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PAPER

On degenerate reaction-diffusion epidemic models with mass action or standard incidence mechanism

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Abstract
In this paper, we consider reaction-diffusion epidemic models with mass action or standard incidence mechanism and study the impact of limiting population movement on disease transmissions. We set either the dispersal rate of the susceptible or infected people to zero and study the corresponding degenerate reaction-diffusion model. Our main approach to study the global dynamics of these models is to construct delicate Lyapunov functions. Our results show that the consequences of limiting the movement of susceptible or infected people depend on transmission mechanisms, model parameters and population size.

1. Introduction

Various differential equation epidemic models have been proposed to study the spread of infectious diseases [4, 6, 7, 14, 41], and it has been recognised that population mobility [5, 52, 55] and the spatial heterogeneity of the environment [19, 32] are key factors in disease transmissions. In order to address these issues, many reaction-diffusion epidemic models with non-constant coefficients have been proposed and studied [2, 17, 58, 59]. Specific infectious diseases modelled by diffusive models include malaria [38, 39], rabies [23, 42], dengue fever [53], West Nile virus [27, 31], influenza [40], COVID-19 [18, 25, 51, 57], etc.

In this paper, we consider the following susceptible-infected-susceptible (SIS) diffusive epidemic model, which is a natural extension of the classic ordinary differential equation epidemic model by Kermack and McKendrick [24]:

$$\begin{cases} \partial_t S = d_S \Delta S - f(x,S,I) + \gamma(x) I, & x\in\Omega, t>0,\\ \partial_t I = d_I \Delta I + f(x,S,I) - \gamma(x) I, & x\in\Omega, t>0,\\ \partial_\nu S = \partial_\nu I = 0, & x\in\partial\Omega, t>0,\\ S(x,0)=S_0(x),\ I(x,0)=I_0(x), & x\in\Omega. \end{cases} \tag{1.1}$$

Here, the individuals are assumed to live in a bounded domain $\Omega\subset\mathbb{R}^n$ with smooth boundary $\partial\Omega$; $S(x,t)$ and $I(x,t)$ are the density of susceptible and infected individuals at position $x\in\Omega$ and time t, respectively; d_S and d_I are the movement rates of susceptible and recovered individuals, respectively; γ is the disease recovery rate; $f(x,S,I)$ describes the interaction of susceptible and infected people; ν is the unit outward normal of $\partial\Omega$ and the homogeneous Neumann boundary conditions mean that the individuals cannot cross the boundary.

In the pioneering work by Allen *et al.* [2], model (1.1) with standard incidence mechanism, $f(x, S, I) = \beta(x)SI/(S+I)$, has been proposed and studied (β is called the disease recovery rate). In [2], the authors define a basic reproduction number $\mathcal{R}_0$ and show that the model has a unique endemic equilibrium (EE) (i.e. positive equilibrium) if $\mathcal{R}_0 > 1$. Most importantly, they show that the disease component of the EE approaches zero as d_S approaches zero if $\beta - \gamma$ changes sign. Biologically, assuming that the population eventually stabilises at the EE, this result indicates that the disease can be eliminated by controlling the mobility of susceptible individuals if there are places that are of low risk (i.e. $\beta(x) < \gamma(x)$). In contrast, if the dispersal rate of infected people is limited, [45] shows that the disease cannot be completely eliminated. In [13, 62], the authors considered model (1.1) with mass action mechanism, $f(x, S, I) = \beta(x)SI$, which is algebraically simpler but mathematically more challenging. For this model, assuming again that the population eventually stabilises at EE solutions, it has been shown that lowering the movement of susceptible people can eliminate the disease only when the size of the population is below some critical number, solely determined by γ/β [9, 62], while infected individuals may concentrate on certain hot spots when limiting their movement [9, 47, 62]. It is important to note that stability of the EE solutions for models with standard incidence or mass action mechanism is only known in a few cases: either the population movement rate is uniform (i.e. $d_S = d_I$) or the ratio β/γ is constant [13, 46]. For more related works, we refer the interested readers to [10–12, 22, 26, 28–30, 34, 35, 46, 48–50, 54, 56, 60] and the references therein.

For the corresponding ordinary differential equation epidemic model of (1.1):

$$\begin{cases} S' = -f(S, I) + \gamma I, \\ I' = f(S, I) - \gamma I, \end{cases}$$

the global dynamics is determined by the basic reproduction number, which can be interpreted as the average number of infected individuals generated by one infectious individual in an otherwise susceptible population: if it is greater than one, the solution converges to an EE and the disease persists; if it is less than one, the solution converges to a disease free equilibrium and the infected individuals go to extinction. Here, the basic reproduction number is $\mathcal{R}_0^1 = N\beta/\gamma$ if $f(S, I) = \beta SI$ and $\mathcal{R}_0^2 = \beta/\gamma$ if $f(S, I) = \beta SI/(S+I)$.

In this paper, we revisit model (1.1) and study the impact of limiting population movement on disease transmissions. Different from most of the aforementioned studies, we will work on the global dynamics of the time-dependent model (1.1) with either $d_S = 0$ or $d_I = 0$ rather than consider the asymptotic profiles of the EE as $d_S \to 0$ or $d_I \to 0$. Intuitively, if the disease evolves at a faster time scale than the control of population movement, and the population eventually stabilises at an EE, then the asymptotic profiles of the EE solutions may reflect the effect of the control strategy. However, if the control of population movement happens in a faster time scale and the solution of model (1.1) converges to that of the corresponding degenerate system as $d_S \to 0$ or $d_I \to 0$, then the global dynamics of the degenerate system will better tell the impact of the control strategies.

There are several recent efforts on degenerate reaction-diffusion population models (see [15, 16, 33, 36, 37, 43, 63] and the references cited therein). In particular, the two works [9, 33] have partial results on model (1.1) with $d_S = 0$, which was interpreted as the situation of a total lock down for the susceptible population. Note that when either $d_S = 0$ or $d_I = 0$, system (1.1) becomes degenerate and hence there is a lack of some compactness of the solution operator, which induces many challenges in the study of the large time behaviour of the solutions.

Our main approach to study the global dynamics of the degenerate system (1.1) (i.e. $d_S = 0$ or $d_I = 0$) is to construct Lyapunov functions (except for the case $d_S = 0$ in Section 3.1) and combine these with delicate analysis. In particular, in Section 3.2, we present a Lyapunov function that one may easily draw a false conclusion using it and the convergence result based on it is very quite unusual (see Remark 3.8 and Figure 2). In the proof of Theorem 4.6, we construct a Lyapunov function V that does not satisfy $\dot{V} \leq 0$ (see Eq. 4.25), but we are still able to conclude the convergence of the solution.

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The rest of our paper is organised as follows. In Section 2, we list the assumptions and terminology and present some useful results. In Section 3, we consider the model with mass action mechanism with $d_S=0$ or $d_I=0$. In Section 4, we consider the model with standard incidence mechanism. In Section 5, we run some numerical simulations to illustrate the results. In Section 6, we compare the results from the two different mechanisms and control strategies and discuss the implications for disease control.

2. Preliminaries

Throughout the paper, we make the following assumptions on the parameters:

(A1) The functions β, γ are positive and Hölder continuous on $\bar{\Omega}$;

(A2) The functions S_0, I_0 are nonnegative and continuous on $\bar{\Omega}$ with $I_0 \not\equiv 0$. Moreover, $\int_\Omega (S_0+I_0)dx = N$ for some fixed positive constant N.

Integrating the first two equations of model (1.1) over Ω and summing up them, we find that

$$\frac{d}{dt}\int_\Omega (S(x,t)+I(x,t))dx = 0,$$

which means the total population $\int_\Omega (S(x,t)+I(x,t))dx$ remains a constant for all $t \geq 0$. By assumption (A2), the total population is N.

For convenience, we introduce a few definitions and notations. Set $r := \gamma/\beta$ and $R := \beta/\gamma$. For a real-valued continuous function h on $\overline{\Omega}$, let $h_M := \sup_{x\in\Omega} h(x)$, $h_m := \inf_{x\in\Omega} h(x)$ and $\bar{h} = \int_\Omega h(x)dx/|\Omega|$.

Let $\{e^{t\Delta}\}_{t\geq 0}$ be the analytic c_0-semigroup on $L^p(\Omega)$, $1 \leq p < \infty$, generated by the Laplace operator Δ on Ω subject to the homogeneous Neumann boundary conditions on $\partial\Omega$. Let $\mathrm{Dom}_p(\Delta)$ be the domain of the infinitesimal generator Δ of $\{e^{t\Delta}\}_{t\geq 0}$ on $L^p(\Omega)$. Then $\mathrm{Dom}_p(\Delta) = \{u \in W^{2,p}(\Omega) | \partial_\nu u = 0 \text{ on } \partial\Omega\}$ for $p \in (1,\infty)$ and $\mathrm{Dom}_1(\Delta) \subset \{u \in W^{2,1}(\Omega) | \partial_\nu u = 0 \text{ on } \partial\Omega\}$ (see [3]). When $\{e^{t\Delta}\}_{t\geq 0}$ is considered as an analytic c_0-semigroup on $C(\bar{\Omega})$, the domain of its infinitesimal generator Δ is given by

$$\mathrm{Dom}_\infty(\Delta) := \left\{u \in \cap_{p\geq 1}\mathrm{Dom}_p(\Delta) |\ \Delta u \in C(\overline{\Omega})\right\}.$$

Let $\mathcal{Z} := \left\{u \in L^1(\Omega) : \int_\Omega u dx = 0\right\}$ be a Banach subspace of $L^1(\Omega)$. Then $\{e^{t\Delta}\}_{t\geq 1}$ leaves $\mathcal{Z}$ invariant and by [61, Lemma 1.3], there is a positive real number $C_0 > 0$ such that

$$\|e^{t\Delta}u\|_{L^1(\Omega)} \leq C_0 e^{-\sigma_1 t}\|u\|_{L^1(\Omega)}, \quad \forall\, u \in \mathcal{Z}, \tag{2.1}$$

where σ_1 is the first positive eigenvalue of $-\Delta$, subject to the homogeneous boundary condition on $\partial\Omega$. By (2.1), the restriction $\Delta_{|\mathcal{Z}}$ of Δ on $\mathcal{Z} \cap \mathrm{Dom}_1(\Delta)$ is invertible. Hence, for any $0 \leq \alpha < 1$, the fractional power space $\mathcal{Z}^\alpha$ of $-\Delta_{|\mathcal{Z}}$ is well defined (see [20]). Denote by $\{e^{t\Delta_{|\mathcal{Z}}}\}_{t\geq 0}$, the restriction of the $\{e^{t\Delta}\}_{t\geq 0}$ on $\mathcal{Z}$, then it is also an analytic c_0-semigroup. The following estimates on $\mathcal{Z}^\alpha$ will be needed:

Lemma 2.1. [20, Theorem 1.4.3] *For any* $0 < \alpha < 1$, *there is* $C_\alpha > 0$ *such that*

$$\|e^{t\Delta_{|\mathcal{Z}}}z\|_{\mathcal{Z}^\alpha} \leq C_\alpha t^{-\alpha}e^{-\sigma_1 t}\|z\|_{L^1(\Omega)}, \quad \forall\, t > 0,\ z \in \mathcal{Z}, \tag{2.2}$$

and

$$\left\|(e^{t\Delta_{|\mathcal{Z}}} - id)z\right\|_{L^1(\Omega)} \leq C_\alpha t^\alpha \|z\|_{\mathcal{Z}^\alpha}, \quad \forall\, z \in \mathcal{Z}^\alpha. \tag{2.3}$$

The following well-known result will be used after the construction of Lyapunov functions, and we provide a proof for convenience.

