BULL. AUSTRAL. MATH. SOC. 47B47, 47L10, 47B49
VOL. 66 (2002) [477–486]

DERIVATIONS, LOCAL DERIVATIONS AND ATOMIC BOOLEAN SUBSPACE LATTICES

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Let $\mathcal{L}$ be an atomic Boolean subspace lattice on a Banach space X. In this paper, we prove that if $\mathcal{M}$ is an ideal of $\operatorname{Alg}\mathcal{L}$ then every derivation δ from $\operatorname{Alg}\mathcal{L}$ into $\mathcal{M}$ is necessarily quasi-spatial, that is, there exists a densely defined closed linear operator $T : \mathcal{D}(T) \subseteq X \to X$ with its domain $\mathcal{D}(T)$ invariant under every element of $\operatorname{Alg}\mathcal{L}$, such that $\delta(A)x = (TA - AT)x$ for every $A \in \operatorname{Alg}\mathcal{L}$ and every $x \in \mathcal{D}(T)$. Also, if $\mathcal{M} \subseteq \mathcal{B}(X)$ is an $\operatorname{Alg}\mathcal{L}$-module then it is shown that every local derivation from $\operatorname{Alg}\mathcal{L}$ into $\mathcal{M}$ is necessary a derivation. In particular, every local derivation from $\operatorname{Alg}\mathcal{L}$ into $\mathcal{B}(X)$ is a derivation and every local derivation from $\operatorname{Alg}\mathcal{L}$ into itself is a quasi-spatial derivation.

1. INTRODUCTION

Let $\mathcal{A}$ be an (associative) algebra, $\mathcal{M}$ be an $\mathcal{A}$-module, and $\delta : \mathcal{A} \to \mathcal{M}$ be a linear map. Recall that δ is a *derivation* if $\delta(ab) = \delta(a)b + a\delta(b)$ holds for all a and b in $\mathcal{A}$, and that δ is an *inner derivation* if there exists an element m in $\mathcal{M}$ such that $\delta(a) = ma - am$ holds for all a in $\mathcal{A}$. Also, δ is a *local derivation* (respectively, *local inner derivation*) if for each a in $\mathcal{A}$ there is a derivation (respectively, inner derivation) δ_a from $\mathcal{A}$ into $\mathcal{M}$ depending on a, such that $\delta(a) = \delta_a(a)$.

The concept of local derivations was first introduced by Kadison (see [**6**]) who proved that if $\mathcal{A}$ is a von Neumann algebra and $\mathcal{M}$ is a dual $\mathcal{A}$-module, then all norm-continuous local derivations from $\mathcal{A}$ into $\mathcal{M}$ are in fact derivations. In recent years, there has been a growing interest in the study of local derivations of operator algebras (see [**2, 3, 4, 5, 11**]). If X is a Banach space, as usual, we use $\mathcal{B}(X)$ to denote the algebra of all bounded linear operators on X. In [**11**], Larson and Sourour proved that all derivations and all local inner derivations from $\mathcal{B}(X)$ into itself are inner derivations (hence, every local derivation from $\mathcal{B}(X)$ into itself is a derivation). Jing in [**3**] generalised Larson—Sourour's results. He proved that if $\mathcal{A}$ is a reflexive operator algebra on a Banach space X such that both $0_+ \neq 0$ and $X_- \neq X$ in $\operatorname{Lat}\mathcal{A}$, then every derivation $\delta : \mathcal{A} \to \mathcal{A}$ is *spatial*, that is, there

Received 8th May, 2002
Supported by the National Natural Science Foundation of People's Republic China (Grant No. 19971039). The authors would like to thank the referee for many helpful comments.

https://doi.org/10.1017/S0004972700040314 Published online by Cambridge University Press

exists a $T \in \mathcal{B}(X)$ such that $\delta(A) = TA - AT$ for all $A \in \mathcal{A}$, and all norm-continuous local inner derivations from $\mathcal{A}$ into $\mathcal{A}$ are inner derivations when X is additionally reflexive. Moreover, in a recent paper (see [**4**]), he has shown that every local derivation from $\mathcal{A}$ into itself is a derivation.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be finite distributive subspace lattices on the Banach spaces X_1 and X_2, respectively. It was shown by Oreste Panaia in [**13**] that every rank-preserving algebraic isomorphism ϕ of $\operatorname{Alg}\mathcal{L}_1$ onto $\operatorname{Alg}\mathcal{L}_2$ is quasi-spatial, which means that there exists a closed, densely defined, injective linear transformation $T : \mathcal{D}(T) \subset X_1 \to X_2$ with dense range, and with its domain $\mathcal{D}(T)$ invariant under every element of $\operatorname{Alg}\mathcal{L}_1$, such that $\phi(A)Tx = TAx$, for every $x \in \mathcal{D}(T)$ and every $A \in \operatorname{Alg}\mathcal{L}_1$. Recently, Katavolos, Lambrou and Longstaff proved that every algebraic isomorphism between $\operatorname{Alg}\mathcal{P}_1$ and $\operatorname{Alg}\mathcal{P}_2$ is also quasi-spatial, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are two pentagon subspace lattices on Banach spaces (see [**7**]). In addition, we know from [**7, 13**] that the notion of quasi-spatiality of algebraic isomorphisms of operator algebras was introduced by Lambrou in [**9**], where it is proved that the algebraic isomorphisms from $\operatorname{Alg}\mathcal{L}_1$ onto $\operatorname{Alg}\mathcal{L}_2$ are quasi-spatial for any pair $\mathcal{L}_1$, $\mathcal{L}_2$ of atomic Boolean subspace lattices on Banach spaces, but we can not find this reference

