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Additive Maps on Units of Rings

Tamer Koşan, Serap Sahinkaya, and Yiqiang Zhou

Abstract. Let R be a ring. A map $f: R \to R$ is *additive* if $f(a+b) = f(a) + f(b)$ for all elements a and b of R. Here, a map $f: R \to R$ is called *unit-additive* if $f(u+v) = f(u) + f(v)$ for all units u and v of R. Motivated by a recent result of Xu, Pei and Yi showing that, for any field F, every unit-additive map of $\mathbb{M}_n(F)$ is additive for all $n \geq 2$, this paper is about the question of when every unit-additive map of a ring is additive. It is proved that every unit-additive map of a semilocal ring R is additive if and only if either R has no homomorphic image isomorphic to $\mathbb{Z}_2$ or $R/J(R) \cong \mathbb{Z}_2$ with $2 = 0$ in R. Consequently, for any semilocal ring R, every unit-additive map of $\mathbb{M}_n(R)$ is additive for all $n \geq 2$. These results are further extended to rings R such that $R/J(R)$ is a direct product of exchange rings with primitive factors Artinian. A unit-additive map f of a ring R is called *unit-homomorphic* if $f(uv) = f(u)f(v)$ for all units u, v of R. As an application, the question of when every unit-homomorphic map of a ring is an endomorphism is addressed.

1 Introduction

Let R be a ring. A map $f: R \to R$ is called *additive* if $f(a+b) = f(a) + f(b)$ for all elements a and b of R. In 2012, Franca [1] observed that an additive map of the matrix ring $\mathbb{M}_n(F)$ over a field F is completely determined by its action on certain subsets (*e.g.,* the subset consisting of invertible matrices) of the ring $\mathbb{M}_n(F)$. In [11], Xu, Pei, and Yi proved that, for any field F and any $n > 1$, every unit-additive map of $\mathbb{M}_n(F)$ is additive. Here, a map $f: R \to R$ is called *unit-additive* if $f(u+v) = f(u) + f(v)$ for all units u and v of R. This motivates us to consider the question of when every unit-additive map of a ring is additive. In this paper, we first determine the semilocal rings R such that every unit-additive map of R is additive by proving that every unit-additive map of a semilocal ring R is additive if and only if either R has no homomorphic image isomorphic to $\mathbb{Z}_2$ or $R/J(R) \cong \mathbb{Z}_2$ with $2 = 0$ in R. Consequently, for any semilocal ring R, every unit-additive map of $\mathbb{M}_n(R)$ is additive for all $n \geq 2$. This largely extends the main result in [11]. These results are further extended to rings R such that $R/J(R)$ is a direct product of exchange rings with primitive factors Artinian. We also consider a related notion: a map $f: R \to R$ is called *unit-homomorphic* if $f(u+v) = f(u) + f(v)$ and $f(uv) = f(u)f(v)$ for all units u and v of R. As an application, we address the question of when every unit-homomorphic map of a ring is an endomorphism.

Throughout, rings are associative with identity. The Jacobson radical and the set of units of a ring R are denoted by $J(R)$ and $U(R)$, respectively. The $n \times n$ matrix ring

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over R is denoted by $\mathbb{M}_n(R)$. As usual, $\mathbb{Z}_n$ denotes the ring of integers modulo n. A ring R is called *semilocal* if $R/J(R)$ is a semisimple Artinian ring.

2 Semilocal Rings

Definition 2.1 A map $f: R \to R$ is called a *unit-additive map* if f is additive on units of R, *i.e.*, $f(u+v) = f(u) + f(v)$ for all $u, v \in U(R)$.

Notation 2.2 For $a, b \in R$, we write $a \leftrightsquigarrow b$ (or $a \overset{u}{\leftrightsquigarrow} b$, to emphasize the element u) if $a - u, b - u \in U(R)$ for some $u \in U(R)$.

Lemma 2.3 *Let f be a unit-additive map of R. If $a \in R$ and $u \in U(R)$ with $-a \leftrightsquigarrow u$, then $f(a+u) = f(a) + f(u)$.*

Proof As $-a \leftrightsquigarrow u$, there exists $v \in U(R)$ such that $a + v, u - v \in U(R)$. So

$$\begin{aligned} f(a+u) &= f\big((a+v) + (u-v)\big) = f(a+v) + f(u-v) \\ &= f(a+v) + f(u) + f(-v) = \big[f(a+v) + f(-v)\big] + f(u) \\ &= f\big((a+v) - v\big) + f(u) = f(a) + f(u). \end{aligned}$$ ■

The following observation is the key step in the proof of [11, Theorem 4.1].

Lemma 2.4 *If $1 \leftrightsquigarrow x$ for all $x \in R$, then every unit-additive map of R is additive.*

Proof First, we show that $f(a+v) = f(a)+f(v)$ for any $a \in R$ and $v \in U(R)$. In fact, by our assumption, $1 \overset{w}{\leftrightsquigarrow} -v^{-1}a$ for some $w \in U(R)$, so $-a \overset{vw}{\leftrightsquigarrow} v$. So $f(a+v) = f(a)+f(v)$ by Lemma 2.3.