Lemma 2.2. *Suppose that* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *is Hölder continuous and* $\int_0^\infty \phi(t)dt < \infty$. *Then* $\lim_{t\to\infty}\phi(t) = 0$.

Proof. Suppose to the contrary that $\lim_{t\to\infty}\phi(t) \neq 0$. Then there exists $\epsilon > 0$ and an increasing sequence of nonnegative numbers $\{t_k\}$ converging to infinity such that $\phi(t_k) > \epsilon$ for all $k \geq 1$. Restricting to a subsequence if necessary, we assume that $t_{k+1} - t_k > 1$ for each $k \geq 1$. Since $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is Hölder continuous, there exist $\alpha \in (0, 1)$ and $M > 0$ such that $|\phi(t) - \phi(s)| \le M|t - s|^\alpha$ for all $t, s \ge 0$. Let $\delta = \min\left\{\left(\frac{\epsilon}{2M}\right)^{\frac{1}{\alpha}}, 1\right\}$. Then $\phi(t) > \epsilon/2$ for all $t \in [t_k, t_k + \delta]$ and $k \ge 1$. It follows that

$$\int_0^\infty \phi(t)dt \ge \sum_{k\ge 1} \int_{t_k}^{t_k+\delta} \phi(t)dt \ge \sum_{k\ge 1} \delta \frac{\epsilon}{2} = \infty,$$

which is a contradiction. □

We will need the following Hanack-type inequality (see [21]):

Lemma 2.3. *Let u be a nonnegative solution of the following problem on $\Omega \times (0, T)$:*

$$\begin{cases} \partial_t u = \Delta u + a(x, t)u, & x \in \Omega, t > 0, \\ \partial_\nu u = 0, & x \in \partial\Omega, t > 0, \end{cases}$$

where $a \in L^\infty(\Omega \times (0, \infty))$. Then for any $0 < \delta < T$, there exists $C > 0$ depending on δ and $\|a\|_{L^\infty(\Omega\times(0,\infty))}$ such that

$$\sup_{x\in\Omega} u(x, t) \le C \inf_{x\in\Omega} u(x, t), \quad \textit{for all } t \in [\delta, T).$$

We also recall the following well-known result about the elliptic eigenvalue problem.

Lemma 2.4. [8] *Suppose that $d > 0$ and $h \in C(\overline{\Omega})$. Let $\sigma(d, h)$ be the principal eigenvalue of the following elliptic eigenvalue problem:*

$$\begin{cases} d\Delta\varphi + h\varphi = \sigma\varphi, & x \in \Omega, \\ \partial_\nu \varphi = 0, & x \in \partial\Omega. \end{cases}$$

Then $\sigma(d, h)$ is simple, associated with a positive eigenfunction, and given by the variational formula

$$\sigma(d, r) = \sup\left\{\int_\Omega \left[h\varphi^2 - d|\nabla\varphi|^2\right]dx : \varphi \in W^{1,2}(\Omega) \text{ and } \|\varphi\|_{L^2(\Omega)} = 1\right\}. \tag{2.4}$$

Furthermore, the following conclusions hold:

(i) If $h(x) \equiv h$ is a constant function, then $\sigma(d, h) = h$ for all $d > 0$.

(ii) If $h(x)$ is not constant, then the map $(0, \infty) \ni d \mapsto \sigma(d, h)$ is strictly decreasing with

$$\lim_{d\to 0^+} \sigma(d, h) = h_M \quad \text{and} \quad \lim_{d\to\infty} \sigma(d, h) = \overline{h}. \tag{2.5}$$

3. Model with mass action mechanism

3.1. Limiting the movement of susceptible people

First, we consider the impact of limiting the movement of susceptible people on (1.1) with mass action mechanism, that is, the long time behaviour of the following degenerate system:

$$\begin{cases} \partial_t S = -\beta(x)SI + \gamma(x)I, & x \in \bar{\Omega}, t > 0, \\ \partial_t I = d_I \Delta I + \beta(x)SI - \gamma(x)I, & x \in \Omega, t > 0, \\ \partial_\nu I = 0, & x \in \partial\Omega, t > 0, \\ S(x, 0) = S_0(x), \ I(x, 0) = I_0(x), & x \in \bar{\Omega}. \end{cases} \tag{3.1}$$

The following result states that the solution of (3.1) exists and is bounded:

Proposition 3.1. *Suppose that (A1)–(A2) holds and $d_I > 0$. Then (3.1) has a unique nonnegative global solution (S, I), where*

$$S \in C^1([0, \infty), C(\overline{\Omega})) \quad \text{and} \quad I \in C([0, \infty), C(\overline{\Omega})) \cap C^1((0, \infty), \text{Dom}_\infty(\Delta)).$$

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Moreover, there exists $M > 0$ depending on (the L^∞ norm of) initial data such that

$$\|S(\cdot,t)\|_{L^\infty(\Omega)},\ \|I(\cdot,t)\|_{L^\infty(\Omega)} \le M,\ \forall t \ge 0. \tag{3.2}$$

Proof. First, we suppose that the solution exists on some maximal interval $[0, t_{max})$, $t_{\max} \in (0, \infty]$ and prove the boundedness of it. Let $M_1 = \max\{\|S_0\|_{L^\infty(\Omega)},\ r_M\}$. By the first equation of (3.1), $S(x, t) \le M_1$ for all $x \in \bar{\Omega}$ and $0 \le t < t_{\max}$. Since $\int_\Omega Idx \le \int_\Omega (S + I)dx \le N$ for all $t \in [0, t_{\max})$, by [1, Theorem 3.1] (or Lemma 2.3), there exists $M_2 > 0$ depending on $\|I_0\|_{L^\infty(\Omega)}$ and M_1 such that $\|I(\cdot, t)\|_{L^\infty(\Omega)} \le M_2$ for all $0 \le t < t_{\max}$. This proves (3.2).

The integral form of (3.1) is

$$\begin{cases} S(\cdot,t) = e^{-at}S_0 + \int_0^t e^{-a(t-s)}(aS(\cdot,s) - \beta S(\cdot,s)I(\cdot,s) + \gamma I(\cdot,s))ds, & t > 0, \\ I(\cdot,t) = e^{t(d_I\Delta - a)}I_0 + \int_0^t e^{(t-s)(d_I\Delta-a)}(a + \beta S(\cdot,s) - \gamma)I(\cdot,s)ds, & t > 0, \end{cases} \tag{3.3}$$

where $a > 0$ is chosen to be sufficiently large. Using Banach fixed point theory, one can show that (3.3) has a unique nonnegative solution $(S, I) \in [C([0, T], C(\overline{\Omega}))]^2$ for some $T > 0$. By the first equation of (3.3), $S \in C^1([0, T], C(\overline{\Omega}))$. By [44, Theorem 4.3.1 and Corollary 4.3.3], $I \in C([0, T], C(\overline{\Omega})) \cap C^1((0, T], \mathrm{Dom}_\infty(\Delta))$. The a priori bound (3.2) enables us to extend the solution globally. □

We note that the case when the total population is small (i.e. $N < \int_\Omega rdx$) has been addressed in the literature:

Theorem 3.2 ([9, Theorem 2.8-(i)]). *Suppose that* (A1)–(A2) *holds and $d_I > 0$. Let (S, I) be the solution of* (3.1). *If $N < \int_\Omega rdx$, then $(S(x, t), I(x, t)) \to (S^*(x), 0)$ uniformly in $x \in \bar{\Omega}$ as $t \to \infty$, where $S^* \in C(\bar{\Omega})$ and $\int_\Omega S^*dx = N$.*

We are ready to study the asymptotic behaviour of the solution of (3.1).

Theorem 3.3. *Suppose that* (A1)–(A2) *holds and $d_I > 0$. Let (S, I) be the solution of* (3.1). *Then $\|S(\cdot, t) - S^*\|_{L^\infty(\Omega)} \to 0$ as $t \to \infty$ for some $S^* \in C(\overline{\Omega})$, and exactly one of the following two statements hold:*

(i) $S^* = \lambda^*S_0 + (1 - \lambda^*)r$ *for some* $\lambda_* \in C(\bar{\Omega})$ *with* $0 < \lambda_* < 1$ and $\sigma(d_I, \beta\lambda^*(S_0 - r)) \le 0$, $\int_\Omega S^*dx = N$, *and* $\|I(\cdot, t)\|_{L^\infty(\Omega)} \to 0$ *as* $t \to \infty$;

(ii) $S^* = r$ *and* $\|I(\cdot, t) - I^*\|_{L^\infty(\Omega)} \to 0$ *as* $t \to \infty$, *where* $I^* = (N - \int_\Omega rdx)/|\Omega|$ *is a positive constant.*

Moreover, (i) holds if $N \le \int_\Omega rdx$ while (ii) holds if $N > N^*_{S_0,r}$, where $N^*_{S_0,r}$ is defined by

$$N^*_{S_0,r} := \sup\left\{\int_\Omega (\lambda^*S_0 + (1 - \lambda^*)r)dx : \lambda^* \in C(\bar{\Omega}; [0, 1]) \text{ and } \sigma(d_I, \beta\lambda^*(S_0 - r)) \le 0\right\}. \tag{3.4}$$

Proof. Define

$$J(x,t) = \int_0^t I(x,\tau)d\tau, \quad \forall\ (x,t) \in \bar{\Omega} \times [0,\infty).$$

It is clear that $J(x, t)$ is strictly monotone increasing in t for each $x \in \bar{\Omega}$. Hence, we can define

$$\hat{J}(x) := \int_0^\infty I(x,\tau)d\tau = \lim_{t\to\infty} J(x,t) \in (0,\infty], \quad \forall\ x \in \bar{\Omega}.$$

Now, we distinguish two cases.

