod $\pi\mathbb{Q}$ for σ in ∂P, and so we obtain a homomorphism $\psi\colon C_1(\overline{P}, \partial P) \to \mathbb{R}/\pi\mathbb{Q}$. Note that the restriction of ψ to $Z_1(\overline{P}, \partial P)$ does not depend on the choice of the ξ_x. Furthermore, for each affine singular 2-simplex τ, we have $\psi(\partial\tau) = \pm\pi = 0 \mod \pi\mathbb{Q}$, and so ψ descends to a homomorphism $\psi\colon H_1(\overline{P}, \partial P) \to \mathbb{R}/\pi\mathbb{Q}$. It is not hard to check that our $H_1(\overline{P}, \partial P)$ coincides with the usual first (singular) homology group.

We will need the following variant of [BB95, Lemma 7.4] that was implicit in the proof of [NOP22, Proposition 3.4].

Lemma 7.3. *Let G be a group acting almost freely on a* CAT(0) *triangle complex X that is an increasing union of essential G-c.s. Furthermore, assume that there is a vertex v of X with points $\xi_i, \eta_i \in \mathrm{lk}_v^X$, for $i = 1, \dots, n$, such that*

- $d_v^X(\xi_i, \eta_i) = \pi$ *for* $i = 1, \dots, n$, *and*
- $d_v^X(\eta_i, \xi_{i+1}) \geq \pi$ *for* $i = 1, \dots, n-1$ *and*
- $d_v^X(\eta_n, \xi_1) > \pi$.

Then G contains a loxodromic element g with a curved axis.

Proof. Let $Z \subset X$ be an essential G-c.s. containing v, such that lk_v^Z contains ξ_i, η_i, with $d_v^Z(\xi_i, \eta_i) = \pi$, for $i = 1, \dots, n$. By [NOP22, Lemma 5.4] (which was stated in terms of the compact quotient but has

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the same proof for proper and cocompact actions), for any $\varepsilon > 0$, there is a path $\omega = \omega_1 \cdots \omega_{3n}$ in Z, such that

- paths ω_j are local geodesics in Z, and
- there are $g_0 = \mathrm{id}, g_1, \ldots, g_n = g \in G$ such that paths ω_{3i+1} start at $g_i v$, paths ω_{3i} end at $g_i v$, and except for that ω is disjoint from the vertex set Z^0 and transverse to Z^1, and
- the starting direction of ω_{3i+1} is at distance $< \frac{\varepsilon}{2}$ to $g_i\xi_{i+1}$ in $\mathrm{lk}^Z_{g_i v}$, and the ending direction of ω_{3i} is at distance $< \frac{\varepsilon}{2}$ to $g_i\eta_i$ and
- at the remaining breakpoints, ω_j and ω_{j+1} are at angle $> \pi - \varepsilon$ (and $\leq \pi$ since outside Z^0).

Since ω_j are disjoint from Z^0 and transverse to Z^1, by Theorem 2.2, they are geodesics in X. The last two bullet points hold in X as well. In particular, at all the breakpoints ω_j and ω_{j+1} are at angle $> \pi - \varepsilon$. By [BB95, Lemma 2.5], the geodesic γ in X with the same endpoints as ω starts and ends in directions at distance $< (3n-1)\varepsilon$ to $\xi_1, g\eta_n$. Consequently, for ε sufficiently small, by Theorem 2.2, we have that $\bigcup_{l\in\mathbb{Z}} g^l\gamma$ is a curved axis for g. □

Proposition 7.4. *Let G be a group acting almost freely on a* CAT(0) *triangle complex X that is an increasing union of essential G-c.s. and none of whose links are unfoldable. If X is not G-equivariantly isometric (by a possibly nonsimplicial isometry) to a piecewise Euclidean triangle complex X' or X is isometric to such X' but X' is not extrational, then G is virtually cyclic or contains a nonabelian free group.*

Proof. To prove that G is virtually cyclic or contains a nonabelian free group in each case we will show the existence of a curved axis in X (or in a different CAT(0) triangle complex $\overline{X}$) for an element of G, since then the proposition follows from Proposition 6.4.

Assume first that X is not G-equivariantly isometric to a piecewise Euclidean triangle complex X'. Then by [BB95, Proposition 2.11] there is

(i) a point in the interior of a triangle of X with negative Gaussian curvature, or
(ii) a point in the interior of an edge of X with negative sum of geodesic curvatures of some two incident triangles, or
(iii) a vertex v of X with lk^X_v a circle of length $> 2\pi$.

