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Character Density in Central Subalgebras of Compact Quantum Groups

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Abstract. We investigate quantum group generalizations of various density results from Fourier analysis on compact groups. In particular, we establish the density of characters in the space of fixed points of the conjugation action on $L^2(\mathbb{G})$ and use this result to show the weak* density and norm density of characters in $ZL^\infty(\mathbb{G})$ and $ZC(\mathbb{G})$, respectively. As a corollary, we partially answer an open question of Woronowicz. At the level of $L^1(\mathbb{G})$, we show that the center $\mathcal{Z}(L^1(\mathbb{G}))$ is precisely the closed linear span of the quantum characters for a large class of compact quantum groups, including arbitrary compact Kac algebras. In the latter setting, we show, in addition, that $\mathcal{Z}(L^1(\mathbb{G}))$ is a completely complemented $\mathcal{Z}(L^1(\mathbb{G}))$-submodule of $L^1(\mathbb{G})$.

1 Introduction

As in the group setting, irreducible characters play a significant role in harmonic analysis on compact quantum groups [2, 4, 5, 16]. In this note, we investigate the relationship between the irreducible characters of compact quantum groups $\mathbb{G}$ and the central subalgebras of the Banach algebras $L^2(\mathbb{G})$ and $L^1(\mathbb{G})$, and the operator algebras $C(\mathbb{G})$ and $L^\infty(\mathbb{G})$. We characterize the fixed points of the two canonical conjugation actions on $L^2(\mathbb{G})$ as the closed linear span of the characters and quantum characters, respectively, the latter being equal to the center $\mathcal{Z}(L^2(\mathbb{G}))$. We then use these characterizations to establish the weak* density of characters in $ZL^\infty(\mathbb{G}) := \{x \in L^\infty(\mathbb{G}) \mid \Gamma(x) = \Sigma\Gamma(x)\}$ and norm density in $ZC(\mathbb{G}) := \{x \in C(\mathbb{G}) \mid \Gamma(x) = \Sigma\Gamma(x)\}$, thereby partially answering an open question of Woronowicz (see [16, Proposition 5.11]), and generalizing the partial solution of Lemeux in the Kac setting [11, Theorem 1.4]. For any compact quantum group whose dual has the central almost completely positive approximation property in the sense of [5, Definition 3], we show that $\mathcal{Z}(L^1(\mathbb{G}))$ is the closed linear span of the characters. We establish the same result for arbitrary compact Kac algebras by showing that $\mathcal{Z}(L^1(\mathbb{G}))$ is a completely complemented $\mathcal{Z}(L^1(\mathbb{G}))$-submodule of $L^1(\mathbb{G})$.

2 Compact Quantum Groups

A *locally compact quantum group* is a quadruple $\mathbb{G} = (L^\infty(\mathbb{G}), \Gamma, \varphi, \psi)$, where $L^\infty(\mathbb{G})$ is a Hopf–von Neumann algebra with a co-associative co-multiplication $\Gamma\colon L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{G})$, and φ and ψ are fixed (normal faithful semifinite) left and right Haar weights on $L^\infty(\mathbb{G})$, respectively [9, 10]. For every locally compact quantum

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group $\mathbb{G}$ there exists a *left fundamental unitary operator* W on $L^2(\mathbb{G},\varphi)\otimes_2 L^2(\mathbb{G},\varphi)$ and a *right fundamental unitary operator* V on $L^2(\mathbb{G},\psi)\otimes_2 L^2(\mathbb{G},\psi)$ implementing the co-multiplication Γ via

$$\Gamma(x) = W^*(1\otimes x)W = V(x\otimes 1)V^*, \quad x\in L^\infty(\mathbb{G}).$$

Both unitaries satisfy the *pentagonal relation*; that is,

$$W_{12}W_{13}W_{23} = W_{23}W_{12} \quad \text{and} \quad V_{12}V_{13}V_{23} = V_{23}V_{12}.$$

At the level of the Hilbert spaces,

$$W^*\Lambda_{\varphi\otimes\varphi}(x\otimes y) = \Lambda_{\varphi\otimes\varphi}\big(\Gamma(y)(x\otimes 1)\big) \quad \text{and} \quad V\Lambda_{\psi\otimes\psi}(a\otimes b) = \Lambda_{\psi\otimes\psi}\big(\Gamma(a)(1\otimes b)\big)$$

for $x,y\in\mathcal{N}_\varphi$ and $a,b\in\mathcal{N}_\psi$. By [10, Proposition 2.11], we can identify $L_2(\mathbb{G},\varphi)$ and $L_2(\mathbb{G},\psi)$, so we will simply use $L^2(\mathbb{G})$ for this Hilbert space throughout the paper. The *reduced quantum group C^*-algebra* of $L^\infty(\mathbb{G})$ is defined as

$$C_0(\mathbb{G}) := \overline{\{(\mathrm{id}\otimes\omega)(W) \mid \omega\in\mathcal{T}(L^2(\mathbb{G}))\}}^{\|\cdot\|}.$$

We say that $\mathbb{G}$ is *compact* if $C_0(\mathbb{G})$ is a unital C^*-algebra, in which case we denote $C_0(\mathbb{G})$ by $C(\mathbb{G})$. For compact quantum groups it follows that φ is finite and right invariant. In particular, $\varphi=\psi$.

We let R and $(\tau_t)_{t\in\mathbb{R}}$ denote the *unitary antipode* and *scaling group* of $\mathbb{G}$, respectively. The unitary antipode satisfies

$$(R\otimes R)\circ\Gamma = \Sigma\circ\Gamma\circ R, \tag{2.1}$$

where $\Sigma: L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{G})\to L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{G})$ denotes the flip map. The *antipode* of $\mathbb{G}$ is $S = R\tau_{-i/2}$, and is a closed densely defined operator on $L^\infty(\mathbb{G})$, whose domain we denote by $\mathcal{D}(S)$.

Let $L^1(\mathbb{G})$ denote the predual of $L^\infty(\mathbb{G})$. Then the pre-adjoint of Γ induces an associative completely contractive multiplication on $L^1(\mathbb{G})$, defined by

$$\star\colon L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{G})\ni f\otimes g\longmapsto f\star g = \Gamma_*(f\otimes g)\in L^1(\mathbb{G}).$$

There is a canonical $L^1(\mathbb{G})$-bimodule structure on $L^\infty(\mathbb{G})$, given by

$$\langle f\star x, g\rangle = \langle x, g\star f\rangle \quad \text{and} \quad \langle x\star f, g\rangle = \langle x, f\star g\rangle, \quad x\in L^\infty(\mathbb{G}),\ f,g\in L^1(\mathbb{G}).$$

We say that $\mathbb{G}$ is *co-amenable* if $L^1(\mathbb{G})$ has a bounded left (equivalently, right or two-sided) approximate identity (*cf.* [3, Theorem 3.1]).