Case 1. $\hat{J}(x_0) < \infty$ for some $x_0 \in \overline{\Omega}$. By (3.2), we have $\sup_{t\ge 0} \|\beta S(\cdot, t) - \gamma\|_\infty < \infty$. So by Lemma 2.3, there is a positive constant c_1 such that

$$I_M(\cdot,t) \le c_1 I_m(\cdot,t), \quad \forall\ t \ge 1. \tag{3.5}$$

It follows that

$$\hat{J}(x) = J(x,1) + \int_1^\infty I(x,\tau)d\tau \le J(x,1) + c_1\int_1^\infty I(x_0,\tau)d\tau \le J(x,1) + c_1 J(x_0), \quad \forall\ x \in \bar{\Omega}.$$

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This shows that $\hat{J}(x) < \infty$ for all $x \in \bar{\Omega}$ and $\|\hat{J}\|_{L^\infty(\Omega)} < \infty$. Moreover, by (3.5),

$$\|\hat{J} - J(\cdot, t)\|_{L^\infty(\Omega)} \le c_1 \int_t^\infty I(x_0, \tau)d\tau \to 0 \quad \text{as} \quad t \to \infty.$$

Hence, $\hat{J} \in C(\bar{\Omega})$. Next, observing that

$$\partial_t(S - r) = \partial_t S = -\beta(S - r)I, \qquad \forall\, (x, t) \in \bar{\Omega} \times (0, \infty),$$

we have

$$S(x, t) - r(x) = (S_0(x) - r(x))e^{-\beta(x)J(x,t)}, \quad \forall\, (x, t) \in \bar{\Omega} \times (0, \infty). \tag{3.6}$$

This implies

$$S(\cdot, t) \to S^* := r + e^{-\beta\hat{J}}(S_0 - r) \text{ uniformly on } \bar{\Omega} \text{ as } t \to \infty.$$

Let $\lambda^* = e^{-\beta\hat{J}}$. Then $0 < \lambda^* < 1$, $\lambda^* \in C(\bar{\Omega})$ and $S^* = \lambda^* S_0 + (1 - \lambda^*)r$. Let $\varphi^* > 0$ be the eigenfunction associated with $\sigma(d_I, \beta(S^* - r))$ satisfying $\varphi^*_M = 1$. Then it holds that

$$\begin{aligned}\frac{d}{dt}\int_\Omega \varphi^* I dx = & d_I \int_\Omega \varphi^* \Delta I dx + \int_\Omega \beta(S - r)\varphi^* I dx \\ = & \sigma(d_I, \beta(S^* - r)) \int_\Omega \varphi^* I dx + \int_\Omega \beta(S - S^*)\varphi^* I dx \\ \ge & \Big(\sigma(d_I, \beta(S^* - r)) - \beta_M \|S - S^*\|_{L^\infty(\Omega)}\Big) \int_\Omega \varphi^* I dx.\end{aligned}$$

Hence, we have

$$N \ge \int_\Omega \varphi^* I(x, t)dx \ge e^{\int_0^t (\sigma(d_I, \beta(S^*-r)) - \beta_M\|S - S^*\|_{L^\infty(\Omega)})d\tau} \int_\Omega \varphi^* I_0 dx, \quad \forall\, t > 0.$$

This implies that

$$\frac{\beta_M}{t}\int_0^t \|S(\cdot, \tau) - S^*\|_{L^\infty(\Omega)}d\tau + \frac{\ln(N) - \ln(\|\varphi^* I_0\|_{L^1(\Omega)})}{t} \ge \sigma(d_I, \beta(S^* - r)), \quad t > 0.$$

Observing that the left-hand side of this inequality tends zeo as $t \to \infty$, we conclude that $\sigma(d_I, \beta(S^* - r)) \le 0$.

Finally, by (3.2) and the parabolic estimates and the Sobolev emdedding theorem, I is Hölder continuous on $\bar{\Omega} \times [1, \infty)$. Hence, by Lemma 2.2 and the fact that $\hat{J}(x_0) < \infty$, we obtain from (3.5) that

$$\|I(\cdot, t)\|_{L^\infty(\Omega)} \le c_1 I(x_0, t) \to 0 \quad \text{as} \quad t \to \infty,$$

which in turn implies that

$$\int_\Omega S^* dx = \lim_{t\to\infty}\int_\Omega S dx = \lim_{t\to\infty}\int_\Omega (S + I)dx = N.$$

This completes the proof of (i).

Case 2. $\hat{J}(x) = \infty$ for all $x \in \bar{\Omega}$. Fix $x_1 \in \bar{\Omega}$. Then $\int_1^t I(x_1, \tau)d\tau \to \infty$ as $t \to \infty$. This together with (3.5)–(3.6) implies that

$$\|S(\cdot, t) - r\|_{L^\infty(\Omega)} \le \|S_0 - r\|_{L^\infty(\Omega)} e^{-\frac{\beta_m}{c_1}\int_1^t I(x_1,\tau)d\tau} \to 0 \quad \text{as} \quad t \to \infty.$$

As a result,

$$\lim_{t\to\infty}\int_\Omega I dx = N - \int_\Omega r dx. \tag{3.7}$$

This shows that, in the current case, we must have that $N \ge \int_\Omega r dx$. By (3.2), the parabolic estimates and the Sobolev embedding theorem, the orbit $\{I(\cdot, t)\}_{t\ge 1}$ is precompact in $C(\bar{\Omega})$. Noticing $S(\cdot, t) \to r$ in

$C(\bar{\Omega})$ as $t \to \infty$, the ω limit set $\omega(S_0, I_0) = \cap_{t\geq 1}\overline{\cup_{s\geq t}\{(S(\cdot,t), I(\cdot,s))\}}$ is well defined, where the completion is in $[C(\bar{\Omega})]^2$. Fix $(S^*, I^*) \in \omega(S_0, I_0)$. Since $S(\cdot,t) \to r$ in $C(\bar{\Omega})$ as $t \to \infty$, then $S^* = r^*$. Next, we show that $I^*(x) = (N - \int_\Omega r)/|\Omega|$ for all $x \in \Omega$. To this end, since $\omega(I_0, S_0)$ is invariant under the semiflow of the solution operator induced by (3.1) and the orbit $\{I(\cdot,t)\}_{t\geq 1}$ is precompact in $C(\bar{\Omega})$, we can employ standard parabolic regularity arguments to the equation of $I(\cdot,t)$, and coupled with the fact that $S(\cdot,t) \to r$ in $C(\bar{\Omega})$ as $t \to \infty$, to conclude that there is a bounded entire solution $(\tilde{S}(x,t), \tilde{I}(x,t))$ of (3.1) fulfilling $\tilde{I}(\cdot,0) = I^*$ and $\tilde{S}(\cdot,t) = r$ for all $t \in \mathbb{R}$. Hence, $\tilde{I}(x,t)$ satisfies

$$\begin{cases} \partial_t \tilde{I} = d_I \Delta \tilde{I}, & x \in \Omega,\ t \in \mathbb{R}, \\ \partial_\nu \tilde{I} = 0, & x \in \partial\Omega,\ t \in \mathbb{R}, \\ \tilde{I}(x,0) = I^*(x), & x \in \bar{\Omega}. \end{cases}$$

So, $\tilde{I}(\cdot,t) = \int_\Omega I^*(x)dx/|\Omega|$ for all $t \in \mathbb{R}$. However by (3.7), $\int_\Omega I^* dx = \big(N - \int_\Omega r dx\big)/|\Omega|$. Hence, I^* is the constant function $\big(N - \int_\Omega r dx\big)/|\Omega|$. This shows that $\omega(S_0, I_0) = \big\{r, \big(N - \int_\Omega r dx\big)/|\Omega|\big\}$ and completes the proof of (ii).

It is easy to see that if $N \leq \int_\Omega r dx$, then (i) holds. Note that if (i) holds then $N = \int_\Omega (\lambda^* S_0 + (1 - \lambda^*)r)dx$ for some $\lambda^* \in C(\overline{\Omega})$ satisfying $0 < \lambda^* < 1$ and $\sigma(d_I, \beta\lambda^*(S_0 - r)) \leq 0$. Hence, we must have $N \leq N^*_{S_0,r}$. So alternative (ii) must hold whenever $N > N^*_{S_0,r}$. □

Proposition 3.4. *Let $N^*_{S_0,r}$ be defined as in Theorem 3.3. The following statements hold.*

(1) *It holds that $N^*_{S_0,r} \geq \int_\Omega r dx$.*

(2) *It holds that $N^*_{S_0,r} \leq \int_\Omega \max\{S_0, r\}dx$. Hence, if $N > \int_\Omega \max\{S_0, r\}dx$, then alternative (ii) of Theorem 3.3 holds.*

(3) *The strict inequality $N^*_{S_0,r} < \int_\Omega \max\{S_0, r\}dx$ holds if $\|(S_0 - r)_+\|_{L^\infty(\Omega)} > 0$ (Therefore, in general, the condition $N > N^*_{S_0,r}$ is weaker than $N > \int_\Omega \max\{S_0, r\}dx$).*

Proof. (1) Taking $\lambda^* \equiv 0$ in (3.4), we have the desired result.

(2) For any $\lambda^* \in C(\bar{\Omega}; [0,1])$, we have $\int_\Omega (\lambda^* S_0 + (1 - \lambda^*)r)dx \leq \max\{S_0, r\}$. By the definition of $N^*_{S_0,r}$, we have $N^*_{S_0,r} \leq \int_\Omega \max\{S_0, r\}dx$.

(3) Suppose that $\|(S_0 - r)_+\|_{L^\infty(\Omega)} > 0$ and we prove $N^*_{S_0,r} < \int_\Omega \max\{S_0, r\}dx$ by contradiction. Suppose to the contrary that there is $\lambda^*_k \in C(\overline{\Omega}; [0,1])$ satisfying $\sigma\big(d_I, \beta\lambda^*_k(S_0 - r)\big) \leq 0$ for each $k \geq 1$ such that

$$\int_\Omega \big(\lambda^*_k S_0 + \big(1 - \lambda^*_k\big)r\big)dx \to \int_\Omega \max\{S_0, r\}dx \quad \text{as} \quad k \to \infty.$$

Since $\|\lambda^*_k\|_{L^\infty(\Omega)} \leq 1$, possibly after passing to a subsequence and using the Banach–Alaoglu theorem, there is $\lambda^* \in L^\infty(\Omega)$ satisfying $0 \leq \lambda^* \leq 1$ almost everywhere on Ω, such that $\lambda^*_k \to \lambda^*$ weakly star in $L^\infty(\Omega)$ as $k \to \infty$. So we have

$$\int_\Omega \big(\lambda^*_k S_0 + \big(1 - \lambda^*_k\big)r\big)dx \to \int_\Omega (\lambda^* S_0 + (1 - \lambda^*)r)dx \quad \text{as} \quad k \to \infty.$$

It follows that

$$\int_\Omega (\lambda^* S_0 + (1 - \lambda^*)r)dx = \int_\Omega \max\{S_0, r\}dx,$$

which yields that $\max\{S_0, r\} = \lambda^* S_0 + (1 - \lambda^*)r$ almost everywhere on Ω. So, $\beta\lambda^*(S_0 - r) = \beta\big(\max\{S_0, r\} - r\big)$ almost everywhere on Ω. Therefore, by the assumption $\|(S_0 - r)_+\|_\infty > 0$,

$$\int_\Omega \beta\lambda^*(S_0 - r)dx = \int_\Omega \beta\big(\max\{S_0, r\} - r\big)dx = \int_{\{S_0 > r\}} \beta(S_0 - r)dx > 0. \tag{3.8}$$

However, since $\sigma\big(d_I, \beta\lambda^*_k(S_0 - r)\big) \leq 0$, we have $\int_\Omega \beta\lambda^*_k(S_0 - r))dx \leq 0$ for any $k \geq 1$ by Lemma 2.4. Letting $k \to \infty$, we get $\int_\Omega \beta\lambda^*(S_0 - r)dx \leq 0$, which contradicts with (3.8). □

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We complement Theorem 3.3 with a corollary:

Corollary 3.5. *Suppose that (A1)–(A2) holds, $N > \int_\Omega r dx$, and $d_I > 0$. If either β is constant or $S_0 - r$ has a constant sign, then alternative (ii) of Theorem 3.3 holds for any solution of (3.1).*

Proof. Let $N^*_{S_0,r}$ be defined as in (3.4). (1) If β is constant, then for any $\lambda^* \in C(\overline{\Omega}; [0, 1])$, $\sigma(d_I, \beta\lambda^*(S_0 - r)) \leq 0$ implies that $\int_\Omega \lambda^*(S_0 - r)dx \leq 0$. In this case,

$$\int_\Omega (\lambda^* S_0 + (1 - \lambda^*)r)dx = \int_\Omega \lambda^*(S_0 - r)dx + \int_\Omega rdx \leq \int_\Omega rdx, \quad \forall\, \lambda^* \in C(\overline{\Omega}; [0, 1]).$$

So, we have $N^*_{S_0,r} = \int_\Omega rdx$.

(2) If $S_0 \leq r$, then $N^*_{S_0,r} = \int_\Omega rdx$.