In case (iii), or, more generally, if lk^X_v has a component C that is a circle of length $> 2\pi$, let ξ_1, η_1 be points at distance π in C, and let η_2, ξ_2 be their antipodal points. Applying Lemma 7.3 with $n = 2$, we obtain a curved axis. In cases (i) and (ii), by [NOP22, Lemma 5.5], there is a CAT(0) triangle complex $\overline{X}$, obtained from X by a G-equivariant subdivision and a G-equivariant replacement of the smooth Riemannian metrics, with a vertex $u \in \overline{X}$ whose $\mathrm{lk}^{\overline{X}}_u$ is either

- a circle of length $> 2\pi$ or
- a graph obtained from a family of disjoint circles $C_1, C_2, \ldots$ of length 2π by glueing them along a nontrivial arc b of length $< \pi$.

The first bullet point brings us to case (iii). In the case of the second bullet point, let $\xi_1, \xi_2 \in C_1 \setminus b$ and $\eta_1, \eta_2 \in C_2 \setminus b$ be points at distance $\frac{\pi}{2}$ from the endpoints of b, with $d^{\overline{X}}_u(\xi_1, \eta_1) = d^{\overline{X}}_u(\xi_2, \eta_2) = \pi$. Applying Lemma 7.3 with $n = 2$, we obtain a curved axis in $\overline{X}$, as desired.

Thus, without loss of generality, we can assume that X is a piecewise Euclidean triangle complex. If X is not rational, then by [BB95, Proposition 7.7], applied to an essential G-c.s., there is a closed locally injective edge-path β in some lk^X_v whose length is not commensurable with π. In particular, by [BB95, Lemma 6.1(iii)], there are points ξ, η in lk^X_v at distance $> \pi + \delta$, for some $\delta > 0$ (one could apply [NOP22, Corollary 1.7] to find such ξ, η in β, but it does not simplify the argument). Let β_- (respectively, β_+) be the shortest path from ξ (respectively, η) to β. Since the length of β is not commensurable with π, there is a path $\beta_-\beta_0\beta_+$ with β_0 factoring through the universal cover of β whose length equals $(2n-1)\pi + \delta'$ for some $n \in \mathbb{N}$ and $0 \leq \delta' \leq \delta$. Choosing $\xi_1 = \xi, \eta_1, \xi_2, \ldots, \eta_n$ as consecutive points at distance π along that path, we have $d^X_v(\xi_1, \eta_n) > \pi$. Applying Lemma 7.3, we obtain a curved axis.

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Finally, if X is not extrational, let P be a patch of X with nontrivial $\psi = \psi(P)$. Since P is planar, there is an element in $H_1(\overline{P}, \partial P)$ represented by a piecewise affine path α in $\overline{P}$ with endpoints in ∂P and $\psi(\alpha) \neq 0$. Let α be shortest among such paths, which exists since $\mathrm{Stab}(P)$ acts cocompactly on $\overline{P}$. Note that then α does not intersect ∂P except at its endpoints, since, otherwise, we could decompose it into two shorter paths, with ψ nontrivial on at least one of them. Thus, the image of α in X (for which we keep the same notation) is a local geodesic in X that intersects the branching locus E exactly at its endpoints x, x'. Let e (respectively, e') be the segment of lk_x^X (respectively, $\mathrm{lk}_{x'}^X$) containing the point corresponding to the direction of α. By the shortness condition, we have that e (respectively, e') has endpoints at distance $\geq \frac{\pi}{2}$ from x (respectively, x'), and so is of length $\geq \pi$. They cannot both have length π, since then we would have $\psi(\alpha) = 0$, so assume, without loss of generality, that the length l of e is $> \pi$.

If $l > 2\pi$, then it is easy to find points $\eta_1, \xi_1, \xi_2, \eta_2$ lying on l in that order and satisfying the hypothesis of Lemma 7.3 with $n = 2$. If $2\pi > l > \pi$ or $l = 2\pi$ and the endpoints of e are distinct, then the construction of such points is given in the proof of [BB95, Lemma 7.6]. It remains to consider the case where $l = 2\pi$ and where both endpoints of e are equal to a vertex y. Let Γ_2 be the graph obtained from lk_x^X by removing e. If Γ_2 contains a point z at distance $> \pi$ from y, then it is easy to find points $\xi_1, \eta_1, \xi_2, \eta_2$ on a geodesic from z to the midpoint of e satisfying the hypothesis of Lemma 7.3 with $n = 2$. Otherwise, Γ_2 is a clover, contradicting the assumption that lk_x^X is not unfoldable. □

8. Sheared geodesics

In this section, we prove that the following piecewise geodesics have distinct endpoints. We will use some vocabulary from Section 4.