Let $L^1_*(\mathbb{G})$ be the subspace of $L^1(\mathbb{G})$ defined by

$$L^1_*(\mathbb{G}) = \big\{f\in L^1(\mathbb{G}) : \exists\, g\in L^1(\mathbb{G}) \text{ s.t. } g(x) = f^*\circ S(x) \quad \forall x\in\mathcal{D}(S)\big\},$$

where $f^*(x) = \overline{f(x^*)}$, $x\in L^\infty(\mathbb{G})$. It is known from [15, §1.13] that $L^1_*(\mathbb{G})$ is a dense subalgebra of $L^1(\mathbb{G})$. There is an involution on $L^1_*(\mathbb{G})$, given by $f^o = f^*\circ S$, such that $L^1_*(\mathbb{G})$ becomes a Banach *-algebra under the norm $\|f\|_* = \max\{\|f\|,\|f^o\|\}$.

A *unitary co-representation* of $\mathbb{G}$ is a unitary $U\in L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(H)$ satisfying $(\Gamma\otimes\mathrm{id})(U) = U_{13}U_{23}$. Every unitary co-representation gives rise to a representation of $L^1(\mathbb{G})$ via

$$L^1(\mathbb{G})\ni f\mapsto (f\otimes\mathrm{id})(U)\in\mathcal{B}(H).$$

In particular, the left fundamental unitary W gives rise to the *left regular representation* $\lambda: L^1(\mathbb{G}) \to \mathcal{B}(L^2(\mathbb{G}))$ defined by $\lambda(f) = (f \otimes \mathrm{id})(W)$, $f \in L^1(\mathbb{G})$, that is an injective, completely contractive homomorphism from $L^1(\mathbb{G})$ into $\mathcal{B}(L^2(\mathbb{G}))$. Then $L^\infty(\widehat{\mathbb{G}}) := \{\lambda(f) : f \in L^1(\mathbb{G})\}''$ is the von Neumann algebra associated with the dual quantum group $\widehat{\mathbb{G}}$ of $\mathbb{G}$. When $\mathbb{G}$ is compact with normalized Haar state φ, the following holds [16]: every irreducible co-representation u^α is finite-dimensional and is unitarily equivalent to a sub-representation of W, and every unitary co-representation of $\mathbb{G}$ can be decomposed into a direct sum of irreducible co-representations. We let $\mathrm{Irr}(\mathbb{G}) := \{u^\alpha\}$ denote a complete set of representatives of irreducible co-representations of $\mathbb{G}$ that are pairwise inequivalent. Slicing by vector functionals $\omega_{ij} = \omega_{e_j,e_i}$ relative to an orthonormal basis of H_α, we obtain elements $u^\alpha_{ij} = (\mathrm{id} \otimes \omega_{ij})(u^\alpha) \in L^\infty(\mathbb{G})$ satisfying

$$\Gamma(u^\alpha_{ij}) = \sum_{k=1}^{n_\alpha} u^\alpha_{ik} \otimes u^\alpha_{kj}, \quad 1 \le i, j \le n_\alpha.$$

The linear space $\mathcal{A} := \mathrm{span}\{u^\alpha_{ij} \mid \alpha \in \mathrm{Irr}(\mathbb{G})\ 1 \le i, j \le n_\alpha\}$ forms a unital Hopf $*$-algebra that is dense in $C(\mathbb{G})$.

For every $\alpha \in \mathrm{Irr}(\mathbb{G})$ there exists a positive invertible matrix $F^\alpha \in M_{n_\alpha}(\mathbb{C})$ such that the corresponding "F–matrices" implement the left Haar weight of the dual $\widehat{\mathbb{G}}$. Without loss of generality, we can assume that $F^\alpha = \mathrm{diag}(\lambda^\alpha_1, \dots, \lambda^\alpha_{n_\alpha})$ [6, Proposition 2.1]. Since $\mathrm{tr}(F^\alpha) = \mathrm{tr}(F^\alpha)^{-1}$, it follows that

$$\sum_{i=1}^{n_\alpha} \lambda^\alpha_i = \sum_{i=1}^{n_\alpha} \frac{1}{\lambda^\alpha_i} = \mathrm{tr}(F^\alpha) =: d_\alpha,$$

where d_α is the *quantum dimension* of u^α. If $\mathbb{G}$ is a *compact Kac algebra*, meaning φ is tracial, then $d_\alpha = n_\alpha$ and $F^\alpha = 1_{n_\alpha}$ for all $\alpha \in \mathrm{Irr}(\mathbb{G})$. For every α there exists a conjugate representation $\overline{\alpha}$ on $\overline{H}_\alpha$, such that

$$\lambda^{\overline{\alpha}}_i = (\lambda^\alpha_i)^{-1} \quad \text{and} \quad u^{\overline{\alpha}}_{ij} = \sqrt{\frac{\lambda^\alpha_i}{\lambda^\alpha_j}}\, u^{\alpha^*}_{ij}$$

(see [13, Proposition 1.4.6]).

In the general setting, the Peter–Weyl orthogonality relations are as follows:

$$\varphi\big((u^\beta_{kl})^* u^\alpha_{ij}\big) = \delta_{\alpha\beta}\delta_{ik}\delta_{jl}\frac{1}{\lambda^\alpha_i d_\alpha}, \quad \varphi\big(u^\beta_{kl}(u^\alpha_{ij})^*\big) = \delta_{\alpha\beta}\delta_{ik}\delta_{jl}\frac{\lambda^\alpha_j}{d_\alpha}.$$

From this it follows that $\{\sqrt{d_\alpha \lambda^\alpha_i}\Lambda_\varphi(u^\alpha_{ij}) \mid \alpha \in \mathrm{Irr}(\mathbb{G}), 1 \le i, j \le n_\alpha\}$ is an orthonormal basis for $L^2(\mathbb{G})$.

For an element $x \in L^\infty(\mathbb{G})$, we let $x \cdot \varphi$ and $\varphi \cdot x$ denote the elements in $L^1(\mathbb{G})$ given by $\langle x \cdot \varphi, y\rangle = \varphi(yx)$ and $\langle \varphi \cdot x, y\rangle = \varphi(xy)$, $y \in L^\infty(\mathbb{G})$. If $x = u^\alpha_{ij}$, we denote $u^\alpha_{ij} \cdot \varphi$ by φ^α_{ij}. By the density of $\mathcal{A}$ in $C(\mathbb{G})$, it follows that $L^\infty(\mathbb{G}) \cdot \varphi$ is dense in $L^1(\mathbb{G})$. Let

$$\mathcal{I}_\varphi := \big\{ f \in L^1(\mathbb{G}) \;\big|\; \exists\, M > 0 : |\langle f, x^*\rangle| \le M \|\Lambda_\varphi(x)\| \,\forall x \in L^\infty(\mathbb{G}) \big\}.$$

Then $\mathcal{I}_\varphi$ is a dense left ideal in $L^1(\mathbb{G})$ containing $L^\infty(\mathbb{G})\cdot\varphi$ such that for every $f\in\mathcal{I}_\varphi$, there exists a unique $a(f)\in L^2(\mathbb{G})$ satisfying $\langle a(f),\Lambda_\varphi(x)\rangle=\langle f,x^*\rangle$ for all $x\in L^\infty(\mathbb{G})$. If $x\in L^\infty(\mathbb{G})$, then $a(x\cdot\varphi)=\Lambda_\varphi(x)$.