Here, motivated by the results mentioned above, we consider the derivations and the local derivations of the reflexive operator algebras on Banach spaces with atomic Boolean invariant subspace lattices. For the statements of our results we give the following definition concerning quasi-spatiality of derivations of operator algebras, which seems to be known.

Definition 1.1: Let X be a Banach space, $\mathcal{A} \subseteq \mathcal{B}(X)$ be a subalgebra, and $\mathcal{M} \subseteq \mathcal{B}(X)$ be an $\mathcal{A}$-module. A derivation $\delta : \mathcal{A} \to \mathcal{M}$ is called quasi-spatial if there exists a densely defined closed linear operator $T : \mathcal{D}(T) \subseteq X \to X$ with its domain $\mathcal{D}(T)$ invariant under every element of $\mathcal{A}$, such that $\delta(A)x = (TA - AT)x$ holds for every $A \in \mathcal{A}$ and every $x \in \mathcal{D}(T)$.

Let $\mathcal{L}$ be an atomic Boolean subspace lattices on a Banach space X. We shall show that if $\mathcal{M}$ is an ideal of $\operatorname{Alg}\mathcal{L}$ then every derivation δ from $\operatorname{Alg}\mathcal{L}$ into $\mathcal{M}$ is necessarily quasi-spatial. Also, if $\mathcal{M} \subseteq \mathcal{B}(X)$ is an $\operatorname{Alg}\mathcal{L}$-module then it is shown that every local derivation from $\operatorname{Alg}\mathcal{L}$ into $\mathcal{M}$ is necessary a derivation. In particular, every local derivation from $\operatorname{Alg}\mathcal{L}$ into $\mathcal{B}(X)$ is a derivation and every local derivation from $\operatorname{Alg}\mathcal{L}$ into itself is a quasi-spatial derivation.

2. Notation and preliminaries

Throughout X will denote a fixed real or complex Banach space, with topological dual X^*. The terms *operator* acting on X and *subspace* of X will mean bounded linear map of X into itself and norm-closed linear manifold of X, respectively. For $T \in \mathcal{B}(X)$, denote by T^* the adjoint of T, by $G(T)$ the graph of T, that is $G(T) = \{(x, Tx) : x \in X\}$, and by I the identity operator on X. For $x \in X$ and $f^* \in X^*$, the operator $x \otimes f^*$ is defined by $y \mapsto f^*(y)x$ for $y \in X$. This operator has rank one if and only if both x and f^* are nonzero. Note that the adjoint of $x \otimes f^*$ is the operator $f^* \otimes \widehat{x}$ which is given by $g^* \mapsto g^*(x)f^*$ for $g^* \in X^*$, where $\widehat{x}$ is the image of x under the canonical map of X into X^{**} (the second dual of X). For any non-empty subset $L \subseteq X$, $L^\perp$ denotes its annihilator, that is, $L^\perp = \{f^* \in X^* : f^*(x) = 0 \text{ for all } x \in L\}$; and dually, for any non-empty subset $F \subseteq X^*$, $F_\perp$ denotes its pre-annihilator, that is, $F_\perp = \{x \in X : f^*(x) = 0 \text{ for all } f^* \in F\}$. For every family $\{L_\gamma\}_{\gamma\in\Gamma}$ of subspaces of X, we have $\left(\bigvee_{\gamma\in\Gamma} L_\gamma\right)^\perp = \bigcap_{\gamma\in\Gamma} L_\gamma^\perp$ and $\bigvee_{\gamma\in\Gamma} L_\gamma^\perp \subseteq \left(\bigcap_{\gamma\in\Gamma} L_\gamma\right)^\perp$; indeed, it is easy to verify that $\left(\bigcap_{\gamma\in\Gamma} L_\gamma\right)^\perp$ is the weak star closure of $\bigvee_{\gamma\in\Gamma} L_\gamma^\perp$. Here '$\vee$' and '$\cap$' denote 'norm-closed linear span' and 'set theoretic intersection'.

