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REFLEXIVE REPRESENTATIONS AND BANACH C^*-MODULES

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ABSTRACT. Suppose $\mathcal{A}$ is a unital C^*-algebra and $m: \mathcal{A} \longrightarrow B(X)$ is unital bounded algebra homomorphism where $B(X)$ is the algebra of all operators on a Banach space X. When X is a Hilbert space, a problem of Kadison [9] asks whether m is similar to a $*$-homomorphism. Haagerup [5] has shown that the answer is positive when $m(\mathcal{A})$ has a cyclic vector or whenever m is completely bounded. We use this to show $m(\mathcal{A})$ is reflexive (Alg Lat $m(\mathcal{A}) = m(\mathcal{A})^{-sot}$) whenever X is a Hilbert space. Our main result is that whenever $\mathcal{A}$ is a separable GCR C^*-algebra and X is a reflexive Banach space, then $m(\mathcal{A})$ is reflexive.

Suppose $\mathcal{S}$ is a unital subalgebra of $B(X)$, the algebra of all operators on a Banach space X. The *commutant* $\mathcal{S}'$ of $\mathcal{S}$ is the set of all operators in $B(X)$ that commute with every element of $\mathcal{S}$. Also $\operatorname{Lat}\mathcal{S}$ is the set of invariant (closed linear) subspaces of $\mathcal{S}$, and $\operatorname{AlgLat}\mathcal{S} = \{T \in B(X) : \operatorname{Lat}\mathcal{S} \subset \operatorname{Lat} T\}$.

Suppose $\mathcal{A}$ is a unital C^*-algebra and $m: \mathcal{A} \longrightarrow B(X)$ is a unital bounded homomorphism. If X is a Hilbert space and m is a $*$-homomorphism, then $m(\mathcal{A})$ is a unital C^*-algebra of operators and the von Newmann double commutant theorem [11] implies

$$\operatorname{AlgLat} m(\mathcal{A}) = m(\mathcal{A})^{-sot} \tag{1}$$

$$m(\mathcal{A})'' = m(\mathcal{A})^{-sot}. \tag{2}$$

A problem of R. Kadison [9] asks whether every bounded homorphism from a C^*-algebra into $B(X)$ is similar to a $*$-homomorphism when X is a Hilbert space. An affirmative answer to Kadison's similarity problem would imply that (1) and (2) above hold whenever X is a Hilbert space, without the assumption that m is a $*$-homomorphism. Hence the failure of (1) or (2) when X is a Hilbert space would yield a negative answer to Kadison's similarity problem.

U. Haagerup [5] has shown that Kadison's similarity problem has an affirmative answer when $m(\mathcal{A})$ has a cyclic vector or whenever m is completely bounded, and we use this to show that (1) holds whenever X is a Hilbert space.

In the case that $\mathcal{A}$ is commutative, W. G. Bade [3] showed that (1) holds when the maximal ideal space of $\mathcal{A}$ is Stonian and $m(\{a \in \mathcal{A} : a = a^2\})$ is Bade complete. It was shown by the second author [10] that (1) holds when $\mathcal{A}$ is commutative and m has *weakly compact action* (*i.e.*, for every x in X, the mapping $a \longmapsto m(a)x$ is weakly compact from $\mathcal{A}$ to X). Later, the authors proved [7] (see also [1]) that (1) holds whenever

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$\mathcal{A}$ is commutative and X is an arbitrary Banach space. Also J. Dieudonné [4] gave an example in which $\mathcal{A}$ is commutative, m has weakly compact action, and (2) fails, *i.e.*, $m(\mathcal{A})'' \neq m(\mathcal{A})^{-sot}$.

Our main result in this paper is that when $\mathcal{A}$ is a separable GCR C^*-algebra and X is a reflexive Banach space (*i.e.*, the natural embedding of X into its second dual $X^{\#\#}$ is surjective), then (1) holds.

We begin by showing that Haagerup's results [5] on the similarity problem imply that (1) holds when X is a Hilbert space.