Definition 8.1. A *sheared geodesic* in a piecewise Euclidean triangle complex X is a concatenation $\gamma_1 \cdot \gamma_2 \cdots \gamma_{2k-1} \cdot \gamma_{2k}$ of geodesics, such that (see Figure 5):

- for $i = 1, \ldots, k$, the (possibly trivial) geodesic γ_{2i} lies in the interior of an edge e_i of X and
- for $i = 1, \ldots, k-1$, the geodesic γ_{2i-1} ends and the geodesic γ_{2i+1} starts perpendicularly to e_i in triangles of X that are distinct, and the geodesic γ_{2k-1} ends perpendicularly to e_k in a triangle.

Proposition 8.2. *Let X be a piecewise Euclidean triangle complex that is* $\mathrm{CAT}(0)$. *Let γ in X be a sheared geodesic. Then γ is not a closed path.*

The proof will use the following two building blocks.

Lemma 8.3. *Let X be a piecewise Euclidean triangle complex that is* $\mathrm{CAT}(0)$. *Let xy be a nontrivial geodesic in X, such that y belongs to the interior of an edge e of X and xy ends in a triangle T. Then for any z in the interior of e, the geodesic xz is nontrivial and ends in T.*

Proof. We have $z \neq x$ since edges are geodesics, and so, in particular, x does not lie in e. If for some z in the interior of e the geodesic xz does not end in T, then, since the geodesic xz varies continuously with z, for some z in the interior of e, the geodesic xz ends in e. Denote by e_1, e_2 the two subedges into which

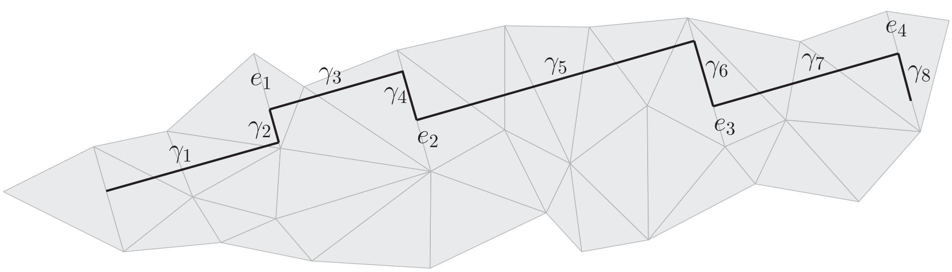

Figure 5. *A sheared geodesic.*

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such z divides e. Suppose that xz ends in e_1. Then the entire e_1 must lie in xz. Moreover, appending xz by e_2, we also obtain a geodesic (Theorem 2.2). Since y lies in $e = e_1 \cdot e_2$, this shows that xy ends in e, which is a contradiction. □

Lemma 8.4. *Let X be a piecewise Euclidean triangle complex that is* CAT(0). *Let xz be a nontrivial geodesic in X, such that z belongs to the interior of an edge e of X and xz ends in a triangle T. Suppose that y belongs to the interior of an edge e' of X and zy is a nontrivial geodesic in X that starts perpendicularly to e in a triangle distinct from T and ends perpendicularly to e' in a triangle T'. Then the geodesic xy is nontrivial and ends in T'.*

Proof. Consider the geodesic triangle xyz. By our assumptions, its Alexandrov angle at z is $> \frac{\pi}{2}$, and so, in particular, $x \neq y$, and its Alexandrov angle at y is $< \frac{\pi}{2}$. Since zy ends in a triangle T' perpendicularly to e', we have that xy ends in T', as desired. □

Proof of Proposition 8.2. Let $\gamma = \gamma_1 \cdot \gamma_2 \cdots \gamma_{2k-1} \cdot \gamma_{2k}$ as in Definition 8.1. For $i = 1, \dots, k$, denote $\gamma_{2i} = y_i z_i$ and denote by T_i the triangle in which γ_{2i-1} ends. Let x be the starting point of γ_1. We prove by induction on $i = 1, \dots, k$, that x and z_i are distinct and that the geodesic xz_i ends in T_i. The proposition follows from this induction hypothesis applied with $i = k$.

For $i = 1$, the induction hypothesis follows from Lemma 8.3. Suppose now that we have established it for some $i = m < k$. Then, by Lemma 8.4, the geodesic xy_{m+1} is nontrivial and ends in T_{m+1}. Thus, by Lemma 8.3, the induction hypothesis holds for $i = m + 1$. □

9. Free

Proof of Proposition 3.2. By Proposition 5.3, we can assume that none of the vertex links of X are unfoldable. Thus, by Proposition 7.4 and Lemma 4.1(i), we can assume that X is piecewise Euclidean and extrational. Let $Z \subset X$ be a thick G-c.s. Note that each patch of X either has no triangle in Z or is contained in Z, in which case, we call it a *Z-patch*.