Now let $a, b \in R$. We can write $b = u + v$ with $u, v \in U(R)$. Then

$$\begin{aligned} f(a+b) &= f\big((a+u) + v\big) = f(a+u) + f(v) \\ &= f(a) + f(u) + f(v) \\ &= f(a) + f(u+v) = f(a) + f(b). \end{aligned}$$ ■

Next, we determine the semilocal rings R such that $1 \leftrightsquigarrow x$ for all $x \in R$. A ring R is said to satisfy the Goodearl–Menal condition if for any $a, b \in R$, there exists $u \in U(R)$ such that $a - u, b - u^{-1} \in U(R)$. The equivalence (iii)$\Leftrightarrow$ (iv) in the next lemma belongs to [6].

Lemma 2.5 *Let R be a semilocal ring. The following are equivalent:*

(i) *$1 \leftrightsquigarrow a$ for all $a \in R$;*
(ii) *$u \leftrightsquigarrow a$ for all $a \in R$ and all $u \in U(R)$;*
(iii) *R satisfies the Goodearl–Menal condition;*
(iv) *R has no factor ring isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$ or $\mathbb{M}_2(\mathbb{Z}_2)$.*

Proof (i)$\Rightarrow$(iv). In $\mathbb{Z}_2$, $1 \not\leftrightsquigarrow 1$. In $\mathbb{Z}_3$, $1 \not\leftrightsquigarrow 2$. In $\mathbb{M}_2(\mathbb{Z}_2)$, $I_2 \not\leftrightsquigarrow A$, where $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. To see this, assume on the contrary that $I_2 \overset{U}{\leftrightsquigarrow} A$, where U is a unit of $\mathbb{M}_2(\mathbb{Z}_2)$. Write

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$U = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. It follows that $U, I_2 - U, A - U$ all have determinant 1. That is,

$$ad + bc = 1, \quad (1+a)(1+d) + bc = 1, \quad ad + (1+b)c = 1.$$

It follows that $c = 0$, $ad = 1$, and $a + d = 1$. This is certainly impossible. Hence, none of $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{M}_2(\mathbb{Z}_2)$ satisfies (i). As condition (i) is inherited by factor rings, (i) implies (iv).

(iii)⇔(iv). This is [6, Theorem 2.2].

(iii)⇒(i). Let $a \in R$. By (iii), there exists $u \in U(R)$ such that $a - u, 1 - u^{-1} \in U(R)$. It follows that $1 \overset{u}{\leftrightsquigarrow} a$.

(ii)⇒(i). This is obvious.

(i)⇒(ii). Let $u \in U(R)$ and $a \in R$. By (i), $1 \overset{v}{\leftrightsquigarrow} u^{-1}a$ for some $v \in U(R)$, so $u \overset{uv}{\leftrightsquigarrow} a$. ■

A ring R is said to satisfy the 2-sum property if every element of R is a sum of two units. One can quickly show that a direct product of rings satisfies the 2-sum property if and only if every direct summand satisfies the 2-sum property, and that a ring R satisfies the 2-sum property if and only if so does $R/J(R)$ (see [2]). On the other hand, Wolfson [10] and Zelinsky [12], independently, showed that the ring of linear transformations of a vector space V over a division ring D satisfies the 2-sum property, except for $\dim(V) = 1$ and $D = \mathbb{Z}_2$. Thus, we have the following lemma.

Lemma 2.6 *A semilocal ring satisfies the* 2*-sum property if and only if no image of R is isomorphic to* $\mathbb{Z}_2$.

Lemma 2.7 *Suppose that R satisfies the* 2*-sum property. If* f *is a unit-additive map of R, then* $f(0) = 0$ *and* $f(-a) = -f(a)$ *for all* $a \in R$.

Proof Write $1 = u + v$ where u, v are units of R. Then

$$\begin{aligned} f(1) &= f(u+v) = f(u) + f(v) = f(1-v) + f(1-u) \\ &= f(1) + f(-v) + f(1) + f(-u), \end{aligned}$$

and so

$$0 = f(-v) + f(-u) + f(1) = f(-v-u) + f(1) = f(-1) + f(1) = f(0).$$

For $w \in U(R)$, we have $0 = f(w - w) = f(w) + f(-w)$, so $f(-w) = -f(w)$. Now let $a \in R$, and write $a = u + v$ where $u, v \in U(R)$. Then

$$f(-a) = f(-u-v) = f(-u)+f(-v) = -f(u)-f(v) = -\big(f(u)+f(v)\big) = -f(a). \ ■$$

Theorem 2.8 *Suppose that* $\mathbb{Z}_2$ *is a homomorphic image of R. Then every unit-additive map of R is additive if and only if* $R/J(R) \cong \mathbb{Z}_2$ *with* $2 = 0$ *in R.*

Proof (⇐) Let f be a unit-additive map of R. For $x \in J(R), 1+x \in U(R)$, so $f(x) = f(1+x)+f(1)$, *i.e.*, $f(1+x) = f(1)+f(x)$. Now let $a, b \in R$. As $R = J(R) \cup (1+J(R))$, we verify that f is additive in three cases.