If $\mathcal{L}$ is a family of subspaces of X, we say that $\mathcal{L}$ is a *subspace lattice on* X if it contains (0) and X, and is closed under the operations $\vee$ and $\cap$, that is, for any family $\{L_\gamma\}_{\gamma\in\Gamma}$ of elements of $\mathcal{L}$, $\bigvee_{\gamma\in\Gamma} L_\gamma \in \mathcal{L}$ and $\bigcap_{\gamma\in\Gamma} L_\gamma \in \mathcal{L}$. For any family $\mathcal{F}$ of subspaces of X, let $\operatorname{Alg}\mathcal{F}$ denote the algebra of all operators on X which leave every subspace in $\mathcal{F}$ invariant. Dually, for any family $\mathcal{A}$ of operators on X, let $\operatorname{Lat}\mathcal{A}$ denote the set of all subspaces of X which are invariant under every operator in $\mathcal{A}$. It is clear that $\operatorname{Alg}\mathcal{F}$ is a unital weakly closed operator algebra and $\operatorname{Lat}\mathcal{A}$ is a subspace lattice. We say that a subspace lattice $\mathcal{L}$ is *reflexive* if $\mathcal{L} = \operatorname{Lat}\operatorname{Alg}\mathcal{L}$, and an operator algebra $\mathcal{A}$ is *reflexive* if $\mathcal{A} = \operatorname{Alg}\operatorname{Lat}\mathcal{A}$.

A subspace lattice $\mathcal{L}$ is called *complemented* if for every $L \in \mathcal{L}$ there is an element $L' \in \mathcal{L}$, a *lattice complement* of L, such that $L \vee L' = X$ and $L \cap L' = (0)$, and *distributive* if the identity $L \cap (M \vee N) = (L \cap M) \vee (L \cap N)$ and its dual hold for all $L, M, N \in \mathcal{L}$. A nonzero complemented and distributive subspace lattice is called a *Boolean subspace lattice.* An nonzero element K in a subspace lattice $\mathcal{L}$ is called an *atom* if, whenever $L \in \mathcal{L}$ such that $(0) \subseteq L \subseteq K$, then either $L = (0)$ or $L = K$. A subspace lattice $\mathcal{L}$ is called *atomic* if each element of $\mathcal{L}$ is the closed linear span of the atoms it contains. It is well known that an atomic Boolean subspace lattice is completely distributive (see [**8**, **12**]). From [**12**] we know that a subspace lattice is completely distributive if and only if it is strongly reflexive. For the standard definition concerning completely distributive subspace lattices and the alternative characterisations see [**8**, **12**]. From these two references we can find the following lemma which is crucial to this paper.

LEMMA 2.1. *If $\mathcal{L}$ is a subspace lattice on X, then the rank one operator $x \otimes f^* \in \operatorname{Alg}\mathcal{L}$ if and only if there exists a $L \in \mathcal{L}$ such that $x \in L$, and $f^* \in L_-^\perp$, where $L_- = \bigvee\{M \in \mathcal{L} : M \not\supseteq L\}$ and $L_-^\perp$ means $(L_-)^\perp$.*

In the above lemma, if $\mathcal{L}$ is an atomic Boolean subspace lattice, then the subspace L can be taken to be an atom of $\mathcal{L}$ since a nonzero $K \in \mathcal{L}$ is an atom if and only if $K_- \neq X$ (see [**12**]), in which case L_- is the same as L' (see [**12**]), the (unique) lattice complement of L. It is worth noting that the equations $(0)_+ = (0)$ and $X_- = X$ are always true in an atomic Boolean subspace lattice $\mathcal{L}$, so the associated reflexive algebra $\operatorname{Alg}\mathcal{L}$ is not in the class of reflexive algebras introduced in [3, 4]. Here $(0)_+ = \bigcap\{L \in \mathcal{L} : L \neq (0)\}$.

Finally, for an atomic Boolean subspace lattice $\mathcal{L}$ on X, $\mathcal{L}_A$ will denote the set of atoms of $\mathcal{L}$, $\mathcal{L}_A^\perp$ the set $\{K_-^\perp : K \in \mathcal{L}_A\}$ and $\mathcal{R}_\mathcal{L}$ the set of rank one operators in $\operatorname{Alg}\mathcal{L}$. Also, let $\langle\mathcal{L}_A\rangle$ and $\langle\mathcal{L}_A^\perp\rangle$ be the (not necessarily closed) linear span of $\mathcal{L}_A$ and $\mathcal{L}_A^\perp$, respectively.