THEOREM 1. *If X is a Hilbert space, then (1) holds.*

PROOF. Suppose $T \in \operatorname{AlgLat}\big(m(\mathcal{A})\big)$. To show that $T \in m(\mathcal{A})^{-sot}$, we need to show that every strong-operator neighbourhood of T intersects $m(\mathcal{A})$. Suppose $\varepsilon > 0$ and $\{x_1, \ldots, x_n\} \subset X$. Let M be the norm closure of $m(\mathcal{A})x_1 + \cdots + m(\mathcal{A})x_n$. Define the mapping $\rho: \mathcal{A} \longrightarrow B(M)$ by $\rho(a) = m(a)|M$. It follows that ρ is a bounded unital homomorphism. Let H denote a direct sum of n copies of M. We can identify $B(H)$ with $\mathfrak{M}_n\big(B(M)\big)$. Define a bounded unital homomorphism $\rho_n: \mathfrak{M}_n(\mathcal{A}) \longrightarrow B(H)$ by $\rho_n\big((a_{ij})\big) = \big(\rho(a_{ij})\big)$. Then $x = (x_1, \ldots, x_n)$ is a cyclic vector for $\rho_n\big(\mathfrak{M}_n(\mathcal{A})\big)$. Hence, by Haagerup's result, ρ_n is similar to a $*$-homomorphism, which imples that ρ is completely bounded. It follows from Haagerup [5] that ρ is similar to a $*$-homomorphism. Hence $\operatorname{AlgLat}\big(\rho(\mathcal{A})\big) = \rho(\mathcal{A})^{-sot}$. However, $M \in \operatorname{Lat}\big(m(\mathcal{A})\big)$ and $T \in \operatorname{AlgLat}\big(m(\mathcal{A})\big)$; thus $T|M \in \operatorname{AlgLat}\big(\rho(\mathcal{A})\big)$. Therefore there is an element b in $\mathcal{A}$ such that $\|[T - m(b)]x_k\| = \|T|M - \rho(b)]x_k\| < \varepsilon$ for $1 \leq k \leq n$.This shows that $T \in m(\mathcal{A})^{-sot}$. Hence (1) holds. ∎

We call a unital C^*-algebra $\mathcal{A}$ *strongly reflexive* if (1) holds for every Banach space X and every bounded unital homomorphism m. The results in [7] say that every commutative C^*-algebra is strongly reflexive.

LEMMA 2. *The following are true.*

i. *If $\mathcal{A}$ is a C^*-algebra and n is a positive integer, then $\mathcal{A}$ is strongly reflexive if and only if $\mathfrak{M}_n(\mathcal{A})$ is strongly reflexive.*

ii. *A finite direct sum of C^*-algebras is strongly reflexive if and only if each summand is strongly reflexive.*

PROOF. (i). Suppose $m: \mathfrak{M}_n(\mathcal{A}) \longrightarrow B(X)$ is a bounded unital homomorphism. We can assume that m is an isometry. Let $\{e_{ij}\}$ be the standard matrix units in $\mathfrak{M}_n(\mathcal{A})$. Let $X_i = m(e_{ii})(X)$ for $1 \leq i \leq n$. Then $m(e_{ij})$maps X_j isometrically onto X_i for $1 \leq i, j \leq n$. Hence we can assume that X is a direct sum of n copies of a Banach space Y, and we can identify $B(X)$ with $\mathfrak{M}_n\big((B(Y)\big)$ in such a way that $m(e_{ij}) = e_{ij}$ for $1 \leq i, j \leq n$. Define $\rho: \mathcal{A} \longrightarrow B(Y)$ by $\rho(a)e_{11} = m(ae_{11})$. It follows that $m\big((a_{ij})\big) = \big(\rho(a_{ij})\big)$ for every matrix (a_{ij}) in $\mathfrak{M}_n(\mathcal{A})$. Next suppose that $M \in \operatorname{Lat} m\big(\mathfrak{M}_n(\mathcal{A})\big)$. Then $M = m(e_{11})(M) + m(e_{22})(M) + \cdots + m(e_{nn})(M)$. Furthermore, since $m\big(\mathfrak{M}_n(\mathcal{A})\big)$ contains the matrix units, it follows that $m(e_{11})(M)$, $m(e_{22})(M), \ldots, m(e_{nn})(M)$ are all the same subspace N of Y. Thus M is a direct sum of n copies of N. It is clear that $N \in \operatorname{Lat} \rho(\mathcal{A})$. Conversely, if

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$N \in \operatorname{Lat}\rho(\mathcal{A})$, and M is a direct sum of n copies of N, then $M \in \operatorname{Lat} m\big(\mathfrak{M}_n(\mathcal{A})\big)$. Hence $\operatorname{Lat} m\big(\mathfrak{M}_n(\mathcal{A})\big)$ is precisely $\{N \oplus N \oplus \cdots \oplus N : N \in \operatorname{Lat}\rho(\mathcal{A})\}$.

It follows that $\operatorname{AlgLat} m\big(\mathfrak{M}_n(\mathcal{A})\big) = \mathfrak{M}_n\Big(AlgLat\big(\rho(\mathcal{A})\big)\Big)$. Therefore $\operatorname{AlgLat}\rho(\mathcal{A}) = \rho(\mathcal{A})^{-sot}$ if and only if $\operatorname{AlgLat} m\big(\mathfrak{M}_n(\mathcal{A})\big) = m\big(\mathfrak{M}_n(\mathcal{A})\big)^{-sot}$. It is clear that (i) holds.