By left invariance of the Haar state, the map $I: L^2(\mathbb{G})\ni\Lambda_\varphi(x)\mapsto\Lambda_{\varphi\otimes\varphi}(\Gamma(x))\in L^2(\mathbb{G})\otimes_2 L^2(\mathbb{G})$ is an isometry. Composing its adjoint $I^*: L^2(\mathbb{G})\otimes_2 L^2(\mathbb{G})\to L^2(\mathbb{G})$ with the canonical contraction $L^2(\mathbb{G})\otimes^\gamma L^2(\mathbb{G})\to L^2(\mathbb{G})\otimes_2 L^2(\mathbb{G})$, where $\otimes^\gamma$ denotes the projective tensor product, we obtain a Banach algebra structure on $L^2(\mathbb{G})$. On elementary tensors, the multiplication is given by

$$\Lambda_\varphi(x)\otimes\Lambda_\varphi(y)\longmapsto a\big((x\cdot\varphi)\star(y\cdot\varphi)\big),\quad x,y\in L^\infty(\mathbb{G}).$$

Moreover, there exists a contractive homomorphic injection $b: L^2(\mathbb{G})\to L^1(\mathbb{G})$ satisfying $b(a(f))=f$ for all $f\in\mathcal{I}_\varphi$. We refer the reader to [7, §6.2] for details in the Kac case, the proofs carrying over verbatim to general compact $\mathbb{G}$.

As in the case of compact groups, the irreducible characters of $\mathbb{G}$ play an important role in the harmonic analysis. For $\alpha\in\mathrm{Irr}(\mathbb{G})$, we let

$$\chi^\alpha:=(\mathrm{id}\otimes\mathrm{tr})(u^\alpha)=\sum_{i=1}^{n_\alpha}u^\alpha_{ii}\in L^\infty(\mathbb{G})$$

be the *character* of α, and we let

$$\chi^\alpha_q:=(\mathrm{id}\otimes F^\alpha)(u^\alpha)=\sum_{i=1}^{n_\alpha}\lambda^\alpha_i u^\alpha_{ii}\in L^\infty(\mathbb{G})$$

be the *quantum character* of α. The characters (as well as the quantum characters) satisfy the decomposition relations

$$\chi^\alpha\chi^\beta=\sum_{\gamma\in\mathrm{Irr}(\mathbb{G})}N^\gamma_{\alpha\beta}\chi^\gamma, \tag{2.2}$$

where $N^\gamma_{\alpha\beta}$ is the multiplicity of γ in the tensor product representation $\alpha\otimes\beta$ (see [13, Proposition 1.4.3]). It follows that $\chi^{\overline{\alpha}}=\chi^{\alpha^*}$, $\alpha\in\mathrm{Irr}(\mathbb{G})$, so that $\overline{\mathrm{span}}\{\chi^\alpha\mid\alpha\in\mathrm{Irr}(\mathbb{G})\}$ is a C^*-subalgebra of $C(\mathbb{G})$. Letting $\varphi^\alpha_q:=\chi^\alpha_q\cdot\varphi$ be the $L^1(\mathbb{G})$ elements corresponding to the quantum characters of $\mathbb{G}$, it follows from the orthogonality relations that

$$\langle\varphi^\alpha_q\star f,u^{\beta*}_{kl}\rangle=\langle f\star\varphi^\alpha_q,u^{\beta*}_{kl}\rangle=\langle f,u^{\beta*}_{kl}\rangle\frac{\delta_{\alpha\beta}}{d_\alpha}$$

for all $f\in L^1(\mathbb{G})$ and $\beta\in\mathrm{Irr}(\mathbb{G})$. In particular,

$$\varphi^\alpha_q\star\varphi^\alpha_q=\frac{1}{d_\alpha}\varphi^\alpha_q,\quad \alpha\in\mathrm{Irr}(\mathbb{G}).$$

By weak* density of $\mathcal{A}$ in $L^\infty(\mathbb{G})$, it follows that $\overline{\mathrm{alg}}\{\varphi^\alpha_q\mid\alpha\in\mathrm{Irr}(\mathbb{G})\}$ is a closed ideal in $\mathcal{Z}(L^1(\mathbb{G}))$, the center of $L^1(\mathbb{G})$. Below we establish the reverse inclusion for a large class of compact quantum groups.

From the Peter–Weyl relations, one easily sees that

$$\big\langle\Lambda_\varphi(\chi^\alpha),\Lambda_\varphi(\chi^\beta)\big\rangle=\delta_{\alpha\beta}=\big\langle\Lambda_\varphi(\chi^\alpha_q),\Lambda_\varphi(\chi^\beta_q)\big\rangle,\quad\alpha,\beta\in\mathrm{Irr}(\mathbb{G}), \tag{2.3}$$

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so spatially, there is no difference between the subspaces of $L^2(\mathbb{G})$ generated by $\{\Lambda_\varphi(\chi^\alpha)\}$ and $\{\Lambda_\varphi(\chi^\alpha_q)\}$. The multiplicative structure of these spaces is quite different, however, as we now investigate.

3 Central Subalgebras

Let $\mathbb{G}$ be a compact quantum group, and let $\beta_2 : L^1(\mathbb{G}) \to \mathcal{B}(L^2(\mathbb{G}))$ be the conjugation representation of $\mathbb{G}$, defined by

$$\beta_2(f) = (f \otimes \mathrm{id})\big(W(1 \otimes U^*)W(1 \otimes U)\big), \quad f \in L^1(\mathbb{G}),$$

where $U = \widehat{J}J$, and J and $\widehat{J}$ are the conjugate linear isometries arising from the GNS constructions of φ and $\widehat{\varphi}$, respectively. One can easily verify that

$$W(1 \otimes U^*)W(1 \otimes U) \in L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(L^2(\mathbb{G}))$$

is a unitary co-representation of $\mathbb{G}$, so that β_2 is indeed a homomorphism. Let $ZL^2(\mathbb{G})$ denote the set of fixed vectors under β_2, *i.e.*, those $\xi \in L^2(\mathbb{G})$ satisfying $\beta_2(f)\xi = \langle f, 1\rangle\xi$ for all $f \in L^1(\mathbb{G})$. We call $ZL^2(\mathbb{G})$ the space of *central vectors* of $L^2(\mathbb{G})$, and in what follows we study its connection to the center $\mathcal{Z}(L^2(\mathbb{G}))$. We begin with a few lemmas.

Lemma 3.1 *Let $\mathbb{G}$ be a compact quantum group. Then for $x \in \mathcal{A}$,*

$$\beta_2(\varphi)\Lambda_\varphi(x) = \sum_{i=1}^n \lambda(y_i \cdot \varphi)\Lambda_\varphi(x_i) = \sum_{i=1}^n \Lambda_\varphi\big((\varphi \otimes \mathrm{id})(y_i \otimes 1)\Gamma(x_i)\big),$$

where $\Gamma(x) = \sum_{i=1}^n x_i \otimes y_i$.