(3) If $S_0 \geq r$, then $N^*_{S_0,r} \leq \int_\Omega S_0 dx$.

In cases (1)–(3), we have $N^*_{S_0,r} \leq \max\left\{\int_\Omega rdx, \int_\Omega S_0 dx\right\}$. By hypothesis (A2), we always have $N > \int_\Omega S_0 dx$. Therefore, if either of these scenarios holds, $N > \int_\Omega rdx$ implies $N > N^*_{S_0,r}$. The conclusion now follows from Theorem 3.3. □

Remark 3.6. *Let $N^*_{S_0,r}$ be defined as in (3.4). Alternative (ii) of Theorem 3.3 always holds when $N > \int_\Omega rdx$ if $N^*_{S_0,r} \leq \max\left\{\int_\Omega S_0 dx, \int_\Omega rdx\right\}$. Sufficient conditions ensuring the validity of the latter inequality are given in Corollary 3.5. It remains an open problem to know whether alternative (i) of Theorem 3.3 may hold for some initial data satisfying $\max\left\{\int_\Omega S_0 dx, \int_\Omega rdx\right\} < N < N^*_{S_0,r}$. Note that it is possible to construct examples of positive and continuous functions S_0 satisfying $\max\left\{\int_\Omega S_0 dx, \int_\Omega rdx\right\} < N^*_{S_0,r}$. Whenever such S_0 is fixed, we can always select I_0 to be small enough such that $N = \int_\Omega (S_0 + I_0)dx$ satisfies $\max\left\{\int_\Omega S_0 dx, \int_\Omega rdx\right\} < N < N^*_{S_0,r}$.*

3.2. Limiting the movement of infected people

We consider the impact of limiting the movement of infected people on (1.1) with mass action mechanism, that is, the long time behaviour of the following degenerate system:

$$\begin{cases} \partial_t S = d_S \Delta S - \beta(x)SI + \gamma(x)I, & x \in \Omega, t > 0, \\ \partial_t I = \beta(x)SI - \gamma(x)I, & x \in \bar{\Omega}, t > 0, \\ \partial_\nu S = 0, & x \in \partial\Omega, t > 0, \\ S(x, 0) = S_0(x),\ I(x, 0) = I_0(x), & x \in \bar{\Omega}. \end{cases} \tag{3.9}$$

In the following result, we show the global existence of the solution of (3.9). We remark that the S component of the solution is globally bounded while the I component may blow up at $t = \infty$.

Proposition 3.7. *Suppose that* (A1)–(A2) *holds and $d_S > 0$. Then (3.9) has a unique nonnegative global solution (S, I), where*

$$S \in C([0, \infty), C(\overline{\Omega})) \cap C^1((0, \infty), \mathrm{Dom}_\infty(\Delta)) \quad \text{and} \quad I \in C^1([0, \infty), C(\overline{\Omega})).$$

Moreover, $\|S(\cdot, t)\|_{L^\infty(\Omega)} \leq \max\{\|S_0\|_{L^\infty(\Omega)},\ r_M\}$ for all $t \geq 0$.

Proof. Suppose that a nonnegative solution exists. Let $M_1 = \max\{\|S_0\|_{L^\infty(\Omega)},\ r_M\}$. By the first equation of (3.9) and the comparison principle, $0 \leq S(x, t) \leq M_1$ for all $x \in \bar{\Omega}$ and $t > 0$, which implies that

$$0 \leq I(x, t) = I_0(x)e^{\int_0^t (\beta(x)S(x,s) - \gamma(x))ds} \leq I_0(x)e^{\beta_M M_1 t}, \quad \forall x \in \bar{\Omega}, t \geq 0.$$

This gives $\|I(\cdot, t)\|_{L^\infty(\Omega)} \leq \|I_0\|_{L^\infty(\Omega)}e^{\beta_M M_1 t}$ for all $t \geq 0$. The local existence of the solution of (3.9) can be proved similar to Proposition 3.1. The a prior bound of the solution ensures the global existence of it. □

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To study the asymptotic behaviour of the solutions of (3.9), we use the following Lyapunov function

$$V(S,I)=\int_\Omega\left(\frac{1}{2}S^2+rI\right)dx.$$

If (S,I) is the solution of (3.9), it is easy to check that

$$\frac{d}{dt}V(S,I)=-d_S\int_\Omega|\nabla S|^2dx-\int_\Omega\beta(S-r)^2Idx. \tag{3.10}$$

Remark 3.8. *We point out that it is very easy to draw a false conclusion using the above Lyapunov function: first by the term $\int_\Omega\beta(S-r)^2Idx$ in (3.10), one may conclude that either $S\to r$ or $I\to 0$ as $t\to\infty$; then by the term $\int_\Omega|\nabla S|^2dx$, $\nabla S\to 0$ and so $I\to 0$ if r is not constant. We will show that this intuition is indeed false in Theorem 3.11. Actually, it is possible that S converges to some constant $\bar{S}$ with $r_m\le\bar{S}\le r_M$ and I converges to some measure supported at $\{x\in\bar{\Omega}:r(x)=\bar{S}\}$.*

To conclude that $\int_\Omega|\nabla S|^2dx\to 0$ or $\int_\Omega\beta(S-r)^2Idx\to 0$ as $t\to\infty$, we will need the following lemma.

Lemma 3.9. *Suppose that (A1)–(A2) holds and $d_S>0$. Let (S,I) be the solution of (3.9). Then the following conclusions hold:*

(i) The mapping $[1,\infty)\ni t\mapsto S(\cdot,t)-\overline{S(\cdot,t)}\in L^1(\Omega)$ is Hölder continuous. Furthermore, if $n=1$, then the mapping $[1,\infty)\ni t\mapsto S(\cdot,t)-\overline{S(\cdot,t)}\in W^{1,2}(\Omega)$ is also Hölder continuous.

(ii) If $n=1$, the mapping $[1,\infty)\ni t\mapsto\int_\Omega\beta(S-r)^2Idx$ is Hölder continuous.

Proof. Setting $Z=S(\cdot,t)-\overline{S(\cdot,t)}$ and $F(\cdot,t)=\beta(r-S(\cdot,t))I(\cdot,t)-\overline{\beta(r-S(\cdot,t))I(\cdot,t)}$, it holds that $Z(\cdot,t),F(\cdot,t)\in\mathcal{Z}$ for all $t\ge 0$ and

$$\begin{cases}\partial_tZ=d_S\Delta Z+F(\cdot,t), & t>0,\ x\in\Omega,\\ \partial_\nu Z=0, & t>0,\ x\in\partial\Omega.\end{cases}$$

Hence by the variation of constant formula, we have

$$Z(\cdot,t)=e^{td_S\Delta_{|\mathcal{Z}}}Z(\cdot,0)+\int_0^t e^{(t-\tau)\Delta_{|\mathcal{Z}}}F(\cdot,\tau)d\tau,\quad\forall\, t>0.$$

By Proposition 3.7 with $M:=\max\{\|S_0\|_{L^\infty(\Omega)},r_M\}$, it holds that

$$\|F(\cdot,t)\|_{L^1(\Omega)}\le 2\beta_MM\|I(\cdot,t)\|_{L^1(\Omega)}\le M_1:=2NM\beta_M,\quad\forall\, t\ge 0. \tag{3.11}$$

(i) Fix $0\le\tilde{\alpha}<\tilde{\alpha}+\alpha<1$. For any $t\ge 1$ and $h>0$, by (2.1)–(2.3) and (3.11),

$$\begin{aligned}&\|Z(t+h)-Z(t)\|_{\mathcal{Z}^{\tilde{\alpha}}}\\ &\le\left\|\left(e^{hd_S\Delta_{|\mathcal{Z}}}-\mathrm{id}\right)e^{td_S\Delta_{|\mathcal{Z}}}Z_0\right\|_{\mathcal{Z}^{\tilde{\alpha}}}+\int_0^t\left\|\left(e^{hd_S\Delta_{|\mathcal{Z}}}-\mathrm{id}\right)e^{(t-\tau)d_S\Delta_{|\mathcal{Z}}}F(\cdot,\tau)\right\|_{\mathcal{Z}^{\tilde{\alpha}}}d\tau\\ &\quad+\int_0^h\left\|e^{(h-\tau)d_S\Delta_{|\mathcal{Z}}}F(\cdot,t+\tau)\right\|_{\mathcal{Z}^{\tilde{\alpha}}}d\tau\\ &\le C_\alpha d_S^\alpha h^\alpha\left(\left\|e^{td_S\Delta_{|\mathcal{Z}}}Z_0\right\|_{\mathcal{Z}^{\tilde{\alpha}+\alpha}}+\int_0^t\left\|e^{(t-\tau)d_S\Delta_{|\mathcal{Z}}}F(\cdot,\tau)\right\|_{\mathcal{Z}^{\tilde{\alpha}+\alpha}}d\tau\right)\\ &\quad+C_{\tilde{\alpha}}\int_0^h\left(d_S(h-\tau)\right)^{-\tilde{\alpha}}e^{-d_S\sigma(h-\tau)}\|F(\cdot,t+\tau)\|_{L^1(\Omega)}d\tau\end{aligned}$$

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$$\leq C_\alpha C_{\tilde{\alpha}+\alpha} d_S^{-\tilde{\alpha}} h^\alpha \Big(t^{-(\tilde{\alpha}+\alpha)} e^{-\sigma t d_S} \|Z_0\|_{L^1(\Omega)} + \int_0^t (t-\tau)^{-(\tilde{\alpha}+\alpha)} e^{-d_S\sigma(t-\tau)} \|F(\cdot,\tau)\|_{L^1(\Omega)} d\tau \Big)$$
$$+ C_{\tilde{\alpha}} d_S^{-\tilde{\alpha}} M_1 \int_0^h \tau^{-\tilde{\alpha}} e^{-d_S\sigma\tau} d\tau$$
$$\leq C_\alpha C_{\alpha+\tilde{\alpha}} d_S^{-\tilde{\alpha}} h^\alpha \Big(\|Z_0\|_{L^1(\Omega)} + M_1 \int_0^\infty \tau^{-(\tilde{\alpha}+\alpha)} e^{-d_S\sigma\tau} d\tau \Big) + C_{\tilde{\alpha}} d_S^{-\tilde{\alpha}} M_1 \int_0^h \tau^{-\tilde{\alpha}} d\tau$$
$$\leq M_{\alpha,\tilde{\alpha}} \big(h^\alpha + h^{1-\tilde{\alpha}} \big). \tag{3.12}$$

In particular, if $\tilde{\alpha} = 0$, we get that

$$\|Z(t+h) - Z(t)\|_{L^1(\Omega)} \leq M_{\alpha,\tilde{\alpha}}(h^\alpha + h), \quad \forall\, t \geq 1, \ h > 0,$$

which yields the first assertion of (i). On the other, if $n = 1$, choosing $\alpha \in (\frac{3}{4}, 1)$, it follows from [20, Theorem 1.6.1] that $\mathcal{Z}^\alpha$ is continuously embedded in $W^{1,2}(\Omega)$. Hence, the last assertion of (i) also follows from (3.12).