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Case 1: $a, b \in J(R)$. Then

$$\begin{aligned} f(a+b) &= f\big((1+a)+(1+b)\big) = f(1+a)+f(1+b) \\ &= f(1)+f(a)+f(1)+f(b) = f(a)+f(b). \end{aligned}$$

Case 2: $a \in J(R)$ and $b \in 1+J(R)$. Write $b = 1+y$ with $y \in J(R)$. So $f(a+y) = f(a)+f(y)$ by case 1. Thus,

$$\begin{aligned} f(a+b) &= f\big(1+(a+y)\big) = f(1)+f(a+y) \\ &= f(1)+f(a)+f(y) = f(a)+\big[f(1)+f(y)\big] \\ &= f(a)+f(1+y) = f(a)+f(b). \end{aligned}$$

Case 3: $a, b \in 1+J(R)$. Then $f(a+b) = f(a)+f(b)$ as f is unit-additive.

($\Rightarrow$) By the hypothesis, $R/I \cong \mathbb{Z}_2$ for an ideal I of R. If $I = 0$, then $R = \mathbb{Z}_2$. Hence, we can assume that $I \neq 0$.

We next show that $I = J(R)$. Assume on the contrary that $I \neq J(R)$. Then $1+I \neq U(R)$. Note that $R = I \cup (1+I)$. Define $f\colon R \to R$ by $f(x) = 2$ for $x \in I$, $f(1+x) = 1$ for $x \in I$ with $1+x \in U(R)$, and $f(1+x) = 2$ for $x \in I$ with $1+x \notin U(R)$. Then, for $u, v \in U(R)$, $u = 1+x$, and $v = 1+y$, where $x, y \in I$, so

$$f(u+v) = f(2+x+y) = 2 = 1+1 = f(1+x)+f(1+y) = f(u)+f(v).$$

That is, f is a unit-additive map of R. As $1+I \neq U(R)$, there exists $z \in I$ such that $1+z \notin U(R)$. Thus, $f(1+z) = 2 \neq 1+2 = f(1)+f(z)$, so f is not additive. This contradiction shows that $I = J(R)$. It remains to show that $2 = 0$ in R. Note that $R = J(R) \cup (1+J(R))$. Define $f\colon R \to R$ by $f(x) = 2$ and $f(1+x) = 1$ for $x \in J(R)$. Then for $u, v \in J(R)$, $u = 1+x$, and $v = 1+y$, where $x, y \in J(R)$, so $f(u+v) = f(2+x+y) = 2 = 1+1 = f(u)+f(v)$. Hence, f is a unit-additive map of R, so is additive. Thus, $1 = f(1) = f(1+0) = f(1)+f(0) = 1+2$, so $2 = 0$ follows. ■

The following definition is a key ingredient needed.

Definition 2.9 A ring R is said to satisfy condition $(*)$ if, for any $a \in R$ and any $b \in U(R)$, there exist units u, v such that $a+b-u, a+v, b-u-v \in U(R)$.

Obviously, a ring with $(*)$ satisfies the 2-sum property.

Lemma 2.10 *If a ring R satisfies $(*)$, then every unit-additive map f of R is additive.*

Proof We first show that $f(a+b) = f(a)+f(b)$ for any $a \in R$ and any $b \in U(R)$. By the hypothesis, there exist units u, v such that $a+b-u, a+v, b-u-v \in U(R)$. Then by Lemma 2.7,

$$\begin{aligned} f(a+b)-f(a)-f(b) &= f(a+b)+f(-a)+f(-b) \\ &= f\big((a+b-u)+u\big)+f\big((-a-v)+v\big)+f(-b) \\ &= f(a+b-u)+f(u)+f(-a-v)+f(v)+f(-b) \\ &= \big[f(a+b-u)+f(-a-v)\big]+f(u)+f(v)+f(-b) \\ &= f(b-u-v)+f(u)+f(v)+f(-b) \end{aligned}$$

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$$\begin{aligned}&= \big[f(b-u-v)+f(-b)\big]+f(u)+f(v)\\&= f(-u-v)+f(u)+f(v)\\&= f(-u)+f(-v)+f(u)+f(v)\\&= \big[f(-u)+f(u)\big]+\big[f(-v)+f(v)\big]\\&= f(0)+f(0)=0+0=0.\end{aligned}$$

So $f(a+b)=f(a)+f(b)$.

Now let $x,y\in R$, and write $y=u+v$ where u,v are units of R. Then

$$f(x+y)=f(x+u+v)=f(x+u)+f(v)=f(x)+f(u)+f(v)=f(x)+f(y).$$

So f is additive. ∎

Lemma 2.11 (i) *A ring R satisfies $(*)$ if and only if $R/J(R)$ satisfies $(*)$.*
(ii) *A ring direct product $\prod R_i$ satisfies $(*)$ if and only if each R_i satisfies $(*)$.*

Proof (i) ($\Rightarrow$) Let $x\in R/J(R)$ and $y\in U(R/J(R))$. Write $x=\bar a$ and $y=\bar b$. Then $a\in R$ and $b\in U(R)$. By the hypothesis, there exist $u,v\in U(R)$ such that $a+b-u, a+v, b-u-v\in U(R)$. Thus, $\bar u,\bar v, x+y-\bar u, x+\bar v, y-\bar u-\bar v\in U(R/J(R))$.