Proof We let $\sigma : L^2(\mathbb{G}) \otimes L^2(\mathbb{G}) \to L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$ denote the flip map. If $a \in \mathcal{A}$, we have

$$\begin{aligned}
\big\langle \beta_2(\varphi)\Lambda_\varphi(x), \Lambda_\varphi(a)\big\rangle &= \big\langle W\sigma V\sigma\Lambda_{\varphi\otimes\varphi}(1 \otimes x), \Lambda_{\varphi\otimes\varphi}(1 \otimes a)\big\rangle \\
&= \big\langle \sigma\Lambda_{\varphi\otimes\varphi}(\Gamma(x)), \Lambda_{\varphi\otimes\varphi}(\Gamma(a))\big\rangle = \varphi \otimes \varphi\big(\Gamma(a^*)\Sigma\Gamma(x)\big) \\
&= \varphi \otimes \varphi\big(\Gamma(a^*)\Sigma\Gamma(x)\big) = \sum_{i=1}^n \varphi \otimes \varphi\big(\Gamma(a^*)y_i \otimes x_i\big) \\
&= \sum_{i=1}^n y_i \cdot \varphi\big((\mathrm{id} \otimes \varphi)(\Gamma(a^*)(1 \otimes x_i))\big).
\end{aligned}$$

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Since $S((\mathrm{id}\otimes\varphi)(\Gamma(a^*)(1\otimes x_i))) = (\mathrm{id}\otimes\varphi)((1\otimes a^*)\Gamma(x_i))$ and $S(S(y)^*)^* = y$ for all $y \in \mathcal{A}$, we have $(\mathrm{id}\otimes\varphi)(\Gamma(a^*)(1\otimes x_i)) = S((\mathrm{id}\otimes\varphi)(\Gamma(x_i^*)(1\otimes a)))^*$. Continuing,

$$\begin{aligned}\langle \beta_2(\varphi)\Lambda_\varphi(x), \Lambda_\varphi(a)\rangle &= \sum_{i=1}^n y_i\cdot\varphi\big(S((\mathrm{id}\otimes\varphi)(\Gamma(x_i^*)(1\otimes a)))^*\big)\\ &= \sum_{i=1}^n \overline{(y_i\cdot\varphi)^o\big((\mathrm{id}\otimes\varphi)(\Gamma(x_i^*)(1\otimes a))\big)} = \sum_{i=1}^n \overline{\varphi\big(((y_i\cdot\varphi)^o\otimes\mathrm{id})(\Gamma(x_i^*))a\big)}\\ &= \sum_{i=1}^n \big\langle \Lambda_\varphi\big(((y_i\cdot\varphi)^{o^*}\otimes\mathrm{id})\Gamma(x_i)\big), \Lambda_\varphi(a)\big\rangle\\ &= \sum_{i=1}^n \big\langle \big((y_i\cdot\varphi)^{o^*}\otimes\mathrm{id}\big)(W^*)\Lambda_\varphi(x_i), \Lambda_\varphi(a)\big\rangle\\ &= \sum_{i=1}^n \big\langle \lambda\big((y_i\cdot\varphi)^o\big)^*\Lambda_\varphi(x_i), \Lambda_\varphi(a)\big\rangle = \sum_{i=1}^n \big\langle \lambda(y_i\cdot\varphi)\Lambda_\varphi(x_i), \Lambda_\varphi(a)\big\rangle.\end{aligned}$$

This establishes the first formula. The second follows from the general relation

$$((y\cdot\varphi)^{o^*}\otimes\mathrm{id})\Gamma(x) = (\varphi\otimes\mathrm{id})((S(y)\otimes 1)\Gamma(x))$$

valid for all $x, y \in \mathcal{D}(S)$, as is easily verified. ■

Lemma 3.2 *Let $\mathbb{G}$ be a compact quantum group, and $x \in \mathcal{A}$. Then $x\cdot\varphi \in L^1_*(\mathbb{G})$ with $(x\cdot\varphi)^o = S(x)^*\cdot\varphi$.*

Proof First note that $\varphi = \varphi\circ S$ on $\mathcal{D}(S)$. Then for $y \in \mathcal{D}(S)$ we have

$$\begin{aligned}\langle S(x)^*\cdot\varphi, y\rangle &= \varphi(yS(x)^*) = \overline{\varphi(S(x)y^*)} = \overline{\varphi\big(S(x)S(S(y)^*)\big)}\\ &= \overline{\varphi\big(S(S(y)^*x)\big)} = \overline{\varphi\big(S(y)^*x\big)} = \overline{\langle x\cdot\varphi, S(y)^*\rangle}.\end{aligned}$$

Thus, $x\cdot\varphi \in L^1_*(\mathbb{G})$ with $(x\cdot\varphi)^o = S(x)^*\cdot\varphi$. ■

Lemma 3.3 *Let $\mathbb{G}$ be a compact quantum group. Then*

$$\big\{ d_\alpha\sqrt{\lambda_i^\alpha\lambda_j^\alpha}\,\lambda(\varphi_{ij}^\alpha) \mid \alpha\in\mathrm{Irr}(\mathbb{G}), 1\le i,j\le n_\alpha\big\}$$

forms a set of matrix units for the von Neumann algebra $L^\infty(\widehat{\mathbb{G}})$. In particular, for every $\xi\in L^2(\mathbb{G})$, we have

$$\xi = \sum_{\alpha\in\mathrm{Irr}(\mathbb{G})} d_\alpha\Lambda_\varphi(\chi_q^\alpha)\star\xi,$$

where the sum converges in $L^2(\mathbb{G})$.

Proof Let $e_{ij}^\alpha := d_\alpha\sqrt{\lambda_i^\alpha\lambda_j^\alpha}\,\lambda(\varphi_{ij}^\alpha)$ for $1\le i,j\le n_\alpha$ and $\alpha\in\mathrm{Irr}(\mathbb{G})$. By Lemma 3.2, $\varphi_{ij}^\alpha\in L^1_*(\mathbb{G})$ and $\varphi_{ij}^{\alpha^o} = S(u_{ij}^\alpha)^*\cdot\varphi = u_{ji}^\alpha\cdot\varphi = \varphi_{ji}^\alpha$. Since λ is involutive on $L^1_*(\mathbb{G})$

[10, Proposition 2.4], we have $e_{ij}^{\alpha *} = e_{ji}^{\alpha}$. Then

$$\langle \varphi_{ij}^{\alpha} \star \varphi_{kl}^{\beta}, u_{mn}^{\gamma *} \rangle = \sum_{r=1}^{n_\gamma} \langle \varphi_{ij}^{\alpha} \otimes \varphi_{kl}^{\beta}, u_{mr}^{\gamma *} \otimes u_{rn}^{\gamma *} \rangle = \sum_{r=1}^{n_\gamma} \varphi(u_{mr}^{\gamma *} u_{ij}^{\alpha}) \varphi(u_{rn}^{\gamma *} u_{kl}^{\beta})$$
$$= \frac{\delta_{\alpha\gamma}\delta_{\beta\gamma}\delta_{im}\delta_{jk}\delta_{nl}}{\lambda_k^{\alpha}\lambda_i^{\alpha} d_\alpha^2} = \frac{\delta_{\alpha\beta}\delta_{kj}}{\lambda_k^{\alpha} d_\alpha} \langle \varphi_{il}^{\alpha}, u_{mn}^{\gamma *} \rangle.$$

It follows that

$$e_{ij}^{\alpha} e_{kl}^{\beta} = d_\alpha d_\beta \sqrt{\lambda_i^{\alpha}\lambda_j^{\alpha}} \sqrt{\lambda_k^{\beta}\lambda_l^{\beta}} \lambda(\varphi_{ij}^{\alpha} \star \varphi_{kl}^{\beta}) = \delta_{\alpha\beta}\delta_{kj} e_{il}^{\alpha}.$$