(ii) Suppose $n = 1$. Let $G(t) := \int_\Omega \beta(S - r)^2 I dx$ for $t \geq 0$. Since $n = 1$, $W^{1,2}(\Omega)$ is compactly embedded in $C(\overline{\Omega})$. Since the mapping $[1,\infty) \ni t \mapsto Z(\cdot,t) \in W^{1,2}(\Omega)$ is Hölder continuous by (i), the mapping $[1,\infty) \ni t \mapsto Z(\cdot,t) \in C(\overline{\Omega})$ is also Hölder continuous. This together with the fact that

$$\sup_{t>0} \left| \frac{d}{dt} \int_\Omega S(x,t)dx \right| = \sup_{t>0} \left| \int_\Omega \beta(S(x,t) - r(x)) I(x,t) dx \right| \leq MN\beta_M$$

implies that the mapping $[1,\infty) \ni t \mapsto S(\cdot,t) \in C(\overline{\Omega})$ is also Hölder continuous. Thus, there exist $0 < \tau < 1$ and $c > 0$ such that

$$\|S(\cdot,t+h) - S(\cdot,t)\|_{L^\infty(\Omega)} \leq c|h|^\tau, \quad t \geq 1.$$

So for any $t \geq 1$ and $h > 0$, we have

$$|G(t+h) - G(t)| \leq \beta_M \|(S(\cdot,t+h) - r)^2 - (S(\cdot,t) - r)^2\|_\infty \int_\Omega I(x,t+h)dx$$
$$+ \beta_M \|(S(\cdot,t) - r)^2\|_{L^\infty(\Omega)} \int_\Omega |I(x,t+h) - I(x,t)| dx$$
$$\leq 2cMNh^\tau \beta_M + M^2\beta_M \int_\Omega \int_0^h \beta |S(x,t+s) - r(x)| I(x,t+s) ds dx$$
$$\leq 2cMNh^\tau \beta_M + M^3\beta_M^2 \int_\Omega \int_0^h I(x,t+s) ds dx$$
$$= 2cMNh^\tau \beta_M + M^3\beta_M^2 \int_0^h \int_\Omega I(x,t+s) dx ds$$
$$\leq 2cMNh^\tau \beta_M + M^3\beta_M^2 Nh \leq \big(c + M^2\beta_M\big) M_1 (h^\tau + h),$$

which yields the desired result. □

Lemma 3.10. *Suppose that $n = 1$, (A1)–(A2) holds and $d_S > 0$. Let (S, I) be the solution of (3.9). Then, $\|\nabla S(\cdot,t)\|_{L^2(\Omega)} \to 0$ and $\int_\Omega \beta(S(x,t) - r)^2 I(x,t)dx \to 0$ as $t \to \infty$.*

Proof. Integrating (3.10) over $(0,\infty)$ and by $\int_\Omega I dx \leq N$ and Proposition 3.7, we have

$$\int_0^\infty \|\nabla S(\cdot,t)\|_{L^2(\Omega)}^2 dt < \infty \text{ and } \int_0^\infty \int_\Omega \beta(S(x,t) - r)^2 I(x,t) dx dt < \infty. \tag{3.13}$$

Then the claim follows from Lemmas 2.2 and 3.9. □

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We are ready to state the main result concerning the global dynamics of (3.9). Motivated by the biological meaning and expression of $\mathcal{R}_0^1$, as in [2, 62], we call H^+ and H^- as the high-risk and low-risk sites, respectively, where

$$H^+=\left\{x\in\bar\Omega:\frac{N}{|\Omega|}\beta(x)-\gamma(x)>0\right\}\ \text{ and }\ H^-=\left\{x\in\bar\Omega:\frac{N}{|\Omega|}\beta(x)-\gamma(x)<0\right\}.$$

Define $\tilde r_m=\inf_{x\in\{I_0>0\}} r(x)$ and $\mathcal{M}:=\left\{x\in\overline{\{I_0>0\}}:r(x)=\tilde r_m\right\}$. Biologically, $\mathcal{M}$ consists with all the points of the highest risk relative to I_0. We will show that the infected people will concentrate on $\mathcal{M}$ when limiting their movement.

Theorem 3.11. *Suppose that* (A1)–(A2) *holds and* $d_S>0$. *Let* (S,I) *be the solution of* (3.9). *Then, we have*

$$\lim_{t\to\infty}\|S(\cdot,t)-\overline{S(\cdot,t)}\|_{L^p(\Omega)}=0,\quad \forall\, p\in[1,\infty). \tag{3.14}$$

If in addition $n=1$, *then the limit in* (3.14) *also holds for* $p=\infty$,

$$\lim_{t\to\infty}\|I(\cdot,t)\|_{L^\infty(K\cup\{I_0=0\})}=0 \tag{3.15}$$

for any compact set $K\subset H^-$ *if* H^- *is not empty, and the following conclusions hold:*

(i) If $H^+\cap\{I_0>0\}=\emptyset$, *then* $\|S(\cdot,t)-N/|\Omega|\|_{L^\infty(\Omega)}\to 0$ *and* $\int_\Omega I(x,t)dx\to 0$ *as* $t\to\infty$;

(ii) If $H^+\cap\{I_0>0\}\neq\emptyset$, *then there is a sequence* $\{t_k\}_{k\ge1}$ *converging to infinity such that* $\|S(\cdot,t_k)-\tilde r_m\|_{L^\infty(\Omega)}\to 0$, $\int_\Omega I(x,t_k)dx\to N-|\Omega|\tilde r_m$ *and* $I(\cdot,t_k)\to 0$ *as* $k\to\infty$ *almost everywhere on* $\{x\in\Omega:r(x)\neq\tilde r_m\}$. *In particular, if* $\mathcal{M}=\{x_1,\cdots,x_L\}\subset\{I_0>0\}$, *then* $I(\cdot,t_k)\to(N-|\Omega|\tilde r_m)\sum_{i=1}^L c_i\delta_{x_i}$ *weakly as* $k\to\infty$, *where* $0\le c_i\le 1$, $\sum_{i=1}^L c_i=1$, *and* δ_{x_i} *is the Dirac measure centred at* x_i.

Proof. By the Poincaré inequality, there is a positive constant $\lambda_0>0$ such that

$$\|S(\cdot,t)-\overline{S(\cdot,t)}\|_{L^2(\Omega)}\le\lambda_0\|\nabla S(\cdot,t)\|_{L^2(\Omega)},\quad\forall\, t>0. \tag{3.16}$$

Hence using Hölder's inequality and recalling (3.10), we get that

$$\begin{aligned}\int_0^\infty\|S(\cdot,t)-\overline{S(\cdot,t)}\|^2_{L^1(\Omega)}dt\le&|\Omega|\int_0^\infty\|S(\cdot,t)-\overline{S(\cdot,t)}\|^2_{L^2(\Omega)}dt\\ \le&\lambda_0^2|\Omega|\int_0^\infty\|\nabla S(\cdot,t)\|^2_{L^2(\Omega)}dt<\infty.\end{aligned}$$

Therefore, by Lemmas 2.2 and 3.9, we obtain $\|S(\cdot,t)-\overline{S(\cdot,t)}\|^2_{L^1(\Omega)}\to 0$ as $t\to\infty$. By $\sup_{t\ge1}\|S(\cdot,t)\|_\infty<\infty$ and Hölder's inequality, (3.14) holds.

From this point, we shall suppose that $n=1$ and complete the proof of the theorem. In view of Lemma 3.10 and inequality (3.16), we have

$$\lim_{t\to\infty}\|S(\cdot,t)-\overline{S(\cdot,t)}\|_{W^{1,2}(\Omega)}=0.$$

Since $n=1$, $W^{1,2}(\Omega)$ is compactly embedded into $C(\bar\Omega)$. Therefore,

$$\lim_{t\to\infty}\left\|S(\cdot,t)-\overline{S(\cdot,t)}\right\|_{L^\infty(\Omega)}=0. \tag{3.17}$$

Fix $\varepsilon>0$. By $N=\int_\Omega(S+I)dx$ for all $t\ge0$ and (3.17), there is $t_\varepsilon>0$ such that

$$S(x,t)\le\frac{1}{|\Omega|}\Big(N+\varepsilon-\int_\Omega I(x,t)dx\Big),\quad\forall\,(x,t)\in\bar\Omega\times[t_\varepsilon,\infty). \tag{3.18}$$

Let $K\subset H^-$ be a compact set if H^- is not empty. By the definition of H^-, we have $\min_{x\in K}r(x)>N/|\Omega|$. If $0<\varepsilon\ll1$ is chosen such that $\eta_\varepsilon:=(N+\varepsilon)/|\Omega|-\min_{x\in K}r(x)<0$, then

$$\partial_t I(x,t)\le\beta(x)\left(\frac{N+\varepsilon}{|\Omega|}-\min_{x\in K}r(x)\right)I(x,t)\le\beta_m\eta_\varepsilon I(x,t),\quad (x,t)\in K\times[t_\varepsilon,\infty).$$

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It follows that

$$\|I(\cdot,t)\|_{L^\infty(K)} \le e^{(t-t_\varepsilon)\beta_m\eta_\varepsilon}\|I(\cdot,t_\varepsilon)\|_{L^\infty(K)} \to 0, \quad \text{as } t\to\infty.$$

This together with the fact that $I(x,t)=0$ for all $t\ge 0$ and $x\in\{I_0=0\}$ completes the proof of (3.15).

To prove (i)–(ii), we claim that

$$\limsup_{t\to\infty} \|I(\cdot,t)\|_{L^1(\Omega)} \le (N-|\Omega|\tilde r_m)_+ \,. \tag{3.19}$$

Since $\int_{\{I_0=0\}} I(x,t)dx=0$ for all $t\ge 0$, it suffices to show $\limsup_{t\to\infty}\|I(\cdot,t)\|_{L^1(\{I_0>0\})}\le (N-|\Omega|\tilde r_m)_+$. To see this, observe from (3.18) that

$$\partial_t I \le \frac{\beta}{|\Omega|}\left((N-|\Omega|\tilde r_m)_+ + \varepsilon - \|I(\cdot,t)\|_{L^1(\{I_0>0\})}\right) I, \quad \forall\, t\ge t_\varepsilon,\ x\in\{I_0>0\}, \tag{3.20}$$

where $\tilde r_m = \inf_{x\in\{I_0>0\}} r(x)$. Let

$$F(t) := \frac{\int_{\{I_0>0\}}\beta I(x,t)dx}{\int_{\{I_0>0\}} I(x,t)dx}, \quad \forall\, t\ge 1. \tag{3.21}$$

Then $F(t):[1,\infty)\to\mathbb{R}_+$ is Locally Lipschitz continuous with $\beta_m\le F(t)\le\beta_M$, $t\ge 1$. Integrating (3.20) over $\{I_0>0\}$, for any $t>t_\varepsilon$, we get

$$\begin{aligned}\frac{d}{dt}\|I\|_{L^1(\{I_0>0\})} \le& \frac{1}{|\Omega|}\left((N-|\Omega|\tilde r_m)_+ + \varepsilon - \|I\|_{L^1(\{I_0>0\})}\right)\int_{\{I_0>0\}}\beta I dx\\ =&\frac{F(t)}{|\Omega|}\left((N-|\Omega|\tilde r_m)_+ + \varepsilon - \|I\|_{L^1(\{I_0>0\})}\right)\|I\|_{L^1(\{I_0>0\})},\end{aligned}$$

where F is defined by (3.21). Let $v(t)$ be the solution of

$$\begin{cases} v'(t) \;=\; \dfrac{F(t)}{|\Omega|}\left((N-|\Omega|\tilde r_m)_+ + \varepsilon - v(t)\right)v(t), \quad t>t_\varepsilon,\\[2ex] v(t_\varepsilon) \;=\; \|I(\cdot,t_\varepsilon)\|_{L^1(\{I_0>0\})}.\end{cases}$$

By the comparison principle, we know that

$$\|I(\cdot,t)\|_{L^1(\{I_0>0\})} \le v(t), \quad \forall t\ge t_\varepsilon.$$

Since $v(t)\to (N-|\Omega|\tilde r_m)_+ + \varepsilon$ as $t\to\infty$ (because $F\ge\beta_m>0$ and $v(t_\varepsilon)>0$) and ε is arbitrarily chosen, (3.19) holds.