($\Leftarrow$) Let $a\in R$ and $b\in U(R)$. Then $\bar a\in R/J(R)$ and $\bar b\in U(R/J(R))$. By the hypothesis, there exist $\bar u,\bar b\in U(R/J(R))$ such that $\bar a+\bar b-\bar u, \bar a+\bar v, \bar b-\bar u-\bar v\in U(R/J(R))$. Thus, $u,v,a+b-u,a+v,b-u-v\in U(R)$.

(ii) This is easily seen. ∎

We point out a needed fact about the ring $R:=\mathbb{M}_2(\mathbb{Z}_2)$: for any non-unit a in R and any unit u in R, either $a \overset{u}{\leftrightsquigarrow} u$ or $a+u\in U(R)$. For example, let $a=\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)$. We have

$$U(R)=\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&1\\1&1\end{pmatrix},\begin{pmatrix}1&0\\1&1\end{pmatrix},\begin{pmatrix}1&1\\1&0\end{pmatrix},\begin{pmatrix}1&1\\0&1\end{pmatrix}\right\},\text{ and}$$

$$\begin{pmatrix}1&1\\0&0\end{pmatrix}\overset{u}{\leftrightsquigarrow}\begin{pmatrix}1&0\\0&1\end{pmatrix}\text{ with } u=\begin{pmatrix}0&1\\1&1\end{pmatrix},\quad \begin{pmatrix}1&1\\0&0\end{pmatrix}\overset{u}{\leftrightsquigarrow}\begin{pmatrix}0&1\\1&0\end{pmatrix}\text{ with } u=\begin{pmatrix}1&0\\1&1\end{pmatrix},$$
$$\begin{pmatrix}1&1\\0&0\end{pmatrix}\overset{u}{\leftrightsquigarrow}\begin{pmatrix}1&1\\1&0\end{pmatrix}\text{ with } u=\begin{pmatrix}0&1\\1&1\end{pmatrix},\quad \begin{pmatrix}1&1\\0&0\end{pmatrix}\overset{u}{\leftrightsquigarrow}\begin{pmatrix}1&1\\0&1\end{pmatrix}\text{ with } u=\begin{pmatrix}1&0\\1&1\end{pmatrix},$$
$$\begin{pmatrix}1&1\\0&0\end{pmatrix}+\begin{pmatrix}0&1\\1&1\end{pmatrix}=\begin{pmatrix}1&0\\1&1\end{pmatrix}\in U(R),\quad \begin{pmatrix}1&1\\0&0\end{pmatrix}+\begin{pmatrix}1&0\\1&1\end{pmatrix}=\begin{pmatrix}0&1\\1&1\end{pmatrix}\in U(R).$$

The following observation is crucial to proving our main result.

Lemma 2.12 *Let R be a semilocal ring. Then R satisfies $(*)$ if and only if R satisfies the 2-sum property.*

Proof We just need to show the sufficiency. Because of Lemmas 2.6 and 2.11, we can assume that R is a simple Artinian ring not isomorphic to $\mathbb{Z}_2$. We verify that, for any $a\in R$ and any $b\in U(R)$, there exist $u,v\in U(R)$ such that $a+b-u, a+v, b-u-v\in U(R)$. We proceed with three cases.

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Case 1: $R = \mathbb{Z}_3$. If $a = 0$, take $u = 2b$ and $v = b$. If $a \neq 0$, take $u = b$ and $v = a$.

Case 2: $R = \mathbb{M}_2(\mathbb{Z}_2)$. First assume that a is not a unit. Then either $a + b \in U(R)$ or $a \leftrightsquigarrow b$. If $a + b \in U(R)$, write $a + b = x + y$ with units x and y, and take $u = x$ and $v = b$. If $a \leftrightsquigarrow b$, write $a = c + d$ and $b = c + d'$ with units c, d, d' and take $u = d$ and $v = d$.

If a is a unit, write $a = x + y$ with units x and y, and take $u = b$ and $v = x$.

Case 3: R is not isomorphic to $\mathbb{Z}_3$ and $\mathbb{M}_2(\mathbb{Z}_2)$. Then by Lemma 2.5, $-a \leftrightsquigarrow b$. Write $-a = c - d$ and $b = c + d'$ with units c, d, d' and take $u = d$ and $v = -d$. ∎

Now we are ready to present the main result in this section.

Theorem 2.13 *Let R be a semilocal ring. The following are equivalent:*

(i) *every unit-additive map of R is additive;*

(ii) *R has no image isomorphic to* $\mathbb{Z}_2$*, or* $R/J(R) \cong \mathbb{Z}_2$ *with* $2 = 0$ *in R.*

Proof (i)⇒(ii) This follows from Theorem 2.8.