Since X is rational, and Z is a G-c.s., there is $q \in \mathbb{N}$, such that for each Z-patch P and each vertex $v \in \partial P$, the length of $\mathrm{lk}_v^{\overline{P}}$ is a multiplicity of $\frac{\pi}{q}$. For each Z-patch P, we define the homomorphism $\psi' = \psi'(P)\colon H_1(\overline{P}, \partial P) \to \mathbb{R}/\frac{\pi}{q}\mathbb{Z}$ in the same way as ψ, but replacing $\pi\mathbb{Q}$ by $\frac{\pi}{q}\mathbb{Z}$. We have $\psi = \psi'$ mod $\pi\mathbb{Q}$. Since ψ is trivial, the image of ψ' is contained in $\pi\mathbb{Q}/\frac{\pi}{q}\mathbb{Z}$. Since there are finitely many G-orbits of Z-patches, and since each $H_1(\overline{P}, \partial P)$ is finitely generated as a Stab(P)-module, there is $q' \in \mathbb{N}$, such that the image of each ψ' is contained in $\frac{\pi}{q'}\mathbb{Z}/\frac{\pi}{q}\mathbb{Z}$. Consequently, for any Z-patch P, any geodesic xy in $\overline{P}$ disjoint from ∂P, except at its endpoints, that is at angle $\in \frac{\pi}{q'}\mathbb{Z}$ from ∂P at x, is also at angle $\in \frac{\pi}{q'}\mathbb{Z}$ from ∂P at y. Without loss of generality, assume that q' is even.

We need the following variant of the Liouville measure μ from [BB95, Section 3]. Let S be the set of all the directions ξ at an angle $\theta(\xi) \in \frac{\pi}{q'}\mathbb{Z} \cap (-\frac{\pi}{2}, \frac{\pi}{2})$ from a direction normal to E in the links lk_x^Z for all the points $x \in Z$ that lie in the interior of an edge e of E. The *Liouville measure $d\mu(\xi)$* on S is given as $\cos\theta(\xi)dx$, where dx is the volume element on e. Let $V \subset S$ be the full measure subset of S of directions ξ, such that each geodesic ray γ in Z with starting direction ξ is disjoint from Z^0. Let $F\colon V \to \mathcal{P}(V)$ be the map defined by $\eta \in F(\xi)$ for $\eta \in \mathrm{lk}_x^Z$ if there exists a geodesic yz in Z with starting direction ξ, intersecting E only in y and x and with η being the direction at x of xz. Since G acts on Z properly and cocompactly, we have that each $F(\xi)$ is finite. We can thus define a Markov chain with states V and transition probabilities $\frac{1}{|F(\xi)|}$ from ξ to each $\eta \in F(\xi)$. By (the calculation in) [BB95, Proposition 3.3], the measure μ is stationary for this Markov chain. Thus, the space $V^{\mathbb{Z}}$ can be equipped with Markov measure μ^* invariant under the shift (see, e.g. [Wal82, Example (8), page 21]). Since Z is a G-c.s., the quotient $V^{\mathbb{Z}}/G$ by the diagonal action of G is of finite measure. Note that the shift map descends to $V^{\mathbb{Z}}/G$ and is still measure preserving.

Let e be an edge of Z lying in three distinct triangles T_a, T_b, T_c of Z. Let $V^{ab} \subset V^{\mathbb{Z}}$ be the set of $(\xi_i)_i$, such that

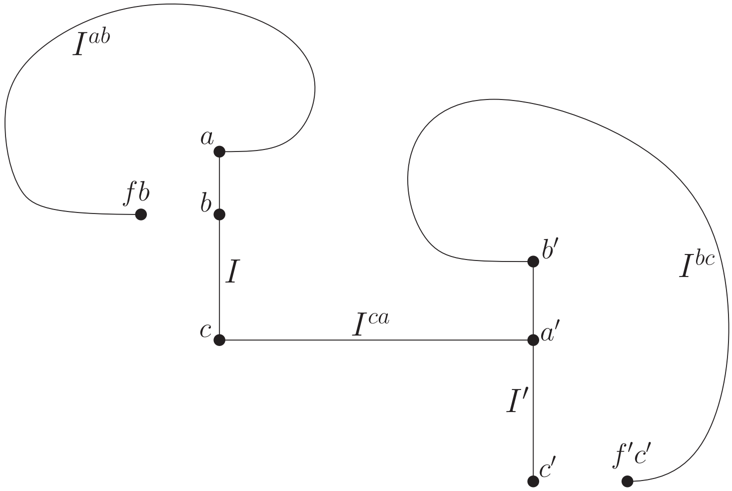

Figure 6. *The graph* Γ.