(i) Suppose that $H^+\cap\{I_0>0\}=\emptyset$. Then we have $(N-|\Omega|\tilde r_m)_+=0$. By (3.19), $\|I(\cdot,t)\|_{L^1(\Omega)}\to 0$ as $t\to\infty$. This together with the fact that $\int_\Omega Sdx = N-\int_\Omega Idx$ for all $t>0$, yields $\int_\Omega S(x,t)dx\to N$ as $t\to\infty$. It then follows from (3.17) that $\|S(\cdot,t)-N/|\Omega|\|_{L^\infty(\Omega)}\to 0$ as $t\to\infty$.

(ii) Suppose that $H^+\cap\{I_0>0\}\neq\emptyset$. Then we have $N/|\Omega|>\tilde r_m$. Since $\int_\Omega Sdx = N-\int_\Omega Idx$, we conclude from (3.19) that

$$\liminf_{t\to\infty}\int_\Omega S(x,t)dx \ge |\Omega|\tilde r_{\min},$$

which in view of (3.17) implies that

$$\liminf_{t\to\infty}\min_{x\in\overline\Omega} S(x,t) \ge \tilde r_m. \tag{3.22}$$

Now, we claim that

$$\liminf_{t\to\infty}\min_{x\in\overline\Omega} S(x,t) = \tilde r_m. \tag{3.23}$$

We proceed by contradiction. Suppose to the contrary that (3.23) is false. Thanks to (3.22), there exist $0<\tilde\varepsilon\ll 1$ and $\tilde t_0>0$ such that

$$S(x,t)\ge \tilde r_m + \tilde\varepsilon, \quad \forall\, t\ge\tilde t_0.$$

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Since r is continuous on $\bar{\Omega}$ and $\tilde{r}_m = \inf_{x\in\{I_0>0\}} r(x)$, there is an open set $\mathcal{O} \subset \{I_0 > 0\}$ such that $r(x) < \tilde{r}_m + \tilde{\varepsilon}/2$ for all $x \in \mathcal{O}$. Hence, we have

$$\partial_t I(x,t) = \beta(S-r)I \geq \frac{\tilde{\varepsilon}}{2}\beta_m I(x,t), \quad t > \tilde{t}_0,\ x \in \mathcal{O}.$$

An integration of this inequality yields

$$N \geq \int_{\mathcal{O}} I(x,t)dx \geq e^{\frac{\tilde{\varepsilon}}{2}\beta_m(t-\tilde{t}_0)} \int_{\mathcal{O}} I(x,\tilde{t}_0)dx, \quad t > \tilde{t}_0.$$

This is clearly impossible since $\int_{\mathcal{O}} I(x,\tilde{t}_0)dx > 0$. Therefore, (3.23) must hold.

By (3.23), there is a sequence $\{t_k\}_{k\geq 1}$ converging to infinity such that $\min_{x\in\overline{\Omega}} S(x,t_k) \to \tilde{r}_m$ as $k \to \infty$. Hence, since

$$\begin{aligned}
\|\tilde{r}_m - S(\cdot,t_k)\|_{L^\infty(\Omega)} \leq & |\tilde{r}_m - \overline{S(\cdot,t_k)}| + \|S(\cdot,t_k) - \overline{S(\cdot,t_k)}\|_{L^\infty(\Omega)} \\
\leq & |\tilde{r}_m - \min_{x\in\overline{\Omega}} S(x,t_k)| + |\min_{x\in\overline{\Omega}} S(x,t_k) - \overline{S(\cdot,t_k)}| + \|S(\cdot,t_k) - \overline{S(\cdot,t_k)}\|_{L^\infty(\Omega)} \\
\leq & |\tilde{r}_m - \min_{x\in\overline{\Omega}} S(x,t_k)| + 2\|S(\cdot,t_k) - \overline{S(\cdot,t_k)}\|_{L^\infty(\Omega)}, \quad \forall\, k \geq 1,
\end{aligned}$$

we conclude from (3.17) that $\|S(\cdot,t_k) - \tilde{r}_m\|_{L^\infty(\Omega)} \to 0$ as $k \to \infty$. This in turn implies that $\int_\Omega I(x,t_k)dx \to N - |\Omega|\tilde{r}_m$ as $k \to \infty$. However, by Lemma 3.10, we know that $\int_{\{I_0>0\}} (S(x,t_k) - r)^2 I(x,t_k)dx = \int_\Omega (S(x,t_k) - r)^2 I(x,t_k)dx \to 0$ as $k \to \infty$. Therefore, possibly after passing to a subsequence, $I(\cdot,t_k) \to 0$ as $k \to \infty$ almost everywhere on $\{x \in \Omega : r(x) \neq \tilde{r}_m\}$. Finally, since $\{I(\cdot,t_k)\}$ is bounded in $L^1(\Omega)$ and by Riesz representation theorem, passing to a subsequence if necessary, $I(\cdot,t_k) \to (N - |\Omega|\tilde{r}_m)\mu$ weakly as $k \to \infty$ for some probability Radon measure μ. Since

$$\int_{\{I_0>0\}} \beta(S(x,t_k) - r)^2 I(x,t_k)dx \to \left(N - |\Omega|\tilde{r}_m\right) \int_{\{I_0>0\}} \beta(r_m - r)^2 d\mu = 0 \ \ as\ k \to \infty,$$

μ is supported in $\overline{\{I_0 > 0\}} \cap \mathcal{M}$. In particular if $\mathcal{M} = \{x_1, \cdots, x_L\} \subset \{I_0 > 0\}$, then $\mu = \sum_{i=1}^L c_i\delta_{x_i}$ for some $0 \leq c_i \leq 1$ with $\sum_{i=1}^L c_i = 1$. □

Remark 3.12. *We conjecture that Theorem 3.11 holds for any $n \geq 1$ and $\|S(\cdot,t) - \tilde{r}_m\|_{L^\infty(\Omega)} \to 0$ as $t \to \infty$ in (ii).*

4. Model with standard incidence mechanism

4.1. Limiting the movement of susceptible people

First, we consider the impact of limiting the movement of susceptible people on (1.1) with standard incidence mechanism by setting $d_S = 0$, i.e.

$$\begin{cases}
\partial_t S = -\beta(x)\dfrac{SI}{S+I} + \gamma(x)I, & x \in \bar{\Omega}, t > 0, \\
\partial_t I = d_I \Delta I + \beta(x)\dfrac{SI}{S+I} - \gamma(x)I, & x \in \Omega, t > 0, \\
\partial_\nu I = 0, & x \in \partial\Omega, t > 0, \\
S(x,0) = S_0(x),\ I(x,0) = I_0(x), & x \in \bar{\Omega}.
\end{cases} \tag{4.1}$$

Proposition 4.1. *Suppose that* (A1)–(A2) *holds and $d_I > 0$. Then* (4.1) *has a unique nonnegative global solution (S, I), where*

$$S \in C^1([0,\infty), C(\overline{\Omega})) \quad \text{and} \quad I \in C([0,\infty), C(\overline{\Omega})) \cap C^1((0,\infty), \mathrm{Dom}_\infty(\Delta)).$$

Moreover, there exists $M > 0$ depending on (the L^∞ norm of) initial data such that

$$\|I(\cdot,t)\|_{L^\infty(\Omega)} \leq M,\ \forall t \geq 0. \tag{4.2}$$

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Proof. If we define $SI/(S+I)=0$ when $(S,I)=(0,0)$, then $SI/(S+I)$ is Lipschitz in the first quadrant. So the existence and uniqueness of nonnegative local solution (S,I) can be proved using the Banach fixed point theorem, where $S\in C^1([0,T),C(\overline{\Omega}))$ and $I\in C([0,T),C(\overline{\Omega}))\cap C^1((0,T),\mathrm{Dom}_\infty(\Delta))$ for some $T>0$.

Since the right-hand side of the second equation of (4.1) has linear growth rate (i.e. $\beta SI/(S+I)-\gamma I\le CI$ for some constant $C>0$) and $\int_\Omega Idx\le N$ for all $t>0$, by [1, Theorem 3.1], there exists $M>0$ such that $\|I(\cdot,t)\|_{L^\infty(\Omega)}\le M$ for all $t>0$. By the first equation of (4.1), we have $\partial_t S\le\gamma I$, which implies

$$\|S(\cdot,t)\|_{L^\infty(\Omega)}\le\|S_0\|_{L^\infty(\Omega)}+M\gamma_M t,\quad t\ge 0.$$

Hence, we can extend the solution globally. □

We define H^+, H^0, and H^- as the high-risk, moderate-risk and low-risk sites, respectively, where $H^+=\left\{x\in\bar{\Omega}:\beta(x)-\gamma(x)>0\right\}$, $H^0=\left\{x\in\bar{\Omega}:\beta(x)-\gamma(x)=0\right\}$ and $H^-=\left\{x\in\bar{\Omega}:\beta(x)-\gamma(x)<0\right\}$. The following result has appeared in the literature.

Theorem 4.2 ([34, Theorem 2.5, Lemma 5.6]). *Suppose that* (A1)–(A2) *holds and* $d_I>0$. *Let* (S,I) *be the solution of* (4.1). *Then the following conclusions hold:*

(i) If H^- is nonempty, then $\lim_{t\to\infty}\|S(\cdot,t)-S^*\|_{L^\infty(\Omega)}=0$ and $\lim_{t\to\infty}\|I(\cdot,t)\|_{L^\infty(\Omega)}=0$, where $S^*\in C(\bar{\Omega})$ satisfies $S^*>0$ on $H^-\cup H^0$.

(ii) If $\beta(x)>\gamma(x)$ for all $x\in\bar{\Omega}$, then $\lim_{t\to\infty}\|S(t,\cdot)-\gamma I^*/(\beta-\gamma)\|_{L^\infty(\Omega)}=0$ and $\lim_{t\to\infty}\|I(t,\cdot)-I^*\|_{L^\infty(\Omega)}=0$, where I^* is a constant given by

$$I^*:=\frac{N}{\int_\Omega\frac{\beta}{\beta-\gamma}dx}.\tag{4.3}$$

Remark 4.3. *The only case not covered by Theorem 4.2 is when $\beta\ge\gamma$ and H^0 is not empty, which we will deal with later in this section. In the case $H^-\ne\emptyset$ and $\mathcal{R}_0>1$, it has been shown in* [34, Theorem 2.5] *that $J^*:=\{x\in\bar{\Omega}:S^*(x)=0\}$ is a subset of H^+ such that both J^* and $\Omega\backslash J^*$ have positive measure, where $\mathcal{R}_0$, defined as*

$$\mathcal{R}_0=\sup\left\{\frac{\int_\Omega\beta\phi^2dx}{\int_\Omega(d_I|\nabla\phi|^2+\gamma\phi^2)dx}:\phi\in H^1(\Omega)\backslash\{0\}\right\},\tag{4.4}$$

is the basic reproduction number of the diffusive epidemic model (1.1) *with standard infection incidence mechanism and diffusion rates $d_S>0$ and $d_I>0$.*

Theorem 4.4. *Suppose that (A1)–(A2) holds and $d_I>0$. Let (S,I) be the solution of* (4.1). *If $\beta(x)\ge\gamma(x)$ for all $x\in\bar{\Omega}$ and H^0 is nontrivial, then the following conclusions hold:*

(i) If H^0 has positive measure, then there exists $M>0$ depending on (the L^∞ norm of) initial data such that

$$\|S(\cdot,t)\|_{L^\infty(\Omega)}\le M,\quad\text{for all }\ t\ge 0.\tag{4.5}$$

Moreover, there is $S^*\in L^\infty(\Omega)$ satisfying $S^*_{|H^0}\in C(H^0)$ and $S^*_{|H^0}>0$ on H^0 such that $\lim_{t\to\infty}\|S(\cdot,t)-S^*\|_{L^\infty(H^0)}=0$, $\lim_{t\to\infty}\|S(\cdot,t)-S^*\|_{L^1(H^+)}=0$, and $\lim_{t\to\infty}\|I(\cdot,t)\|_{L^\infty(\Omega)}=0$. In addition, if $H^+\ne\emptyset$, then $\mathrm{meas}(\{x\in H^+\mid S^*(x)=0\})>0$.