By the above, we know that $\sum_{i=1}^{n_\alpha} e_{ii}^{\alpha} = d_\alpha \lambda(\varphi_q^{\alpha}) \in \mathcal{Z}(L^\infty(\widehat{\mathbb{G}}))$, hence $z_\alpha := d_\alpha \lambda(\varphi_q^{\alpha})$ is a central projection in $L^\infty(\widehat{\mathbb{G}})$ acting as the identity on the factor $\{e_{ij}^{\alpha} \mid 1 \leq i, j, \leq n\} \cong M_{n_\alpha}(\mathbb{C})$. Thus,

$$L^\infty(\widehat{\mathbb{G}}) = \bigoplus_{\alpha \in \mathrm{Irr}(\mathbb{G})} z_\alpha L^\infty(\widehat{\mathbb{G}}) \cong \bigoplus_{\alpha \in \mathrm{Irr}(\mathbb{G})} M_{n_\alpha}(\mathbb{C}). \qquad \blacksquare$$

We now show that the central vectors in $L^2(\mathbb{G})$ are precisely the span of the characters, generalizing the well-known fact for compact groups.

Proposition 3.4 *Let $\mathbb{G}$ be a compact quantum group. Then*

$$ZL^2(\mathbb{G}) = \overline{\mathrm{span}}\big\{ \Lambda_\varphi(\chi^\alpha) \mid \alpha \in \mathrm{Irr}(\mathbb{G}) \big\}.$$

Proof Since $\varphi^o = \varphi = \varphi^*$, by Lemma 3.1 we have

$$\big\langle \beta_2(\varphi)\Lambda_\varphi(u_{kl}^{\alpha}), \Lambda_\varphi(u_{ij}^{\beta}) \big\rangle = \sum_{n,m=1}^{n_\alpha} \big\langle \Lambda_\varphi\big((\varphi \otimes \mathrm{id})(u_{ln}^{\alpha *} u_{km}^{\alpha} \otimes u_{mn}^{\alpha}) \big), \Lambda_\varphi(u_{ij}^{\beta}) \big\rangle$$
$$= \sum_{n,m=1}^{n_\alpha} \varphi\big((u_{ln}^{\alpha})^* u_{km}^{\alpha} \big) \varphi\big((u_{ij}^{\beta})^* u_{mn}^{\alpha} \big)$$
$$= \delta_{\alpha\beta}\delta_{kl}\delta_{ij} \frac{1}{\lambda_i^{\alpha}\lambda_k^{\alpha} d_\alpha^2}$$
$$= \delta_{kl} \frac{1}{\lambda_k^{\alpha} d_\alpha} \big\langle \Lambda_\varphi(\chi^\alpha), \Lambda_\varphi(u_{ij}^{\beta}) \big\rangle.$$

By density of irreducible coefficients, we obtain

$$\beta_2(\varphi)\Lambda_\varphi(u_{kl}^{\alpha}) = \frac{\delta_{kl}}{\lambda_k^{\alpha} d_\alpha} \Lambda_\varphi(\chi^\alpha). \tag{3.1}$$

Lemma 3.3 implies that $\beta_2(\varphi)$ is a self-adjoint idempotent, so $\beta_2(\varphi)$ is the orthogonal projection onto its fixed points, namely $ZL^2(\mathbb{G})$. Equation (3.1) implies that $\beta_2(\varphi)\Lambda_\varphi(\chi^\alpha) = \Lambda_\varphi(\chi^\alpha)$ for all $\alpha \in \mathrm{Irr}(\mathbb{G})$ so that

$$\overline{\mathrm{span}}\{\Lambda_\varphi(\chi^\alpha) \mid \alpha \in \mathrm{Irr}(\mathbb{G})\} \subseteq ZL^2(\mathbb{G}).$$

Conversely, if $\xi \in ZL^2(\mathbb{G})$ and $\langle \xi, \Lambda_\varphi(\chi^\alpha) \rangle = 0$ for all α, then

$$\big\langle \xi, \Lambda_\varphi(u_{kl}^{\beta}) \big\rangle = \big\langle \xi, \beta_2(\varphi)\Lambda_\varphi(u_{kl}^{\beta}) \big\rangle = \delta_{kl} \frac{1}{\lambda_k^{\alpha} d_\alpha} \big\langle \xi, \Lambda_\varphi(\chi^\alpha) \big\rangle = 0$$

for all $\beta \in \mathrm{Irr}(\mathbb{G})$, $k, l = 1, \dots, n_\beta$. By density, we must have $\xi = 0$. ∎

Proposition 3.5 *Let $\mathbb{G}$ be a compact quantum group. Then*

$$\mathcal{Z}(L^2(\mathbb{G})) = \overline{\mathrm{span}}\{\Lambda_\varphi(\chi_q^\alpha) \mid \alpha \in \mathrm{Irr}(\mathbb{G})\}.$$

Proof Since $\{\varphi_q^\alpha \mid \alpha \in \mathrm{Irr}(\mathbb{G})\} \subseteq \mathcal{Z}(L^1(\mathbb{G}))$, it follows from the definition of convolution in $L^2(\mathbb{G})$ that $\overline{\mathrm{span}}\{\Lambda(\chi_q^\alpha) \mid \alpha \in \mathrm{Irr}(\mathbb{G})\} \subseteq \mathcal{Z}(L^2(\mathbb{G}))$. Conversely, suppose that $\xi \in \mathcal{Z}(L^2(\mathbb{G}))$. Since $b(L^2(\mathbb{G}))$ is dense in $L^1(\mathbb{G})$, it follows that $b(\xi) \in \mathcal{Z}(L^1(\mathbb{G}))$, which in turn makes $\lambda(b(\xi)) \in \mathcal{Z}(L^\infty(\widehat{\mathbb{G}}))$, so that $d_\alpha\lambda(\varphi_q^\alpha)\lambda(b(\xi)) = z_\alpha\lambda(b(\xi)) = c_\alpha z_\alpha = c_\alpha d_\alpha \lambda(\varphi_q^\alpha)$ for each $\alpha \in \mathrm{Irr}(\mathbb{G})$. By injectivity of λ, we obtain $d_\alpha\varphi_q^\alpha \star b(\xi) = c_\alpha d_\alpha \varphi_q^\alpha$. But $\varphi_q^\alpha = b(\Lambda_\varphi(\chi_q^\alpha))$, so injectivity of b implies $d_\alpha\Lambda_\varphi(\chi_q^\alpha) \star \xi = c_\alpha d_\alpha \Lambda_\varphi(\chi_q^\alpha)$. Thus, by Lemma 3.3,

$$\xi = \sum_{\alpha \in \mathrm{Irr}(\mathbb{G})} d_\alpha \Lambda_\varphi(\chi_q^\alpha) \star \xi = \sum_{\alpha \in \mathrm{Irr}(\mathbb{G})} c_\alpha d_\alpha \Lambda_\varphi(\chi_q^\alpha),$$

where the series converges in $L^2(\mathbb{G})$. ∎

Remark 3.6 Let $\beta_2' \colon L^1(\mathbb{G}) \to \mathcal{B}(L^2(\mathbb{G}))$ be the representation defined by

$$\beta_2'(f) = U\beta_2(f)U^* = (f \otimes \mathrm{id})\big((1 \otimes U)W(1 \otimes U^*)W\big), \quad f \in L^1(\mathbb{G}).$$

Then β_2' is another "conjugation representation" on $L^2(\mathbb{G})$ whose fixed points are precisely $\mathcal{Z}(L^2(\mathbb{G}))$. Thus, for non-Kac compact quantum groups, the conjugation representations β_2 and β_2' distinguish the central vectors from the center of the Banach algebra $L^2(\mathbb{G})$.