3. Quasi-spatiality of derivations

Throughout this section, $\mathcal{L}$ will be an atomic Boolean subspace lattice on the Banach space X, $\mathcal{M}$ be an ideal of $\operatorname{Alg}\mathcal{L}$, and $\delta : \operatorname{Alg}\mathcal{L} \to \mathcal{M}$ be a derivation. The main result of this section shows that δ is quasi-spatial.

Lemma 3.1. *For every $K \in \mathcal{L}_A$, there exist two linear maps $T_K : K \to K$ and $S_K : K_-^\perp \to K_-^\perp$, such that*

(i) $\delta(A)x = (T_K A - AT_K)x$, *for $A \in \operatorname{Alg}\mathcal{L}$ and $x \in K$;*

(ii) $\delta(A)^* f^* = (S_K A^* - A^* S_K) f^*$, *for $A \in \operatorname{Alg}\mathcal{L}$ and $f^* \in K_-^\perp$;*

(iii) $\delta(x \otimes f^*) = T_K x \otimes f^* + x \otimes S_K f^*$, *for $x \in K$ and $f^* \in K_-^\perp$.*

Proof: Since $K \cap K_- = (0)$, we can choose fixed nonzero elements $x_K \in K$, $f_K^* \in K_-^\perp$ with $f_K^*(x_K) = 1$.

(i) By Lemma 2.1, $x \otimes f_K^* \in \operatorname{Alg}\mathcal{L}$ for any $x \in K$. Then define a map T_K by

$$T_K x = \delta(x \otimes f_K^*) x_K, \quad x \in K.$$

Clearly, T_K is linear. Since $\mathcal{M}$ is an ideal of $\operatorname{Alg}\mathcal{L}$ and $x_K \in K$, we have $T_K K \subseteq K$. For $A \in \operatorname{Alg}\mathcal{L}$ and $x \in K$, then $Ax \in K$ and

$$\delta(Ax \otimes f_K^*) = \delta(A \cdot x \otimes f_K^*) = \delta(A) \cdot x \otimes f_K^* + A \cdot \delta(x \otimes f_K^*).$$

By letting the two sides of the above equation act on x_K, we obtain $\delta(A)x = (T_K A - AT_K)x$.

(ii) The proof is a dual version of (i). For all $f^* \in K_-^\perp$ we have $x_K \otimes f^* \in \operatorname{Alg}\mathcal{L}$. Define a linear map S_K by

$$S_K f^* = \delta(x_K \otimes f^*)^* f_K^*, \quad f^* \in K_-^\perp.$$

That $S_K K_-^\perp \subseteq K_-^\perp$ follows from the fact that $A^* K_-^\perp \subseteq K_-^\perp$ holds for all $A \in \operatorname{Alg}\mathcal{L}$. For $A \in \operatorname{Alg}\mathcal{L}$ and $f^* \in K_-^\perp$, we have $\delta(x_K \otimes A^* f^*) = \delta(x_K \otimes f^* \cdot A) = \delta(x_K \otimes f^*)A + x_K \otimes f^* \cdot \delta(A)$. Taking the adjoint for each operator in this equation, then

$$\delta(x_K \otimes A^* f^*)^* = A^* \delta(x_K \otimes f^*)^* + \delta(A)^*(f^* \otimes \widehat{x_K}).$$

By applying the above equation to f_K^*, we obtain $\delta(A)^* f^* = (S_K A^* - A^* S_K) f^*$.

(iii) For $x \in K$ and $f^* \in K_-^\perp$, we have by the definitions of T_K and S_K

$$\begin{aligned}\delta(x \otimes f^*) &= \delta(x \otimes f_K^* \cdot x_K \otimes f^*)\\ &= \delta(x \otimes f_K^*) \cdot x_K \otimes f^* + x \otimes f_K^* \cdot \delta(x_K \otimes f^*)\\ &= \delta(x \otimes f_K^*) x_K \otimes f^* + x \otimes \delta(x_K \otimes f^*)^* f_K^*\\ &= T_K x \otimes f^* + x \otimes S_K f^*.\end{aligned}$$

This completes the proof. ∎

For the remainder of this section, for any given $K \in \mathcal{L}_A$, T_K and S_K will denote the linear maps as constructed in Lemma 3.1. Obviously, they depend on the choices made for x_K and f_K^*, so it will be assumed that those choices have been made for each $K \in \mathcal{L}_A$.