(ii)⇒(i) In view of Theorem 2.8, we can assume that R has no image isomorphic to $\mathbb{Z}_2$. So, by Lemma 2.6, R satisfies the 2-sum property. Hence, R satisfies $(*)$ by Lemma 2.12, and so (i) holds by Lemma 2.10. ∎

Corollary 2.14 *If R is a semilocal ring, then every unit-additive map of* $\mathbb{M}_n(R)$ *is additive for all* $n \geq 2$.

Proof If R is semilocal and $n \geq 2$, then $\mathbb{M}_n(R)$ is a semilocal ring with no image isomorphic to $\mathbb{Z}_2$. So the Corollary follows from Theorem 2.13. ∎

3 Exchange Rings with Primitive Factors Artinian

In this section, we extend Theorem 2.13 and Corollary 2.14 to a larger class of rings. For an ideal $K \lhd R$ and $a \in R$, let $\overline{a} = a + K \in R/K$, and so the notation $(\overline{a_{ij}}) \in \mathbb{M}_n(R/K)$ means that $(\overline{a_{ij}}) = (a_{ij} + K)$.

Lemma 3.1 *Let* $\{K_\lambda\}$ *be a chain of ideals of a ring R, and* $K = \cup_\lambda K_\lambda$. *If* $(\overline{a_{ij}}) \in \mathbb{M}_n(R/K)$ *is a unit, then* $(\overline{a_{ij}}) \in \mathbb{M}_n(R/K_\lambda)$ *is a unit for some* λ.

Proof Assume that $(\overline{a_{ij}}) \in \mathbb{M}_n(R/K)$ is a unit. Then there exists $(\overline{b_{ij}}) \in \mathbb{M}_n(R/K)$ such that

$$(\overline{a_{ij}})(\overline{b_{ij}}) = (\overline{b_{ij}})(\overline{a_{ij}}) = \operatorname{diag}\{\bar{1}, \bar{1}, \ldots, \bar{1}\}.$$

Thus, $(a_{ij})(b_{ij}) - I_n$ and $(b_{ij})(a_{ij}) - I_n$ are in $\mathbb{M}_n(K)$. Because $\{K_\lambda\}$ is a chain, there exists some K_λ such that $(a_{ij})(b_{ij}) - I_n$ and $(b_{ij})(a_{ij}) - I_n$ are in $\mathbb{M}_n(K_\lambda)$. Hence,

$$(\overline{a_{ij}})(\overline{b_{ij}}) = (\overline{b_{ij}})(\overline{a_{ij}}) = \operatorname{diag}\{\bar{1}, \bar{1}, \ldots, \bar{1}\}$$

in $\mathbb{M}_n(R/K_\lambda)$. So, $(\overline{a_{ij}}) \in \mathbb{M}_n(R/K_\lambda)$ is a unit. ∎

The notion of an exchange ring was introduced by Warfield [9] via the exchange property of modules. By Goodearl–Warfield [4] or Nicholson [8], a ring R is an exchange ring if and only if for each $a \in R$ there exists $e^2 = e \in R$ such that $e \in aR$ and

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$1 - e \in (1-a)R$. Every semiprimitive exchange ring is an I-ring (*i.e.,* every nonzero right ideal contains a nonzero idempotent), and the class of exchange rings is closed under homomorphic images.

Lemma 3.2 *Let R be an exchange ring with primitive factors Artinian. The following are equivalent:*

(i) *R satisfies $(*)$;*
(ii) *R satisfies the 2-sum property;*
(iii) *R has no homomorphic images isomorphic to $\mathbb{Z}_2$.*

Proof (i)⇒(ii)⇒(iii) These are clear.

(iii)⇒(i) For convenience, for $a \in R$ and $b \in U(R)$ we say that a, b satisfy $(*)$ if there exist units u, v such that $a + b - u, a + v, b - u - v \in U(R)$; otherwise, we say that a, b do not satisfy $(*)$.

Assume on the contrary that R does not satisfy $(*)$. Then there exist $x \in R$ and $y \in U(R)$ such that x, y do not satisfy $(*)$. Thus,

$$\mathcal{F} = \left\{ I \lhd R : \overline{x}, \ \overline{y} \in R/I \text{ do not satisfy } (*) \right\}$$

is not empty. For a chain $\{I_\lambda\}$ of elements of $\mathcal{F}$, let $I = \cup_\lambda I_\lambda$. Then I is an ideal of R. Assume that $\overline{x}, \ \overline{y} \in R/I$ satisfy $(*)$. Then there exist units $\overline{u}, \overline{v}$ in R/I such that

$$\overline{a} + \overline{b} - \overline{u}, \overline{a} + \overline{v}, \overline{b} - \overline{u} - \overline{v} \in U(R/I).$$

Thus, by Lemma 3.1, $\overline{u}, \overline{v}$ and $\overline{a} + \overline{b} - \overline{u}, \overline{a} + \overline{v}, \overline{b} - \overline{u} - \overline{v}$ all are units in R/I_λ for some λ. So $\overline{x}, \overline{y} \in R/I_\alpha$ satisfy $(*)$. This contradiction shows that $I \in \mathcal{F}$. So $\mathcal{F}$ is an inductive set. By Zorn's Lemma, $\mathcal{F}$ has a maximal element, say I. Because every unit of $(R/I)/J(R/I)$ is lifted to a unit of R/I, the maximality of I implies that $J(R/I) = 0$.