In the group setting, W and $(1 \otimes U)W(1 \otimes U^*)$ belong to $L^\infty(G)\overline{\otimes}VN(G)$ and $L^\infty(G)\overline{\otimes}VN(G)'$, respectively, and therefore commute. Hence, for $f \in L^1(G)$, we have

$$\beta_2(f) = \beta_2'(f) = \int_G f(s)\lambda(s)\rho(s)\,ds,$$

where λ and ρ are the left and right regular representations of G, respectively, and ds is the normalized Haar measure on G.

We now establish the corresponding density theorems at the level of $L^\infty(\mathbb{G})$ and $C(\mathbb{G})$. In particular, we show that $ZC(\mathbb{G}) = \{x \in C(\mathbb{G}) \mid \Gamma(x) = \Sigma\Gamma(x)\}$ is precisely the closed linear span of the characters, which partially answers an open question of Woronowicz (see [16, Proposition 5.11]).

Theorem 3.7 *Let $\mathbb{G}$ be a compact quantum group. Then*

$$ZL^\infty(\mathbb{G}) = \{\chi^\alpha \mid \alpha \in \mathrm{Irr}(\mathbb{G})\}''.$$

Proof The inclusion $\{\chi^\alpha \mid \alpha \in \operatorname{Irr}(\mathbb{G})\}'' \subseteq ZL^\infty(\mathbb{G})$ is clear. Let $x \in ZL^\infty(\mathbb{G})$. Then for any $y \in L^\infty(\mathbb{G})$,

$$\begin{aligned}\langle \beta_2(\varphi)\Lambda_\varphi(x), \Lambda_\varphi(y^*)\rangle &= \langle W\sigma V\sigma\Lambda_{\varphi\otimes\varphi}(1\otimes x), \Lambda_{\varphi\otimes\varphi}(1\otimes y^*)\rangle\\ &= \langle \sigma\Lambda_{\varphi\otimes\varphi}(\Gamma(x)), \Lambda_{\varphi\otimes\varphi}(\Gamma(y^*))\rangle\\ &= \varphi\otimes\varphi(\Gamma(y)\Sigma\Gamma(x)) = \varphi\otimes\varphi(\Gamma(yx)) = \varphi(yx)\\ &= \langle \Lambda_\varphi(x), \Lambda_\varphi(y^*)\rangle.\end{aligned}$$

It follows that $\beta_2(\varphi)\Lambda_\varphi(x) = \Lambda_\varphi(x)$. Hence, $\Lambda_\varphi(ZL^\infty(\mathbb{G})) \subseteq ZL^2(\mathbb{G})$ by Proposition 3.4, and

$$ZL^2(\mathbb{G}) = \overline{\Lambda_\varphi(ZL^\infty(\mathbb{G}))}^{\|\cdot\|},$$

as the reverse inclusion is clear.

Note that equations (2.2) and (2.3) entail the traciality of φ on the von Neumann algebra $\{\chi^\alpha \mid \alpha \in \operatorname{Irr}(\mathbb{G})\}''$. In particular, the map $ZL^2(\mathbb{G}) \ni \Lambda_\varphi(x) \mapsto \Lambda_\varphi(x^*) \in ZL^2(\mathbb{G})$ is an isometry. Given, $x, y \in ZL^\infty(\mathbb{G})$, take sequences (x_n) and (y_m) in $\operatorname{span}\{\chi^\alpha \mid \alpha \in \operatorname{Irr}(\mathbb{G})\}$ such that $\Lambda_\varphi(x_n) \to \Lambda_\varphi(x)$ and $\Lambda_\varphi(y_m) \to \Lambda_\varphi(y)$. Then

$$\begin{aligned}\varphi(x^*y) = \langle \Lambda_\varphi(y), \Lambda_\varphi(x)\rangle &= \lim_n\langle \Lambda_\varphi(y_n), \Lambda_\varphi(x)\rangle = \lim_n\lim_m\langle \Lambda_\varphi(y_n), \Lambda_\varphi(x_m)\rangle\\ &= \lim_n\lim_m \varphi(x_m^* y_n) = \lim_n\lim_m \varphi(y_n x_m^*) = \lim_n\lim_m\langle \Lambda_\varphi(x_m^*), \Lambda_\varphi(y_n^*)\rangle\\ &= \lim_n\langle \Lambda_\varphi(x^*), \Lambda_\varphi(y_n^*)\rangle = \varphi(yx^*).\end{aligned}$$

Thus, φ is a faithful trace on $ZL^\infty(\mathbb{G})$, so there is a unique conditional expectation $E: ZL^\infty(\mathbb{G}) \to \{\chi^\alpha \mid \alpha \in \operatorname{Irr}(\mathbb{G})\}''$ satisfying $\Lambda_\varphi(E(x)) = \beta_2(\varphi)\Lambda_\varphi(x) = \Lambda_\varphi(x)$, $x \in ZL^\infty(\mathbb{G})$. Then $E(x) = x$, and $ZL^\infty(\mathbb{G}) \subseteq \{\chi^\alpha \mid \alpha \in \operatorname{Irr}(\mathbb{G})\}''$. ■

Corollary 3.8 *Let $\mathbb{G}$ be a compact quantum group. Then*

$$ZC(\mathbb{G}) = \overline{\operatorname{span}}\{\chi^\alpha \mid \alpha \in \operatorname{Irr}(\mathbb{G})\}.$$

Proof Let $ZL^1(\mathbb{G}) := \overline{\operatorname{span}}\{\varphi^\alpha \mid \alpha \in \operatorname{Irr}(\mathbb{G})\}$. As φ is a normal faithful trace on $ZL^\infty(\mathbb{G})$, it follows that $ZL^\infty(\mathbb{G}) \cong (ZL^1(\mathbb{G}))^*$ completely isometrically and weak*-weak* homeomorphically. Let

$$r: L^\infty(\mathbb{G}) \ni x \longmapsto x|_{ZL^1(\mathbb{G})} \in ZL^\infty(\mathbb{G})$$

be the completely contractive restriction map. The orthogonality relations imply

$$r(u_{ij}^\alpha) = \frac{\delta_{ij}\lambda_i^\alpha}{d_\alpha}\chi^\alpha, \quad \alpha \in \operatorname{Irr}(\mathbb{G}),\ 1 \le i, j \le n_\alpha.$$

In particular, $r(C(\mathbb{G})) \subseteq \overline{\operatorname{span}}\{\chi^\alpha \mid \alpha \in \operatorname{Irr}(\mathbb{G})\}$, and $r(\chi^\alpha) = \chi^\alpha$ for all $\alpha \in \operatorname{Irr}(\mathbb{G})$. Since r is weak*-weak* continuous, by Theorem 3.7 it follows that $r(x) = x$ for all $x \in ZL^\infty(\mathbb{G})$. Hence, if $x \in ZC(\mathbb{G})$, then $x = r(x) \in \overline{\operatorname{span}}\{\chi^\alpha \mid \alpha \in \operatorname{Irr}(\mathbb{G})\}$. Since the reverse inclusion is obvious, we are done. ■