The proof of (ii) is an elementary exercise left to the reader. ■

The next result yields analogues of the preceding lemma for certain infinite direct sums and infinite matrix algebras.

LEMMA 3. *Suppose X is a Banach space, $\mathcal{D}$ is a unital subalgebra of $B(X)$, and $\{P_\lambda\}$ is a bounded net of idempotents in $\mathcal{D}$ converging to 1 in the strong operator topology. If, for each λ, $\operatorname{AlgLat}\big(P_\lambda \mathcal{D} P_\lambda | P_\lambda(X)\big) = \big(P_\lambda \mathcal{D} P_\lambda | P_\lambda(X)\big)^{-sot}$, then $\operatorname{AlgLat}\mathcal{D} = \mathcal{D}^{-sot}$.*

PROOF. Suppose $T \in \operatorname{AlgLat}\mathcal{D}$, $\varepsilon > 0$ and $\mathcal{F}$ is a finite subset of X. We can choose λ so that $\|[P_\lambda T P_\lambda - T]x\| < \varepsilon/2$ for every x in $\mathcal{F}$. Since, for every y in $P_\lambda(X)$, $TP_\lambda y \in [\mathcal{D}P_\lambda y]^-$, it follows that $P_\lambda T P_\lambda | P_\lambda(X) \in \operatorname{AlgLat} P_\lambda \mathcal{D} P_\lambda | P_\lambda(X) = \big(P_\lambda \mathcal{D} P_\lambda | P_\lambda(X)\big)^{-sot}$. Thus there is a D in $\mathcal{D}$ such that $\|[P_\lambda T P_\lambda - P_\lambda D P_\lambda]x\| < \varepsilon/2$ for every x in $\mathcal{F}$. Thus $\|[T - P_\lambda D P_\lambda]x\| < \varepsilon$ for every x in $\mathcal{F}$. Since $P_\lambda D P_\lambda \in \mathcal{D}$, it follows that $T \in \mathcal{D}^{-sot}$. ■

COROLLARY 4. *For each positive integer n, suppose $\mathcal{A}_n$ is a strongly reflexive C*-algebra with identity e_n, and suppose $\mathcal{A}$ is a unital C*-algebra such that $\sum^\oplus \mathcal{A}_n \subset \mathcal{A} \subset \prod^\oplus \mathcal{A}_n$. If X is a Banach space and $m: \mathcal{A} \longrightarrow B(X)$ is a unital bounded homomorphism such that $m(e_1 + \cdots + e_n)$ converges strongly to the identity operator, then*

$$\operatorname{AlgLat} m(\mathcal{A}) = m(\mathcal{A})^{-sot}.$$

Let $\mathfrak{M}_{\infty,0}$ denote the algebra of all the infinite complex matrices with only finitely many non-zero entries. Then $\mathfrak{M}_{\infty,0}(\mathcal{A}) = \mathfrak{M}_{\infty,0} \otimes \mathcal{A}$ can be viewed as the algebra of infinite matrices with elements in $\mathcal{A}$ such that only finitely many entries are non-zero. Let $\mathfrak{M}_\infty(\mathcal{A})$ denote the set of all infinite matrices over $\mathcal{A}$ such that the supremum over $n \geq 1$ of the norms of the $n \times n$ upper left-hand corners is finite. Then $\mathfrak{M}_\infty(\mathcal{A})$ is an $\mathfrak{M}_{\infty,0}(\mathcal{A})$-module. The C*-completion of $\mathfrak{M}_{\infty,0}(\mathcal{A})$ is $\mathcal{A} \otimes \mathcal{K}$, where $\mathcal{K}$ denotes the algebra of compact operators of ℓ^2.

COROLLARY 5. *Suppose $\mathcal{A}$ is a strongly reflexive C*-algebra, and $\mathcal{B}$ is a unital C*-algebra such that $\mathfrak{M}_{\infty,0}(\mathcal{A}) \subset \mathcal{B} \subset \mathfrak{M}_\infty(\mathcal{A})$. If X is a Banach space and $m: \mathcal{B} \longrightarrow B(X)$ is a unital bounded homomorphism such that $P_n = m(e_{11} + \cdots + e_{nn})$ converges strongly to identity operator, then*

$$\operatorname{AlgLat}\big(m(\mathcal{B})\big) = m(\mathcal{B})^{-sot}.$$

We now turn to the case in which X is reflexive, $\mathcal{A}$ is separable and GCR (type I).