We next show that R/I is an indecomposable ring. In fact, if R/I is a decomposable ring, then there exist ideals I_1, I_2 of R such that $I \subsetneqq I_i \subsetneqq R$ $(i = 1, 2)$ and

$$R/I \cong R/I_1 \oplus R/I_2 \quad \text{via} \quad r + I \longmapsto (r + I_1, r + I_2).$$

By the maximality of $I, \overline{x}, \ \overline{y} \in R/I_i$ satisfy $(*)$ for $i = 1, 2$. So, there exist $u + I_1, v + I_1 \in U(R/I_1)$ and $u' + I_2, v' + I_2 \in U(R/I_2)$ such that

$$\begin{aligned}
&(x + I_1) + (y + I_1) - (u + I_1),\\
&(x + I_1) + (v + I_1),\\
&(y + I_1) - (u + I_1) - (v + I_1)
\end{aligned}$$

are units of R/I_1, and

$$\begin{aligned}
&(x + I_2) + (y + I_2) - (u' + I_2),\\
&(x + I_2) + (v' + I_2),\\
&(y + I_2) - (u' + I_2) - (v' + I_2)
\end{aligned}$$

are units of R/I_2. Thus,

$$\begin{aligned}&(u + I_1, u' + I_2),\\ &(v + I_1, v' + I_2),\\ &(x + I_1, x + I_2) + (y + I_1, y + I_2) - (u + I_1, u' + I_2),\\ &(x + I_1, x + I_2) + (v + I_1, v' + I_2),\\ &(y + I_1, y + I_2) - (u + I_1, u' + I_2) - (v + I_1, v' + I_2)\end{aligned}$$

all are units of $R/I_1 \oplus R/I_2$. This shows that $(x+I_1, x+I_2), (y+I_1, y+I_2) \in R/I_1 \oplus R/I_2$ satisfy $(*)$. Hence, because of the ring isomorphism above, $\overline{x}, \overline{y} \in R/I$ satisfy $(*)$. This contradiction shows that R/I is indecomposable.

Thus, R/I is a semiprimitive indecomposable ring that is an exchange ring with primitive factors Artinian. Now by Menal [7, Lemma 1], R/I is a simple Artinian ring. Because R has no homomorphic images isomorphic to $\mathbb{Z}_2$, $R/I \ncong \mathbb{Z}_2$. Hence, by Zelinsky [12, Theorem], R/I satisfies the 2-sum property. Hence, R/I satisfies $(*)$ by Lemma 2.12, contradicting that $I \in \mathcal{F}$. ■

A ring is a *clean ring* if each of its elements is a sum of an idempotent and a unit. It is well known that every clean ring is an exchange ring.

Corollary 3.3 *If R is a clean ring with primitive factors Artinian, and if $2 \in U(R)$, then every unit-additive map of R is additive.*

Proof If $a \in R$ and $\frac{1}{2}(1+a) = e + u$, $e^2 = e$, and $u \in U(R)$, then $a = (2e-1) + 2u$ is a sum of two units (in fact $2e-1$ is an involution). So, by Lemma 3.2, every unit-additive map of R is additive. ■

Theorem 3.4 *Let R be a ring such that $R/J(R)$ is a direct product of exchange rings with primitive factors Artinian. The following are equivalent:*

(i) *every unit-additive map of R is additive;*
(ii) *R has no image isomorphic to $\mathbb{Z}_2$, or $R/J(R) \cong \mathbb{Z}_2$ with $2 = 0$ in R.*

Proof (i)⇒(ii) This is by Theorem 2.8.

(ii)⇒(i) First, by Theorem 2.8, we can assume that R has no homomorphic images isomorphic to $\mathbb{Z}_2$. Second, by Lemma 2.10, it suffices to show that R satisfies $(*)$. So, by Lemma 2.11(i), we can assume that $J(R) = 0$, and hence R is a direct product of exchange rings with primitive factors Artinian. Thus, by Lemma 2.11(ii), we can further assume that R is an exchange ring with primitive factors Artinian. As R has no homomorphic images isomorphic to $\mathbb{Z}_2$, R satisfies $(*)$ by Lemma 3.2. ■

Corollary 3.5 *Let R be an exchange ring with primitive factors Artinian. The following are equivalent:*

(i) *every unit-additive map of R is additive;*
(ii) *R has no image isomorphic to $\mathbb{Z}_2$, or $R/J(R) \cong \mathbb{Z}_2$ with $2 = 0$ in R.*

Corollary 3.6 *Let R be a ring such that $R/J(R)$ is a direct product of simple Artinian rings. The following are equivalent:*

(i) *every unit-additive map of R is additive;*
(ii) *R has no image isomorphic to $\mathbb{Z}_2$, or $R/J(R) \cong \mathbb{Z}_2$ with $2 = 0$ in R.*

A ring R is called *right self-injective* if every R-homomorphism from a right ideal of R into R can be extended to an R-homomorphism from R to R. A ring R is called *strongly π-regular* if, for each $a \in R$, $a^n \in Ra^{n+1} \cap a^{n+1}R$ for some positive integer n. Every one-sided perfect ring (in particular, one-sided Artinian ring) is strongly π-regular. A von Neumann regular ring in which every idempotent is central is called a *strongly regular ring*.