Remark 3.9 For a compact quantum group $\mathbb{G}$, let $C_u(\mathbb{G})$ be its corresponding universal C^*-algebra (see [8] for details). There is a universal co-multiplication $\Gamma_u: C_u(\mathbb{G}) \to C_u(\mathbb{G}) \otimes_{\min} C_u(\mathbb{G})$ satisfying $(\pi\otimes\pi)\circ\Gamma_u = \Gamma\circ\pi$, where

$\pi: C_u(\mathbb{G}) \to C(\mathbb{G})$ is the canonical quotient map. This gives rise to a universal compact quantum group structure on $C_u(\mathbb{G})$. In particular, there is a $*$-algebra $\mathcal{A}_u$ of universal matrix coefficients that is dense in $C_u(\mathbb{G})$, and there are universal characters $\chi_u^\alpha \in C_u(\mathbb{G})$ satisfying $\pi(\chi_u^\alpha) = \chi^\alpha$, $\alpha \in \operatorname{Irr}(\mathbb{G})$. In [16], Woronowicz asks whether $ZC_u(\mathbb{G}) \cap \mathcal{A}_u$ is dense in $ZC_u(\mathbb{G}) = \{x_u \in C_u(\mathbb{G}) \mid \Gamma_u(x_u) = \Sigma\Gamma_u(x_u)\}$. Theorem 3.7, therefore, answers this question, in the affirmative, for all co-amenable compact quantum groups, *i.e.,* those for which $C_u(\mathbb{G}) = C(\mathbb{G})$ (see [3, Theorem 3.1]). This generalizes the partial result of Lemeux in the co-amenable Kac setting [11, Theorem 1.4].

For compact groups G, the standard proof that $\mathcal{Z}(L^1(G))$ is the closed linear span of the characters utilizes a central bounded approximate identity (BAI) for $L^1(G)$. A similar argument applies for any compact quantum group $\mathbb{G}$ for which $L^1(\mathbb{G})$ has a BAI (f_i) in $\operatorname{span}\{\varphi_q^\alpha \mid \alpha \in \operatorname{Irr}(\mathbb{G})\}$, which is bounded in the completely bounded multiplier norm; *i.e.,* the maps $L^1(\mathbb{G}) \ni g \mapsto f_i \star g \in L^1(\mathbb{G})$ are uniformly completely bounded. Any $\mathbb{G}$ whose dual has the central almost completely positive approximation property (ACPAP) in the sense of [5, Definition 3] has this property. Thus,

$$\mathcal{Z}(L^1(\mathbb{G})) = \overline{\operatorname{span}}\{\varphi_q^\alpha \mid \alpha \in \operatorname{Irr}(\mathbb{G})\} \tag{3.2}$$

for any compact $\mathbb{G}$ with the central ACPAP. By [5, Theorem 25], this includes $SU_q(2)$, $q \in [-1,0) \cup (0,1]$, as well as any free orthogonal and unitary quantum groups O_F^+ and U_F^+, for any parameter matrix $F \in GL(n,\mathbb{C})$ (see [5, §1.4,§4.2], for instance). We conjecture that (3.2) is valid for arbitrary compact $\mathbb{G}$. We now provide support for the conjecture by showing that it holds for arbitrary compact Kac algebras. In turn, we generalize a result of Mosak [12, Proposition 1.5 (i)].

Theorem 3.10 *Let $\mathbb{G}$ be a compact Kac algebra. Then $\beta_2(\varphi): L^2(\mathbb{G}) \to L^2(\mathbb{G})$ extends to a completely contractive projection $\beta_1: L^1(\mathbb{G}) \to \mathcal{Z}(L^1(\mathbb{G}))$ satisfying*

$$\beta_1(f \star g) = f \star \beta_1(g), \quad f \in \mathcal{Z}(L^1(\mathbb{G})), g \in L^1(\mathbb{G}).$$

Proof The argument in [14, Lemma 3.2] shows that the map $\Phi: L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ given by

$$\Phi(X) = (\omega_{\Lambda_\varphi(1)} \otimes \mathrm{id})W^*(U^* \otimes 1)X(U \otimes 1)W, \quad X \in L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{G})$$

is a normal, unital, completely positive left inverse to Γ satisfying $\Gamma \circ \Phi = (\Phi \otimes \mathrm{id})(\mathrm{id} \otimes \Gamma)$. Moreover, since $W \in L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\widehat{\mathbb{G}})$ and $U\Lambda_\varphi(1) = \Lambda_\varphi(1)$, we have

$$\varphi\big(\Phi(x \otimes y)\big) = \big(\omega_{\Lambda_\varphi(1)} \otimes \varphi\big)\big((U^*xU \otimes 1)W^*(1 \otimes y)W\big) = \varphi(x)\varphi(y)$$

for all $x, y \in L^\infty(\mathbb{G})$. By normality it follows that $\varphi \otimes \varphi = \varphi \circ \Phi = \varphi \otimes \varphi \circ \Gamma \circ \Phi$, so that $\Gamma \circ \Phi$ is a normal conditional expectation onto $\Gamma(L^\infty(\mathbb{G}))$ preserving $\varphi \otimes \varphi$.

By Lemma 3.1, the map $b \circ \beta_2(\varphi) \circ a: \mathcal{I}_\varphi \to L^1(\mathbb{G})$ satisfies $b \circ \beta_2(\varphi) \circ a(x \cdot \varphi) = \sum_{i=1}^n (y_i \cdot \varphi) \star (x_i \cdot \varphi)$ for $x \in \mathcal{A}$, where $\Gamma(x) = \sum_{i=1}^n x_i \otimes y_i$. Recalling that φ is invariant

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under the unitary antipode, for $y \in L^\infty(\mathbb{G})$ we have

$$\begin{aligned}\big\langle b\circ\beta_2(\varphi)\circ a(x\cdot\varphi), y\big\rangle &= \sum_{i=1}^{n}\big\langle (y_i\cdot\varphi)\star(x_i\cdot\varphi), y\big\rangle = \sum_{i=1}^{n}\big\langle (y_i\cdot\varphi)\otimes(x_i\cdot\varphi), \Gamma(y)\big\rangle\\ &= \sum_{i=1}^{n}\varphi\otimes\varphi\big(\Gamma(y)(y_i\otimes x_i)\big)\\ &= \varphi\otimes\varphi\big(\Gamma(y)\Sigma\Gamma(x)\big) = \varphi\otimes\varphi\big(\Gamma(R(x))(R\otimes R)(\Gamma(y))\big)\\ &= \varphi\otimes\varphi\big(\Gamma\circ\Phi(\Gamma(R(x))(R\otimes R)(\Gamma(y)))\big)\\ &= \varphi\otimes\varphi\big(\Gamma(R(x))\Gamma\circ\Phi((R\otimes R)(\Gamma(y)))\big)\\ &= \varphi\otimes\varphi\big(\Gamma(R(x)\Phi((R\otimes R)(\Gamma(y))))\big)\\ &= \varphi\big(R(x)\Phi((R\otimes R)(\Gamma(y)))\big)\\ &= \varphi\big(R(\Phi((R\otimes R)(\Gamma(y))))x\big)\\ &= \langle x\cdot\varphi, R\circ\Phi\circ(R\otimes R)\circ\Gamma(y)\rangle.\end{aligned}$$