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THEOREM 6. *Suppose X is a reflexive Banach space and $\mathcal{A}$ is a separable GCR C^*-algebra and $m: \mathcal{A} \longrightarrow B(X)$ is bounded unital. Then $m(\mathcal{A})^{-sot} = \mathrm{AlgLat}\big(m(\mathcal{A})\big)$.*

PROOF. Following [8], we can assume that m is an isometry. Since X is reflexive, we can uniquely extend m to a homomorphism $\hat{m}: \mathcal{A}^{\#\#} \longrightarrow B(X)$ that is weak*-*wot* continuous. By [11,3.7], we can represent $\mathcal{A}^{\#\#}$ as a von Neumann algebra on a separable Hilbert space so that the weak operator topology and the weak$*$-topology coincide. Hence there is a projection P in the center of $\mathcal{A}^{\#\#}$ such that $\ker \hat{m} = (1-P)\mathcal{A}^{\#\#}$. Let $H = \operatorname{ran} P$, and let $\mathcal{B} = \mathcal{A}^{\#\#}|H$. Then $\mathcal{B}$ is a von Neumann algebra isomorphic to $\mathcal{A}^{\#\#}/\ker \hat{m}$. Hence we can assume that $\mathcal{A} \subset \mathcal{B} \subset B(H)$, and that there is a unital, isometric, $wot - wot$ continuous homomorphism $\tilde{m}: \mathcal{B} \longrightarrow B(X)$ extending m, and that the unit ball of $\mathcal{A}$ is wot-dense in the unit ball of $\mathcal{B}$.

Since $\mathcal{A}$ is GCR, $\mathcal{B}$ must be a type I von Neumann algebra acting on a separable Hilbert space [11]. Hence, ignoring multiplicities, $\mathcal{B}$ is isomorphic (not unitarily equivalent) to a direct sum of von Neumann algebras $\mathcal{B}_n$, $1 \le n \le \infty$, such that, for some compact Hausdorff space K_n, $\mathcal{B}_n$ is isomorphic to $\mathfrak{M}_n\big(C(K_n)\big)$ for $1 \le n < \infty$ and $\mathcal{B}_\infty$ is isomorphic to $\mathfrak{M}_\infty\big(C(K_\infty)\big)$ so that $e_{11} + e_{22} + \cdots + e_{nn} \longrightarrow 1$ in the weak $*$-topology. Write $\mathcal{B} = \mathcal{B}_\infty \oplus \mathcal{B}_1 \oplus \mathcal{B}_2 \oplus \cdots$, and define a sequence $\{Q_n\}$ of projections by $Q_1 = (e_{11}, 1, 0, 0, 0, \ldots)$, $Q_2 = (e_{11}+e_{22}, 1, 1, 0, 0, \ldots)$, $Q_3 = (e_{11}+e_{22}+e_{33}, 1, 1, 1, 0, 0, 0, \ldots)$, $\ldots$.

It follows from [7] and Lemma 2 that $Q_n\mathcal{B}_nQ_n$ is strongly reflexive for $1 \le n < \infty$. Hence, by Lemma 3 (using $P_n = \tilde{m}(Q_n)$), we conclude that $\mathrm{AlgLat}\,\tilde{m}(\mathcal{B}) = \tilde{m}(\mathcal{B})^{-sot}$. However, the continuity of $\tilde{m}$ implies that $\tilde{m}(\mathcal{B}) \subset \tilde{m}(\mathcal{A})^{-sot} = m(\mathcal{A})^{-sot}$. Since $m(\mathcal{A}) \subset \tilde{m}(\mathcal{B})$ implies $\mathrm{AlgLat}\, m(\mathcal{A}) \subset \mathrm{AlgLat}\,\tilde{m}(\mathcal{B})$, we conclude that $\mathrm{AlgLat}\, m(\mathcal{A}) = m(\mathcal{A})^{-sot}$. ∎

REMARKS. 1. In the preceding theorem we can replace the reflexivity of X with the assumption that m has weakly compact action, since this is what is needed to conclude the existence of the extension $\hat{m}$. Note that C. Akemann, P. G. Dodds, and J. L. B. Gamlen [2], extending the result of *A*. Pełczynski [12], proved that if a Banach space X does not contain a copy of c_0, then m has weakly compact action for every C^*-algebra $\mathcal{A}$. In particular, when X is a reflexive Banach space, m always has weakly compact action.
2. The first author [6] proved an asymptotic version of the von Neumann double commutant theorem (1), and the authors proved [7] that this asymptotic version holds for general Banach spaces when $\mathcal{A}$ is commutative. It would be interesting to know if the asymptotic version of Theorem 6 is true.

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