Suppose that $K_1, \cdots, K_n$ are distinct atoms of $\mathcal{L}$, and $x_i \in K_i$, $f_i^* \in K_{i-}^\perp$ for $i = 1, \cdots, n$. Then, from $\sum_{i=1}^n x_i = 0$ and $\sum_{i=1}^n f_i^* = 0$ we can obtain every x_i and every f_i^* is zero, respectively. For example, let $\sum_{i=1}^n f_i^* = 0$. For any f_i^*, we have by complete distributivity and De Morgans' laws (see [**1**, p. 5])

$$\begin{aligned}f_i^* &\in K_{i-}^\perp \cap \Big(\bigvee_{j \neq i} K_{j-}^\perp\Big) \subseteq K_{i-}^\perp \cap \Big(\bigcap_{j \neq i} K_{j-}\Big)^\perp = \Big(K_{i-} \vee \Big(\bigcap_{j \neq i} K_{j-}\Big)\Big)^\perp\\ &= \Big(K_i' \vee \Big(\bigcap_{j \neq i} K_j'\Big)\Big)^\perp = \Big(K_i' \vee \Big(\bigvee_{j \neq i} K_j\Big)'\Big)^\perp = \Big(\Big(K_i \cap \Big(\bigvee_{j \neq i} K_j\Big)\Big)'\Big)^\perp\\ &= \Big(\Big(\bigvee_{j \neq i} (K_i \cap K_j)\Big)'\Big)^\perp = ((0)')^\perp = X^\perp = (0),\end{aligned}$$

and so $f_i^* = 0$, as desired. Thus every $x \in \langle \mathcal{L}_A \rangle$ has a representation as follows: $x = \sum_{i=1}^n x_i$ with $x_i \in K_i$, $1 \leqslant i \leqslant n$, where $K_1, \cdots, K_n$ are distinct atoms of $\mathcal{L}$. If x is nonzero and each x_i is required to be nonzero, this representation is unique up to permutations of the atoms. Similar statements can apply to the elements of $\langle \mathcal{L}_A^\perp \rangle$. Therefore, the linear maps T_0 and S_0 in the following definition are well-defined.

Definition 3.1: Define $T_0 : \langle \mathcal{L}_A \rangle \to \langle \mathcal{L}_A \rangle$ by $T_0 x = \sum_{i=1}^m T_{K_i} x_i$, where $x = \sum_{i=1}^m x_i$ with $x_i \in K_i$, $1 \leqslant i \leqslant m$, and $K_1, \cdots, K_m$ being distinct atoms of $\mathcal{L}$; and define $S_0 : \langle \mathcal{L}_A^\perp \rangle \to \langle \mathcal{L}_A^\perp \rangle$ by $S_0 f^* = \sum_{j=1}^n S_{L_j} f_j^*$, where $f^* = \sum_{j=1}^n f_j^*$ with $f_j^* \in L_{j-}^\perp$, $1 \leqslant j \leqslant n$, and $L_1, \cdots, L_n$ being distinct atoms of $\mathcal{L}$.

Lemma 3.2. *For $A \in \operatorname{Alg} \mathcal{L}$ and $x \in \langle \mathcal{L}_A \rangle$, we have $\delta(A)x = (T_0 A - A T_0)x$.*

Proof: It is a routine computation by the definition of T_0 and Lemma 3.1 (i). ∎

Lemma 3.3. *For $x \in \langle \mathcal{L}_A \rangle$ and $f^* \in \langle \mathcal{L}_A^\perp \rangle$, we have $f^*(T_0 x) + (S_0 f^*)(x) = 0$.*

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PROOF: We first prove that $f^*(T_0x)+(S_0f^*)(x)=0$ holds for $x \in K$ and $f^* \in K_-^{\perp}$, where $K \in \mathcal{L}_A$ is arbitrary. For, assume that x and f^* are both nonzero. By Lemma 3.1 (iii) we have

$$\delta\big((x\otimes f^*)^2\big) = f^*(x)\delta(x\otimes f^*) = f^*(x)T_0x\otimes f^* + f^*(x)x\otimes S_0f^*;$$

on the other hand,

$$\begin{aligned}\delta\big((x\otimes f^*)^2\big) &= \delta(x\otimes f^*)\cdot x\otimes f^* + x\otimes f^*\cdot\delta(x\otimes f^*)\\ &= f^*(x)T_0x\otimes f^* + (S_0f^*)(x)x\otimes f^* + f^*(T_0x)x\otimes f^* + f^*(x)x\otimes S_0f^*.\end{aligned}$$

Hence $f^*(T_0x)+(S_0f^*)(x)=0$.