Since the span$\{x\cdot\varphi \mid x\in\mathcal{A}\}$ is dense in $L^1(\mathbb{G})$, it follows that the map $b\circ\beta_2(\varphi)\circ a$ has a completely contractive extension to a map $\beta_1\colon L^1(\mathbb{G})\to L^1(\mathbb{G})$ whose adjoint $\beta_1^*\colon L^\infty(\mathbb{G})\to L^\infty(\mathbb{G})$ is given by $\beta_1^*(y) = R\circ\Phi\circ(R\otimes R)\circ\Gamma(y)$ for $y\in L^\infty(\mathbb{G})$. Moreover, since $\beta_2(\varphi)a(x\cdot\varphi) = ab(\beta_2(\varphi)a(x\cdot\varphi))$ for $x\in\mathcal{A}$, and $\beta_2(\varphi)$ is idempotent, we have

$$\beta_1(x\cdot\varphi) = b\beta_2(\varphi)\beta_2(\varphi)a(x\cdot\varphi) = b\beta_2(\varphi)a\circ b\beta_2(\varphi)a(x\cdot\varphi) = \beta_1\circ\beta_1(x\cdot\varphi),$$

which, by density, implies that β_1 is also idempotent.

To establish the module property, fix $g\in L^1(\mathbb{G})$ and $f\in\mathcal{Z}(L^1(\mathbb{G}))$. Then

$$\begin{aligned}R\otimes R\big(\Gamma(f\star x)\big) &= R\otimes R(\Gamma((\mathrm{id}\otimes f)\Gamma(x)))\\ &= R\otimes R\big((\mathrm{id}\otimes\mathrm{id}\otimes f)(\Gamma\otimes\mathrm{id})(\Gamma(x))\big)\\ &= (\mathrm{id}\otimes\mathrm{id}\otimes f)\big((R\otimes R)(\Gamma\otimes\mathrm{id})(\Gamma(x)\big)\\ &= (\mathrm{id}\otimes\mathrm{id}\otimes f)\big(\Sigma\Gamma\otimes\mathrm{id})(R\otimes\mathrm{id})(\Gamma(x)\big) \quad \text{(equation (2.1))}\\ &= \big(\mathrm{id}\otimes\mathrm{id}\otimes R_*(f)\big)(\Sigma\Gamma\otimes\mathrm{id})(R\otimes R)(\Gamma(x))\\ &= \big(\mathrm{id}\otimes\mathrm{id}\otimes R_*(f)\big)(\Sigma\Gamma\otimes\mathrm{id})\big(\Sigma\Gamma(R(x))\big)\\ &= \big(\mathrm{id}\otimes\mathrm{id}\otimes R_*(f)\big)(\mathrm{id}\otimes\Sigma\Gamma)\big(\Sigma\Gamma(R(x))\big) \quad \text{(co-associativity)}\\ &= \big(\mathrm{id}\otimes\mathrm{id}\otimes R_*(f)\big)(\mathrm{id}\otimes\Gamma)\big(\Sigma\Gamma(R(x))\big) \quad \big(f\in\mathcal{Z}(L^1(\mathbb{G}))\big).\end{aligned}$$

Applying the map Φ, we obtain

$$\begin{aligned}\Phi\big(R\otimes R(\Gamma(f\star x))\big) &= \big(\mathrm{id}\otimes R_*(f)\big)(\Phi\otimes\mathrm{id})(\mathrm{id}\otimes\Gamma)\big(\Sigma\Gamma(R(x))\big)\\ &= \big(\mathrm{id}\otimes R_*(f)\big)\Gamma\big(\Phi(\Sigma\Gamma(R(x)))\big)\\ &= \big(\mathrm{id}\otimes R_*(f)\big)\Gamma\big(\Phi(R\otimes R(\Gamma(x)))\big)\\ &= R_*(f)\star\Phi\big(R\otimes R(\Gamma(x))\big).\end{aligned}$$

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Therefore,

$$R \circ \Phi \circ (R \otimes R) \circ \Gamma(f \star x) = R \circ \Phi \circ (R \otimes R) \circ \Gamma(x) \star f = f \star R \circ \Phi \circ (R \otimes R) \circ \Gamma(x)$$

as f is central. The pre-adjoint of the above relation yields the desired module property.

Now, let $f \in L^1(\mathbb{G})$. Taking a sequence $(\Lambda_\varphi(x_n))$ in $L^2(\mathbb{G})$ such that $b(\Lambda_\varphi(x_n))$ converges to f, it follows that $\beta_1(b(\Lambda_\varphi(x_n)))$ converges to $\beta_1(f)$. But

$$\beta_1\big(b(\Lambda_\varphi(x_n))\big) = b\big(\beta_2(\varphi)\Lambda_\varphi(x_n)\big) \in b\big(\mathcal{Z}(L^2(\mathbb{G}))\big),$$

so that $\beta_1(f) \in \overline{b(\mathcal{Z}(L^2(\mathbb{G})))}$, which, by Proposition 3.5, is contained in

$$\overline{\operatorname{span}}\{\varphi^\alpha \mid \alpha \in \operatorname{Irr}(\mathbb{G})\} \subseteq \mathcal{Z}(L^1(\mathbb{G})).$$

Conversely, let $f \in \mathcal{Z}(L^1(\mathbb{G}))$. By Proposition 3.4 we have $\beta_1(\varphi^\alpha) = \varphi^\alpha$ for all $\alpha \in \operatorname{Irr}(\mathbb{G})$, so the module property of β_1 entails

$$\beta_1(f) \star \varphi^\alpha = \beta_1(f \star \varphi^\alpha) = f \star \beta_1(\varphi^\alpha) = f \star \varphi^\alpha, \quad \alpha \in \operatorname{Irr}(\mathbb{G}).$$

Hence, by Lemma 3.3, for every $\xi \in L^2(\mathbb{G})$,

$$\lambda(f)\xi = \sum_{\alpha \in \operatorname{Irr}(\mathbb{G})} d_\alpha \lambda(f \star \varphi^\alpha)\xi = \sum_{\alpha \in \operatorname{Irr}(\mathbb{G})} d_\alpha \lambda(\beta_1(f) \star \varphi^\alpha)\xi = \lambda(\beta_1(f))\xi.$$

By injectivity of λ we then get $f = \beta_1(f)$ so that $\mathcal{Z}(L^1(\mathbb{G})) = \beta_1(L^1(\mathbb{G}))$. ■

Corollary 3.11 *Let $\mathbb{G}$ be a compact Kac algebra. Then*

$$\mathcal{Z}(L^1(\mathbb{G})) = \overline{\operatorname{span}}\{\varphi^\alpha \mid \alpha \in \operatorname{Irr}(\mathbb{G})\}.$$

Remark 3.12 Regarding conjecture (3.2), it would be interesting to first examine the class $SU_q(2n+1)$, $n \geq 1$, as their duals have recently been shown to exhibit central property (T) [1, Corollary 8.9].

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