- we have $\xi_1 \in \mathrm{lk}_x^{T_a}$ for $x \in e$, and ξ_1 is at angle $\frac{\pi}{2}$ from e and
- the geodesic yx from the definition of $\xi_1 \in F(\xi_0)$ ends in T_b.

Note that V^{ab} has positive Markov measure. Thus, by the Poincaré recurrence (see, e.g. [Wal82, Theorem 1.4]), there is $(\xi_i)_i \in V^{ab}$ and $j > 0$ with $(\xi_{i-j})_i \in GV^{ab}$. Consequently, there is a geodesic γ^{ab} in $Z \setminus Z^0$ starting perpendicularly to e in T_a and ending perpendicularly to a translate fe in fT_b, for some $f \in G$.

Denote by a, fb the endpoints of γ^{ab}. Let I^{ab} be the domain of the isometric embedding $\gamma^{ab} : I^{ab} \to Z \setminus Z^0$. Analogously, there is a geodesic $\gamma^{ca} : I^{ca} \to Z \setminus Z^0$ starting perpendicularly to e in T_c and ending perpendicularly to a translate ge in gT_a, with endpoints c, a', for some $g \in G$. Finally, there is a geodesic $\gamma^{bc} : I^{bc} \to Z \setminus Z^0$ starting perpendicularly to ge in T_b and ending perpendicularly to a translate $f'ge$ in $f'gT_c$, with endpoints $b', f'c'$, for some $f' \in G$. Let $\gamma : I \to e$ be the shortest geodesic in e containing all a, b, c in its image (possibly I is a single point), and let $\gamma' : I' \to ge$ be the shortest geodesic in ge containing all a', b', c' in its image.

Let Γ be the metric graph obtained in the following way. We start from the disjoint union of the five intervals $I^{ab}, I^{ca}, I^{bc}, I, I'$, and we identify (see Figure 6):

- points of I and I^{ab} mapping to a under γ and γ^{ab},
- points of I and I^{ca} mapping to c under γ and γ^{ca},
- points of I' and I^{ca} mapping to a' under γ' and γ^{ca} and
- points of I' and I^{bc} mapping to b' under γ' and γ^{bc}.

Note that Γ admits the map $\varphi : \Gamma \to Z$ that is the quotient of $\gamma^{ab} \sqcup \gamma^{ca} \sqcup \gamma^{bc} \sqcup \gamma \sqcup \gamma'$. Let s, t, s', t', be the points in I, I^{ab}, I', I^{bc} mapping under φ to $b, fb, c', f'c'$, respectively.

Let F_2 be the free group on two generators h, h', and let $\widehat{\Gamma}$ be the quotient of the graph $F_2 \times \Gamma$ (which is the disjoint union of F_2 copies of Γ) by the relations $w \times t \sim wh \times s, w \times t' \sim wh' \times s'$, for all $w \in F_2$. Note that $\widehat{\Gamma}$ is a tree with a free action of F_2. Let $\varphi_* : F_2 \to G$ be the homomorphism mapping h, h' to f, f', respectively. Then φ extends to a φ_*-equivariant map $\widehat{\varphi} : \widehat{\Gamma} \to Z$ mapping each $w \times r \in F_2 \times \Gamma$ to $\varphi_*(w)\varphi(r) \in Z$.

Let w be a nontrivial element of F_2, and let $\mathbb{R}_w$ be the axis for w in $\widehat{\Gamma}$. Pick $p \in \mathbb{R}_w$ an endpoint of a translate of one of I^{ab}, I^{ca}, I^{bc} contained in $\mathbb{R}_w$. Let $I_w \subset \mathbb{R}_w$ be the interval between p and wp, and let $\gamma_w : I_w \to Z$ be the restriction of $\widehat{\varphi}$ to I_w. Since T_a, T_b, T_c, were distinct, we have that γ_w is a sheared geodesic. By Proposition 8.2, we have $\widehat{\varphi}(p) \neq \widehat{\varphi}(wp) = \varphi_*(w)\widehat{\varphi}(p)$, and, consequently, $\varphi_*(w)$ is nontrivial. Thus, φ_* is injective, and so G contains a nonabelian free group. □

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