For the general case, combine what has just been shown with the fact that if $x \in K$ and $f^* \in L_-^{\perp}$, where K, L are distinct atoms of $\mathcal{L}$, then $f^*(T_Kx)=0$ and $(S_Lf^*)(x)=0$ since $K \subseteq L_-$. A direct computation gives the required result, and we are done. ∎

LEMMA 3.4. *There exists a linear map $T : \mathcal{D}(T) \subseteq X \to X$ which is the extension of T_0, such that the norm-closure $\overline{G(T_0)}$ of the graph of T_0 is the graph $G(T)$ of T and $\overline{\mathcal{D}(T)} = X$.*

PROOF: To define T we let $\mathcal{D} = \big\{x \in X : (x,y) \in \overline{G(T_0)}, \text{ for some } y \in X\big\}$. Then $\mathcal{D}$ is obviously a linear manifold. Since $\mathcal{L}$ is an atomic Boolean subspace lattice, $\langle\mathcal{L}_A\rangle$ is dense in X. Hence, since $\langle\mathcal{L}_A\rangle \subseteq \mathcal{D}$, $\mathcal{D}$ is dense in X.

For any $x \in \mathcal{D}$, we shall show that there exists a unique $y \in X$, such that $(x,y) \in \overline{G(T_0)}$. For, assume that (x,y_1), $(x,y_2) \in \overline{G(T_0)}$ with y_1, $y_2 \in X$. We then have $(0, y_1 - y_2) \in \overline{G(T_0)}$. Thus, there is a sequence $\{x_n\}_1^\infty$ of elements of $\langle\mathcal{L}_A\rangle$, such that $x_n \to 0$ and $T_0x_n \to y_1 - y_2$. For any $f^* \in \langle\mathcal{L}_A^{\perp}\rangle$, it follows that $f^*(T_0x_n) \to f^*(y_1-y_2)$. On the other hand, from Lemma 3.3 we have $f^*(T_0x_n)+(S_0f^*)(x_n)=0$ for all n. Thus $f^*(T_0x_n)\to 0$ since $(S_0f^*)(x_n)\to 0$, and so $f^*(y_1-y_2)=0$. Thus $y_1=y_2$ once the fact that $\langle\mathcal{L}_A^{\perp}\rangle$ is weak star dense in X^* is proved to be true. Indeed, from [**8, 12**] we know that

$$\begin{aligned}(0) &= \bigcap\{K_- : K \in \mathcal{L} \text{ and } K \neq (0)\}\\ &= \bigcap\{K_- : K \in \mathcal{L},\ K \neq (0) \text{ and } K_- \neq X\}\\ &= \bigcap\{K_- : K \in \mathcal{L} \text{ an atom}\}.\end{aligned}$$

As remarked in the first paragraph of Section 2, the annihilator of the last expression above is the weak star closure of $\langle\mathcal{L}_A^{\perp}\rangle$, as desired.

Up till this point we can define a map $T : \mathcal{D}(T) \subseteq X \to X$ in an obvious way, such that $G(T) = \overline{G(T_0)}$, where $\mathcal{D}(T) = \mathcal{D}$. Clearly, T is linear and extends T_0. The proof is complete. ∎

Now we are in a position to prove the main result of this section.

Theorem 3.1. *Let $\mathcal{L}$ be an atomic Boolean subspace lattice on X, and $\mathcal{M}$ be an ideal of* $\operatorname{Alg}\mathcal{L}$. *Then every derivation δ from* $\operatorname{Alg}\mathcal{L}$ *into $\mathcal{M}$ is necessarily quasi-spatial.*

Proof: Let T be as in Lemma 3.4. Obviously, T is closed and densely defined. It remains to show that $\mathcal{D}(T)$ is invariant under every element of $\operatorname{Alg}\mathcal{L}$ and $\delta(A)x = (TA - AT)x$, for every $A \in \operatorname{Alg}\mathcal{L}$ and every $x \in \mathcal{D}(T)$.

Let $A \in \operatorname{Alg}\mathcal{L}$ and $x \in \mathcal{D}(T)$. Then $(x, Tx) \in \overline{G(T_0)}$ by Lemma 3.4. Thus there exists a sequence $\{x_n\}_1^\infty$ of elements of $\langle \mathcal{L}_A \rangle$, such that $x_n \to x$ and $T_0x_n \to Tx$. Moreover, we have that $Ax_n \to Ax$, $AT_0x_n \to ATx$ and $\delta(A)x_n \to \delta(A)x$. From Lemma 3.2 it follows that $\delta(A)x_n = (T_0A - AT_0)x_n$ for every n, and so $T_0Ax_n \to \delta(A)x + ATx$. We therefore obtain that $(Ax_n, T_0Ax_n) \to (Ax, \delta(A)x + ATx)$. Since $(Ax_n, T_0Ax_n) \in G(T_0)$, $(Ax, \delta(A)x + ATx) \in G(T)$. Hence $Ax \in \mathcal{D}(T)$ which means $A\mathcal{D}(T) \subseteq \mathcal{D}(T)$, and $TAx = \delta(A)x + ATx$, that is $\delta(A)x = (TA - AT)x$. This completes the proof. ∎

Remark 3.1. In this section, the condition that $\mathcal{M}$ is an ideal of $\operatorname{Alg}\mathcal{L}$ is mainly used to guarantee the validity of Lemma 3.3. In addition, $\mathcal{M}$ may be equal to $\operatorname{Alg}\mathcal{L}$.