Corollary 3.7 *Let R be a ring such that $R/J(R)$ is right self-injective strongly π-regular. The following are equivalent:*

(i) *every unit-additive map of R is additive;*
(ii) *R has no image isomorphic to $\mathbb{Z}_2$, or $R/J(R) \cong \mathbb{Z}_2$ with $2 = 0$ in R.*

Proof (i)⇒(ii) This follows from Theorem 2.8.

(ii)⇒(i) By [5, Theorem], R is a finite direct product of matrix rings over strongly regular rings. So the equivalences follow from Theorem 3.4. ∎

We recall some notions from [3, pp. 111–115]. A ring R is called *directly finite* if $ab = 1$ in R implies $ba = 1$ for all $a, b \in R$. An idempotent e in a regular ring R is called an *abelian idempotent* if the ring eRe is abelian. An idempotent e in a regular right self-injective ring is called a *faithful idempotent* if 0 is the only central idempotent orthogonal to e. A regular right self-injective ring is of Type I_f if it is directly finite and it contains a faithful abelian idempotent.

Corollary 3.8 *Let R be a ring such that $R/J(R)$ is a regular right self-injective ring of Type I_f. The following are equivalent:*

(i) *every unit-additive map of R is additive;*
(ii) *R has no image isomorphic to $\mathbb{Z}_2$, or $R/J(R) \cong \mathbb{Z}_2$ with $2 = 0$ in R.*

Proof By [3, Theorem 10.24], R is a direct product of matrix rings over strongly regular rings. So the equivalences follow from Theorem 3.4. ∎

Corollary 3.8 motivates the following question, which we have been unable to answer.

Question 3.9 *Does Corollary 3.8 still hold for a right self-injective ring R?*

4 Applications

Here, we consider a notion related to a unit-additive map.

Definition 4.1 A map $f: R \to R$ is called *unit-homomorphic* if $f(u+v) = f(u)+f(v)$ and $f(uv) = f(u)f(v)$ for all $u, v \in U(R)$.

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The question concerned is: for which rings R is every unit-homomorphic map of R an endomorphism?

Example 4.2 Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $e = (1,0)$ and $e' = (0,1)$.

Define $f: R \to R$ by $f(1) = e$ and $f(a) = 0$ for $1 \neq a \in R$. Then f is unit-homomorphic. Moreover, f preserves multiplication. Because $f(1+e) = 0 \neq e = f(1) + f(e)$, f is not additive.

Define $g: R \to R$ by $g(0) = 0$, $g(1) = e$, $g(e) = 1$, $g(e') = e'$. Then g is unit-homomorphic. Moreover, g preserves addition. Because $g(ee') = g(0) = 0 \neq e' = g(e)g(e')$, g does not preserve multiplication.

Theorem 4.3 *Suppose that $\mathbb{Z}_2$ is a homomorphic image of R. Then every unit-homomorphic map of R is an endomorphism if and only if $R/J(R) \cong \mathbb{Z}_2$ with $2 = 0$ in R.*

Proof ($\Leftarrow$) Let $f: R \to R$ be a unit-homomorphic map. Then f is additive by Theorem 2.8. It remains to show that $f(ab) = f(a)f(b)$ for $a, b \in R$.

As $R/J(R) \cong \mathbb{Z}_2$, $R = J(R) \cup (1 + J(R))$. If $a, b \in 1 + J(R)$, then $f(ab) = f(a)f(b)$ as f is unit-homomorphic. If $a, b \in J(R)$, then

$$\begin{aligned} f(ab) &= f\big((1+(1+a))(1+(1+b))\big) = f\big(1+(1+a)+(1+b)+(1+a)(1+b)\big) \\ &= f(1) + f(1+a) + f(1+b) + f\big((1+a)(1+b)\big) \\ &= f(1)f(1) + f(1+a)f(1) + f(1)f(1+b) + f(1+a)f(1+b) \\ &= \big[f(1) + f(1+a)\big]\big[f(1) + f(1+b)\big] = f(a)f(b). \end{aligned}$$

If one of a, b is in $J(R)$ and the other is in $1 + J(R)$, say $a \in J(R)$ and $b \in 1 + J(R)$, then

$$\begin{aligned} f(ab) &= f\big((1+(1+a))b\big) = f\big(b + (1+a)b\big) \\ &= f(b) + f\big((1+a)b\big) = f(1)f(b) + f(1+a)f(b) \\ &= \big[f(1) + f(1+a)\big]f(b) = f(a)f(b). \end{aligned}$$

($\Rightarrow$) Assume that $R/I \cong \mathbb{Z}_2$ for an ideal I of R. Then $J(R) \subseteq I$, and $U(R) \subseteq 1 + I$ as $R = I \cup (1 + I)$. If $I = 0$, then $R = \mathbb{Z}_2$, so we are done. Hence, we can assume that $I \neq 0$.