(ii) If H^0 has zero measure and $\int_\Omega 1/(\beta-\gamma)dx=\infty$, then there exists $\{t_k\}$ converging to infinity such that $I(\cdot,t_k)\to 0$ in $C(\bar{\Omega})$ and $\int_\Omega S(\cdot,t_k)dx\to N$ as $k\to\infty$.

Proof. (i) By Lemma 2.3, there exists $C>1$ such that

$$\max_{x\in\bar{\Omega}}I(x,t)\le C\min_{x\in\bar{\Omega}}I(x,t),\quad x\in\bar{\Omega},t\ge 1.\tag{4.6}$$

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Fix $x_0 \in H^0$. Let $x \in \bar{\Omega}$. Then $\partial_t S = \frac{(-\beta+\gamma)S+\gamma I}{S+I} I \le \gamma I^2/(S+I)$. We consider the following problem:

$$\begin{cases} \bar{S}' = \gamma(x)\dfrac{I^2(x,t)}{\bar{S}+I(x,t)}, & t>1, \\ \bar{S}(1) = S(x,1). \end{cases} \tag{4.7}$$

Then we have $S(x,t) \le \bar{S}(t)$ for all $t \ge 1$. We claim that $KS(x_0,t)$ is an upper solution of (4.7) if $K>0$ is large enough. To see it, it suffices to check

$$K\partial_t S(x_0,t) = K\gamma(x_0)\frac{I^2(x_0,t)}{S(x_0,t)+I(x_0,t)} \ge \gamma(x)\frac{I^2(x,t)}{KS(x_0,t)+I(x,t)}.$$

Noticing (4.6), we only need to check

$$K\gamma(x_0)\frac{I^2(x,t)/C^2}{S(x_0,t)+\frac{I(x,t)}{C}} \ge \gamma(x)\frac{I^2(x,t)}{KS(x_0,t)+I(x,t)},$$

which is equivalent to $(K^2\gamma(x_0)-C^2\gamma(x))S(x_0,t)+(K\gamma(x_0)-C\gamma(x))I(x,t) \ge 0$. So we can choose K large independent of $x \in \bar{\Omega}$ such that the inequality holds. Hence, we have $KS(x_0,t) \ge \bar{S}(t) \ge S(x,t)$ for all $t \ge 1$. Moreover, interchanging the role of x_0 and x, we have

$$S(x_0,t)/K \le S(x,t) \le KS(x_0,t), \quad \forall x \in H^0,\ t \ge 1. \tag{4.8}$$

By (4.8), $N \ge \int_\Omega S(x,t)dx \ge \int_{H^0} S(x,t)dx \ge \int_{H^0} S(x_0,t)/Kdx = |H^0|S(x_0,t)/K$ for all $t \ge 1$. Therefore, we have $S(x_0,t) \le KN/|H^0|$ and $S(x,t) \le K^2N/|H^0|$ for all $x \in \bar{\Omega}$ and $t \ge 1$.

The convergence of (S,I) can be proved similar to [34, Lemma 5.6], and we include it for completeness. By Proposition 4.1, we have $0 \le S(x,t), I(x,t) \le M$ for all $x \in \bar{\Omega}$ and $t \ge 0$. It follows from the equation of S that

$$\partial_t S = \frac{\gamma I^2}{S+I} \ge \frac{\gamma_m}{2M}\min_{y\in H^0} I^2(y,t), \quad \forall\, x \in H^0, t>0.$$

Integrating the above inequality over $H^0 \times (0,\infty)$ and noticing that H^0 has positive measure, we see that $\int_0^\infty \min_{x\in H^0} I^2(x,t)dx < \infty$. By (4.6), it holds that

$$\int_0^\infty \|I(\cdot,t)\|^2_{L^\infty(\Omega)}dt < \infty. \tag{4.9}$$

By (4.2) and the L^p estimate, I is Hölder continuous on $\bar{\Omega} \times [1,\infty)$. Therefore, Lemma 2.2 and (4.9) imply that $I(x,t) \to 0$ uniformly on $\bar{\Omega}$ as $t \to \infty$.

Since $\partial_t S = \gamma I^2/(S+I)$, $S(x,t)$ is strictly increasing in $t \in (0,\infty)$ for every $x \in H^0$ and

$$\begin{aligned}\int_1^\infty \|S_t(\cdot,t)\|_{L^\infty(H^0)}dt &\le \gamma_M \int_1^\infty \frac{\|I(\cdot,t)\|^2_{L^\infty(\Omega)}}{\min_{x\in H^0} S(x,t)}dt \\ &\le \frac{\gamma_M}{\min_{x\in H^0} S(x,1)}\int_1^\infty \|I(\cdot,t)\|^2_{L^\infty(\Omega)}dt < \infty.\end{aligned} \tag{4.10}$$

Whence, $S(\cdot,t) \to S^*_{|H^0} := S(\cdot,0) + \int_0^\infty S_t(\cdot,t)dt \in C(H^0)$ uniformly on H^0 as $t \to \infty$.

Next, we discuss the convergence of $S(x,t)$ as $t \to \infty$ for $x \in H^+$. To this end, let $\kappa := (\beta-\gamma)/\beta$ and define

$$V(S,I) = \frac{1}{2}\int_\Omega \left(\kappa S^2 + I^2\right) dx.$$

It is easy to check that

$$\frac{d}{dt}V(S,I) = -d_I\int_\Omega |\nabla I|^2 dx - \int_\Omega \gamma\frac{(\kappa S - I)^2}{S+I}I dx. \tag{4.11}$$

Integrating (4.11) over $(0, t)$ and taking $t \to \infty$, we obtain

$$\int_0^\infty \int_\Omega \gamma \frac{(\kappa S - I)^2}{S + I} I dx dt < \infty. \tag{4.12}$$

On the other hand, we have

$$\frac{1}{2}\left(\kappa S^2\right)_t = \gamma \frac{(I - \kappa S)\kappa SI}{S + I} = \gamma \frac{(I - \kappa S)(\kappa S - I + I)I}{S + I} = -\gamma \frac{(I - \kappa S)^2 I}{S + I} + \gamma \frac{(I - \kappa S)}{S + I} I^2.$$

Hence, by (4.9) and (4.12), we have that

$$\begin{aligned}\int_1^\infty \left\| \frac{\left(\kappa S^2\right)_t}{2} \right\|_{L^1(H^+)} dt &\le \int_1^\infty \int_\Omega \gamma \frac{(I - \kappa S)^2 I}{S + I} dx dt + \int_1^\infty \int_\Omega \gamma \frac{|I - \kappa S|}{S + I} I^2 dx dt \\ &\le \int_1^\infty \int_\Omega \gamma \frac{(I - \kappa S)^2 I}{S + I} dx dt + \gamma_M (1 + \|\kappa\|_\infty)|\Omega| \int_1^\infty \|I\|^2_{L^\infty(\Omega)} dt \\ &< \infty.\end{aligned}$$

Therefore, there is a measurable subset $\mathcal{N} \subset H^+$ with meas$(\mathcal{N}) = 0$ such that

$$\int_1^\infty \left| \frac{\left(\kappa S^2\right)_t}{2} \right| dt < \infty, \quad \forall\, x \in H^+ \setminus \mathcal{N}.$$

As a result, we have that $\kappa(x)S^2(x, t) \to \kappa(x)(S_0^2(x) + \int_0^\infty (S^2)_t dt)$ as $t \to \infty$ for $x \in H^+ \setminus \mathcal{N}$. So, $S(x, t) \to S^*_{H^+}(x) := \sqrt{S_0^2(x) + \int_0^\infty (S^2)_t dt}$ as $t \to \infty$ for $x \in H^+ \setminus \mathcal{N}$. Since $\|S(\cdot, t)\|_{L^\infty(\Omega)} \le M$ for all $t \ge 1$, then $S^*_{|H^+} \in L^\infty(H^+)$, and by the Lebesgue dominated theorem and the fact that meas$(\mathcal{N}) = 0$, $\|S(\cdot, t) - S^*_{|H^+}\|_{L^1(H^+)} \to 0$ as $t \to \infty$. Now, taking $S^* := S^*_{|H^0}\chi_{H^0} + S^*_{|H^+}\chi_{H^+}$, then $S^* \in L^\infty(\Omega)$, $S^* \in C(H^0)$, $S^* > 0$ on H^0, and $\|S(\cdot, t) - S^*\|_{L^\infty(H^0)} + \|S(\cdot, t) - S^*\|_{L^1(H^+)} \to 0$ as $t \to \infty$.

Finally, we suppose that $H^+ \neq \emptyset$ and proceed by contradiction to show that

$$\text{meas}(\{x \in H^+ \mid S^*(x) = 0\}) > 0. \tag{4.13}$$

So, suppose to the contrary that (4.12) does not hold. Consider the function

$$F(x, t) = \frac{1}{t} \int_0^t \frac{I(x, s)}{S(x, s) + I(x, s)} ds, \quad \forall\, x \in \Omega,\ t > 0.$$

It is clear that

$$0 \le F(x, t) \le 1, \quad \forall\, x \in \Omega,\ t > 0.$$

Moreover, for each $x \in \Omega$ satisfying $S^*(x) > 0$, it holds that

$$\lim_{t\to\infty} F(x, t) = \lim_{t\to\infty} \frac{I(x, t)}{S(x, t) + I(x, t)} = 0.$$

Hence, since $S^* > 0$ on H^0 and meas$(\{x \in H^+ \mid S^*(x) = 0\}) = 0$, it follows from the Lebesgue dominated convergence theorem that

$$\lim_{t\to\infty} \int_\Omega F(x, t) dx = 0. \tag{4.14}$$

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Let φ be the positive eigenfunction associated with $\sigma(d_I, \beta-\gamma)$ satisfying $\max_{x\in\bar\Omega}\varphi(x)=1$. By the second equation of (4.1) and (4.6), we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega \varphi I dx =& \int_\Omega d_I\varphi\Delta I dx+\int_\Omega\left(\frac{\beta S}{S+I}-\gamma\right)\varphi I dx\\ =& d_I\int_\Omega I(\cdot,t)\Delta\varphi dx+\int_\Omega\left(\frac{\beta S}{S+I}-\gamma\right)\varphi I dx\\ =& \sigma(d_I,\beta-\gamma)\int_\Omega\varphi I dx+\int_\Omega\beta\left(\frac{S}{S+I}-1\right)\varphi I dx\\ =& \sigma(d_I,\beta-\gamma)\int_\Omega\varphi I dx-\int_\Omega\beta\frac{I}{S+I}\varphi I dx\\ \geq& \sigma(d_I,\beta-\gamma)\int_\Omega\varphi I dx-\beta_M\varphi_M\|I\|_{L^\infty(\Omega)}\int_\Omega\frac{I}{S+I}dx\\ \geq& \left(\sigma(d_I,\beta-\gamma)-\frac{\beta_M C\varphi_M}{\varphi_m}\int_\Omega\frac{I}{S+I}dx\right)\int_\Omega\varphi I dx.\end{aligned}$$

By the comparison principle for the ODE, we obtain that

$$\int_\Omega\varphi(x)I(x,t)dx\geq e^{t\left(\sigma(d_I,\beta-\gamma)-M^*\int_\Omega F(x,t)dx\right)}\int_\Omega\varphi(x)I_0(x)dx,\quad t>0, \tag{4.15}$$

where $M^*:=\beta_M C\varphi_M/\varphi_m$. Note that $\sigma(d_I,\beta-\gamma)>0$, since $\beta\geq\gamma$ and $H^+\neq\emptyset$. Hence, it follows from (4.14)–(4.15) that $\int_\Omega\varphi I(\cdot,t)dx\to\infty$ as $t\to\infty$. This contradicts the fact that $\sup_{t\geq 1}\|I(\cdot,t)\|_{L^\infty(\Omega)}<\infty$. Therefore, (4.13) holds.