Remark 3.2. Note that T (as in Lemma 3.4) need not be injective and have dense range. For example, if $\delta : \operatorname{Alg}\mathcal{L} \to \mathcal{M}$ is the zero map, then $\mathcal{D}(T) = X$ and $T = 0$.

4. Local derivations are derivations

In this section, the letter $\mathcal{L}$ still denotes an atomic Boolean subspace lattice on the Banach space X. But, $\mathcal{M} \subseteq \mathcal{B}(X)$ denotes an $\operatorname{Alg}\mathcal{L}$-module, and δ a local derivation from $\operatorname{Alg}\mathcal{L}$ into $\mathcal{M}$. The purpose of this section is to prove that δ is in fact a derivation. Our proof follows by a modification of the arguments of Theorem 3.3 in [4]. Since, as remarked in Section 2, reflexive algebras with atomic Boolean invariant subspace lattices are strictly different from those reflexive algebras considered in [4], the proof is included here.

Let us begin with some lemmas. The first can be found in [4] which is the key to our results.

Lemma 4.1. *Let δ be a local derivation from a Banach algebra $\mathcal{A}$ into an $\mathcal{A}$-module $\mathcal{U}$. Then $\delta(PAQ) = \delta(PA)Q + P\delta(AQ) - P\delta(A)Q$ holds for every $A \in \mathcal{A}$ and every pair P, Q of idempotents in $\mathcal{A}$.*

Lemma 4.2. *For all R_1, R_2 in $\mathcal{R}_\mathcal{L}$ and all A in* $\operatorname{Alg}\mathcal{L}$, *we have*

$$(*) \qquad \delta(R_1AR_2) = \delta(R_1A)R_2 + R_1\delta(AR_2) - R_1\delta(A)R_2.$$

Proof: By Lemma 2.1, write $R_1 = x \otimes f^*$ with $x \in K$, $f^* \in K_-^\perp$, and $R_2 = y \otimes g^*$ with $y \in L$, $g^* \in L_-^\perp$, where K and L are two atoms of $\mathcal{L}$. It suffices to give the proof for the following three cases.

(1) Suppose $f^*(x) \neq 0$ and $g^*(y) \neq 0$. Let $R_1' = R_1/f^*(x)$ and $R_2' = R_2/g^*(y)$. Then both R_1' and R_2' are rank one idempotents, and hence we have $\delta(R_1'AR_2')$

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$= \delta(R_1'A)R_2' + R_1'\delta(AR_2') - R_1'\delta(A)R_2'$ by Lemma 4.1. It follows from the linearity of δ that the equation $(*)$ holds.

(2) Suppose that precisely one of $f^*(x)$ and $g^*(y)$ is zero. Without loss of generality, assume that $f^*(x) = 0$ and $g^*(y) \neq 0$. Since $K \cap K_- = (0)$, $x \notin K_-$. So there exists a $f_1^* \in K_-^\perp$ such that $f_1^*(x) \neq 0$. Then $x \otimes f_1^*$, $x \otimes (f^* + f_1^*) \in \mathcal{R}_\mathcal{L}$. Thus we have by (1)

$$\begin{aligned}
\delta(R_1AR_2) &= \delta(x \otimes (f^* + f_1^*) \cdot A \cdot y \otimes g^*) - \delta(x \otimes f_1^* \cdot A \cdot y \otimes g^*) \\
&= \delta(x \otimes (f^* + f_1^*) \cdot A) \cdot y \otimes g^* + x \otimes (f^* + f_1^*) \cdot \delta(A \cdot y \otimes g^*) \\
&\quad - x \otimes (f^* + f_1^*) \cdot \delta(A) \cdot y \otimes g^* - \delta(x \otimes f_1^* \cdot A) \cdot y \otimes g^* \\
&\quad - x \otimes f_1^* \cdot \delta(A \cdot y \otimes g^*) + x \otimes f_1^* \cdot \delta(A) \cdot y \otimes g^* \\
&= \delta(x \otimes f^* \cdot A) \cdot y \otimes g^* + x \otimes f^* \cdot \delta(A \cdot y \otimes g^*) - x \otimes f^* \cdot \delta(A) \cdot y \otimes g^* \\
&= \delta(R_1A)R_2 + R_1\delta(AR_2) - R_1\delta(A)R_2,
\end{aligned}$$

as desired.