Assume on the contrary that $J(R) \subsetneq I$. Then $U(R) \subsetneq 1 + I$. Define $f: R \to R$ by $f(x) = 2$ for $x \in I$, $f(1+x) = 1$ for $x \in I$ with $1 + x \in U(R)$, and $f(1+x) = 2$ for $x \in I$ with $1 + x \notin U(R)$. Then for $u, v \in U(R)$, $u = 1 + x$ and $v = 1 + y$ where $x, y \in I$, so we have

$$\begin{aligned} f(u+v) &= f(2+x+y) = 2 = 1+1 = f(1+x) + f(1+y) = f(u) + f(v), \\ f(uv) &= f(1+x+y+xy) = 1 = f(1+x)f(1+y) = f(u)f(v). \end{aligned}$$

That is, f is a unit-homomorphic map of R. As $U(R) \subsetneq 1 + I$, there exists $z \in I$ such that $1 + z \notin U(R)$. Thus, $f(1+z) = 2 \neq 1 + 2 = f(1) + f(z)$, so f is not additive. This contradiction shows that $I = J(R)$. It remains to show that $2 = 0$ in R. Note $R = J(R) \cup (1 + J(R))$. Define $f: R \to R$ by $f(x) = 2$ and $f(1+x) = 1$ for $x \in J(R)$.

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Then for $u, v \in J(R)$, $u = 1 + x$ and $v = 1 + y$ where $x, y \in J(R)$, so

$$f(u+v) = f(2+x+y) = 2 = 1+1 = f(u) + f(v),$$
$$f(uv) = f(1+x+y+xy) = 1 = f(u)f(v).$$

Hence, f is a unit-homomorphic map of R, so is an endomorphism. Thus,

$$1 = f(1) = f(1+0) = f(1) + f(0) = 1 + 2,$$

so $2 = 0$ follows. ■

Theorem 4.4 *Let R be a ring such that $R/J(R)$ is a direct product of exchange rings with primitive factors Artinian. Then every unit-homomorphic map of R is an endomorphism if and only if either R has no homomorphic images isomorphic to $\mathbb{Z}_2$ or $R/J(R) \cong \mathbb{Z}_2$ with $2 = 0$ in R.*

Proof (⇒) This follows from Theorem 4.3.

(⇐) Let $f: R \to R$ be a unit-homomorphic map. Then f is additive by Theorem 3.4. It remains to show that $f(ab) = f(a)f(b)$ for $a, b \in R$.

By Theorem 4.3, we can assume that R has no image isomorphic to $\mathbb{Z}_2$. Let $R/J(R)$ be the direct product of rings $\{R_\alpha\}$, where each R_α is an exchange ring with primitive factors Artinian. Then each R_α has no homomorphic images isomorphic to $\mathbb{Z}_2$, and hence it satisfies the 2-sum property by Lemma 3.2. It follows that $R/J(R)$, and hence R satisfies the 2-sum property. Write $a = u + v$ and $b = w + t$ where $u, v, w, t \in U(R)$. Then

$$\begin{aligned} f(ab) &= f(uw + ut + vw + vt) = f(uw) + f(ut) + f(vw) + f(vt) \\ &= f(u)f(w) + f(u)f(t) + f(v)f(w) + f(v)f(t) \\ &= f(u)\big[f(w) + f(t)\big] + f(v)\big[f(w) + f(t)\big] \\ &= \big[f(u) + f(v)\big]\big[f(w) + f(t)\big] = f(a)f(b). \end{aligned}$$

■

Corollary 4.5 *If R is a ring such that $R/J(R)$ is a direct product of exchange rings with primitive factors Artinian, then every unit-homomorphic map of $\mathbb{M}_n(R)$ is an endomorphism for all $n \geq 2$.*

Proof Write $R/J(R) = \prod R_\alpha$, where each R_α is an exchange ring with primitive factors Artinian, and let $S = \mathbb{M}_n(R)$. Then $S/J(S) \cong \mathbb{M}_n(R/J(R)) \cong \prod \mathbb{M}_n(R_\alpha)$, where each $\mathbb{M}_n(R_\alpha)$ is an exchange ring with primitive factors Artinian. As S has no homomorphic images isomorphic to $\mathbb{Z}_2$, every unit-homomorphic map of S is an endomorphism by Theorem 4.4. ■

Corollary 4.6 *If R is an exchange ring with primitive factors Artinian or a semilocal ring, then every unit-homomorphic map of $\mathbb{M}_n(R)$ is an endomorphism for all $n \geq 2$.*

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Department of Mathematics, Gebze Technical University, Gebze/Kocaeli, Turkey
e-mail: mtkosan@gtu.edu.tr ssahinkaya@gtu.edu.tr

Department of Mathematics and Statistics, Memorial University of Newfoundland, St.John's, NL A1C 5S7
e-mail: zhou@mun.ca