(3) Suppose that both $f^*(x)$ and $g^*(y)$ are zero. Since $K \cap K_- = L \cap L_- = (0)$, $x \notin K_-$ and $y \notin L_-$. Then there exist $f_1^* \in K_-^\perp$ and $g_1^* \in L_-^\perp$ such that $f_1^*(x) \neq 0$ and $g_1^*(y) \neq 0$. Thus $x \otimes f_1^*$, $x \otimes (f^* + f_1^*)$, $y \otimes g_1^*$ and $y \otimes (g^* + g_1^*) \in \mathcal{R}_\mathcal{L}$. Clearly,

$$\delta(R_1AR_2) = \delta\big(x \otimes (f^* + f_1^*) \cdot A \cdot y \otimes (g^* + g_1^*)\big) - \delta\big(x \otimes f_1^* \cdot A \cdot y \otimes (g^* + g_1^*)\big) - \delta(x \otimes f^* \cdot A \cdot y \otimes g_1^*).$$

By using (1) and (2), a routine computation similar to that which appeared in (2) implies the validity of the equation $(*)$. This concludes the proof. ∎

Corollary 4.1. *For all R_1 and R_2 in $\mathcal{R}_\mathcal{L}$, we have $\delta(R_1R_2) = \delta(R_1)R_2 + R_1\delta(R_2)$.*

Proof: Since δ is a local derivation, $\delta(I) = 0$. Taking $A = I$ in the equation $(*)$, the desired result immediately follows and the proof is complete. ∎

Lemma 4.3: ([10, Lemma 2.3].) *Let $A \in \mathcal{B}(X)$. Then*

(i) *if $RA = 0$ for every $R \in \mathcal{R}_\mathcal{L}$ then $A = 0$;*
(ii) *if $AR = 0$ for every $R \in \mathcal{R}_L$ then $A = 0$.*

Lemma 4.4. *For every $R \in \mathcal{R}_\mathcal{L}$ and every $A \in \operatorname{Alg}\mathcal{L}$, we have $\delta(AR) = \delta(A)R + A\delta(R)$.*

Proof: Let $R_1 \in \mathcal{R}_\mathcal{L}$ be arbitrary. By Lemma 4.2 we have

$$\delta(R_1AR) = \delta(R_1A)R + R_1\delta(AR) - R_1\delta(A)R.$$

On the other hand, noting that $R_1A \in \mathcal{R}_\mathcal{L}$ if it is nonzero, then by Corollary 4.1

$$\delta(R_1AR) = \delta(R_1A)R + R_1A\delta(R).$$

Equating these two equations we obtain $R_1\delta(AR) = R_1\delta(A)R + R_1A\delta(R)$. It follows from Lemma 4.3 (i) that $\delta(AR) = \delta(A)R + A\delta(R)$. This completes the proof. ∎

It is the time to prove the main result of this section.

THEOREM 4.1. *Let $\mathcal{L}$ be an atomic Boolean subspace lattice on X, and $\mathcal{M} \subseteq \mathcal{B}(X)$ be an $\operatorname{Alg}\mathcal{L}$-module. Then every local derivation δ from $\operatorname{Alg}\mathcal{L}$ into $\mathcal{M}$ is necessary a derivation.*

PROOF: Let A, $B \in \operatorname{Alg}\mathcal{L}$ be arbitrary. For any $R \in \mathcal{R}_{\mathcal{L}}$, we have by Lemma 4.4

$$\begin{aligned}\delta(AB)R + AB\delta(R) &= \delta(ABR) = \delta(A \cdot BR)\\ &= \delta(A)BR + A\delta(BR)\\ &= \delta(A)BR + A\delta(B)R + AB\delta(R).\end{aligned}$$

Therefore $\delta(AB)R = \delta(A)BR + A\delta(B)R$. It follows from Lemma 4.3 (ii) that $\delta(AB) = \delta(A)B + A\delta(B)$. This shows that δ is in fact a derivation, and the proof is complete. ∎

In Theorem 4.1 letting $\mathcal{M} = \mathcal{B}(X)$, then

COROLLARY 4.2. *Let $\mathcal{L}$ be an atomic Boolean subspace lattice on X. Then every local derivation from $\operatorname{Alg}\mathcal{L}$ into $\mathcal{B}(X)$ is a derivation.*

From Theorem 3.1 and Theorem 4.1 we can obtain

COROLLARY 4.3. *Let $\mathcal{L}$ be an atomic Boolean subspace lattice on X and $\mathcal{M}$ be an ideal of $\operatorname{Alg}\mathcal{L}$. Then every local derivation from $\operatorname{Alg}\mathcal{L}$ into $\mathcal{M}$ is a quasi-spatial derivation.*

In particular, we have

COROLLARY 4.4. *Let $\mathcal{L}$ be an atomic Boolean subspace lattice on X. Then every local derivation from $\operatorname{Alg}\mathcal{L}$ into itself is a quasi-spatial derivation.*

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