(ii) Integrating (4.11) over $(0,t)$ and taking $t\to\infty$, we find that

$$\int_0^\infty\int_\Omega|\nabla I|^2 dxdt<\infty. \tag{4.16}$$

By (4.2), the parabolic estimates and the Sobolev embedding theorem, $I\in C^{1+\alpha,1+\alpha/2}(\bar\Omega\times[1,\infty)])$. So by (4.16) and Lemma 2.2, $\|\nabla I(\cdot,t)\|_{L^2(\Omega)}\to 0$ as $t\to\infty$. Moreover, let $\omega_I:=\cap_{t\geq 1}\overline{\cup_{s\geq t}I(\cdot,s))}$, where the completion is in $C(\bar\Omega)$. Then ω_I is well defined, compact and consists with constants.

Suppose to the contrary that $0\notin\omega_I$. By the compactness of ω_I, there exists $\varepsilon_0>0$ such that

$$\liminf_{t\to\infty}\min_{x\in\bar\Omega}I(x,t)>\varepsilon_0. \tag{4.17}$$

By the first equation of (4.1), we have

$$\partial_t S=\frac{(\gamma I-(\beta-\gamma)S)I}{S+I}. \tag{4.18}$$

This combined with (4.17) implies that $\liminf_{t\to\infty}S(x,t)\geq\varepsilon_0\gamma/(\beta-\gamma)$ pointwise for $x\in\Omega\backslash H^0$ as $t\to\infty$. By Fatou's Lemma, we have

$$\int_{\Omega\backslash H^0}\left(\frac{\varepsilon_0\gamma}{\beta-\gamma}+\varepsilon_0\right)dx\leq\int_\Omega\liminf_{t\to\infty}(S+I)dx\leq\liminf_{t\to\infty}\int_\Omega(S+I)dx=N.$$

Since H^0 has measure zero and $\int_\Omega 1/(\beta-\gamma)dx=\infty$, the first term in the above inequality equals infinity. This is a contradiction, and therefore $0\in\omega_I$. Hence, there exists $\{t_k\}$ converging to infinity such that $I(\cdot,t_k)\to 0$ in $C(\bar\Omega)$ as $k\to\infty$. Thus, $\int_\Omega S(\cdot,t_k)dx=N-\int_\Omega I(\cdot,t_k)dx\to N$ as $k\to\infty$. □

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4.2. Limiting the movement of infected people

Then, we consider the impact of limiting the movement of infected people on (1.1) with standard incidence mechanism by setting $d_I = 0$, i.e.

$$\begin{cases} \partial_t S = d_S \Delta S - \beta(x)\dfrac{SI}{S+I} + \gamma(x)I, & x \in \Omega, t > 0, \\ \partial_t I = \beta(x)\dfrac{SI}{S+I} - \gamma(x)I, & x \in \bar{\Omega}, t > 0, \\ \partial_\nu S = 0, & x \in \partial\Omega, t > 0, \\ S(x,0) = S_0(x),\ I(x,0) = I_0(x), & x \in \bar{\Omega}. \end{cases} \tag{4.19}$$

We will establish a prior bound for the solution of (4.19) first.

Proposition 4.5. *Suppose that (A1)–(A2) holds and $d_S > 0$. Then (4.19) has a unique nonnegative global solution (S, I), where*

$$S \in C([0,\infty), C(\overline{\Omega})) \cap C^1((0,\infty), \mathrm{Dom}_\infty(\Delta)) \quad \text{and} \quad I \in C^1([0,\infty), C(\overline{\Omega})).$$

Moreover, there exists $M > 0$ depending on (the L^∞ norm of) initial data such that

$$\|S(\cdot,t)\|_{L^\infty(\Omega)}, \|I(\cdot,t)\|_{L^\infty(\Omega)} \le M, \quad \text{for all } t \ge 0. \tag{4.20}$$

Proof. Similar to Proposition 4.1, (4.19) has a unique local nonnegative solution (S, I), where $S \in C([0,T), C(\overline{\Omega})) \cap C^1((0,T), \mathrm{Dom}_\infty(\Delta))$ and $I \in C^1([0,T), C(\overline{\Omega}))$ for some $T > 0$. It remains to show the boundedness of the solution.

We claim that for any nonnegative integer k there exists $C > 0$ depending on initial data such that $\|S(\cdot,t)\|_{L^{2^k}(\Omega)}, \|I(\cdot,t)\|_{L^{2^k}(\Omega)} \le C$ for all $t \ge 0$. We prove this claim by induction. It is easy to see that the claim holds for $k = 0$. Now we assume that the claim holds for k and will show that it holds for $k+1$. To see it, multiplying both sides of the second equation of (4.19) by $I^{2^{k+1}-1}$ and integrating over Ω, we obtain

$$\begin{aligned} \frac{1}{2^{k+1}}\frac{d}{dt}\int_\Omega I^{2^{k+1}}dx &\le \beta_M \int_\Omega \frac{SI^{2^{k+1}-1}I}{S+I}dx - \gamma_m \int_\Omega I^{2^{k+1}}dx \\ &\le C_0 \int_\Omega S^{2^{k+1}}dx - \frac{\gamma_m}{2}\int_\Omega I^{2^{k+1}}dx, \end{aligned} \tag{4.21}$$

where we have used Young's inequality in the last step.

Multiplying both sides of the first equation of (4.19) by $S^{2^{k+1}-1}$ and integrating over Ω, we obtain

$$\begin{aligned} \frac{1}{2^{k+1}}\frac{d}{dt}\int_\Omega S^{2^{k+1}}dx &= -\frac{2^{k+1}-1}{2^{k+1}}d_S\int_\Omega |\nabla S^{2^k}|^2dx - \int_\Omega \frac{\beta S^{2^{k+1}}I}{S+I}pdx + \int_\Omega \gamma S^{2^{k+1}-1}Idx \\ &\le -C_1\int_\Omega |\nabla S^{2^k}|^2dx + C_2\left(\int_\Omega S^{2^{k+1}}dx + \int_\Omega I^{2^{k+1}}dx\right), \end{aligned} \tag{4.22}$$

where we have used Young's inequality in last step.

Multiplying (4.22) by $C_3 := \dfrac{\gamma_m}{4C_2}$ and summing up with (4.21), we have

$$\frac{d}{dt}\int_\Omega \left(C_3S^{2^{k+1}} + I^{2^{k+1}}\right)dx \le -C_4\int_\Omega |\nabla S^{2^k}|^2dx + C_5\int_\Omega S^{2^{k+1}}dx - C_6\int_\Omega I^{2^{k+1}}dx.$$

By the following interpolation inequality

$$\|u\|^2_{L^2(\Omega)} \le \epsilon\|\nabla u\|^2_{L^2(\Omega)} + C_\epsilon\|u\|^2_{L^1(\Omega)}, \quad \forall u \in H^1(\Omega),$$

we obtain

$$\frac{d}{dt}\int_\Omega \left(C_3S^{2^{k+1}} + I^{2^{k+1}}\right)dx \le C_7\left(\int_\Omega S^{2^k}dx\right)^2 - C_8\int_\Omega \left(C_3S^{2^{k+1}} + I^{2^{k+1}}\right)dx. \tag{4.23}$$

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By the assumption that the claim holds for k and (4.23) there exists $C>0$ such that $\|S(\cdot,t)\|_{L^{2^{k+1}}(\Omega)}, \|I(\cdot,t)\|_{L^{2^{k+1}}(\Omega)} \le C$. This proves the claim.

Fixing $p>N+2$, by the claim and the parabolic L^p estimate, there exists $C>0$ such that $\|S\|_{W_p^{2,1}(\Omega\times(\tau,\tau+1))} < C$ for any $\tau>0$. Since $W_p^{2,1}(\Omega\times(\tau,\tau+1))$ can be embedded into $C(\bar\Omega\times[\tau,\tau+1])$, we obtain the boundedness of S. We rewrite the equation of I as

$$\partial_t I = \frac{((\beta-\gamma)S-\gamma I)I}{S+I}. \tag{4.24}$$

Hence, $\|I(\cdot,t)\|_{L^\infty(\Omega)} \le \max_{0\le s\le t}\{\|I_0\|_{L^\infty(\Omega)},\ \|(\beta+\gamma)S(\cdot,s)\|_{L^\infty(\Omega)}/\gamma_m\}$ for any $t\ge 0$. This proves the boundedness of I. □

We are ready to study the asymptotic behaviour of the solution of (4.19). Let

$$\kappa = \frac{\gamma}{(\beta-\gamma)}\chi_{H^+\cap\{I_0>0\}}$$

and define

$$V(S,I) = \frac{1}{2}\int_\Omega \left(S^2+\kappa I^2\right)dx.$$

It is easy to check that

$$\begin{aligned}\dot V(S,I) =& -d_S\int_\Omega |\nabla S|^2dx + \int_{H^-\cup H^0\cup\{I_0=0\}} S\left(-\beta\frac{SI}{S+I}+\gamma I\right)dx\\ &-\int_{H^+\cap\{I_0>0\}}(\beta-\gamma)_+\frac{(S-\kappa I)^2}{S+I}Idx.\end{aligned} \tag{4.25}$$

The function V does not satisfy $\dot V\le 0$ (the second term on the right-hand side of (4.25) is positive), but it still enables us to conclude the convergence of the solution.

Theorem 4.6. *Suppose that (A1)–(A2) holds and $d_S>0$. Let (S,I) be the solution of (4.19). Then the following statements hold:*

(i) There is a positive number $\overline{S}$ such that

$$\sup_{t\ge0}\left(\|S(\cdot,t)\|_{L^\infty(\Omega)}+\|I(\cdot,t)\|_{L^\infty(\Omega)}\right)\le \overline{S}. \tag{4.26}$$

Furthermore, there exist $t_1>0$ and $\underline{S}>0$, independent of initial data, such that

$$\underline{S}\le \min_{x\in\overline{\Omega}} S(x,t)\le \max_{x\in\overline{\Omega}} S(x,t)\le\overline{S},\quad \forall\, t\ge t_1, \tag{4.27}$$

and

$$\begin{aligned}(R-1)_+\underline{S}\chi_{\{I_0>0\}} &\le \liminf_{t\to\infty} I(x,t)\\ &\le \limsup_{t\to\infty} I(x,t)\le (R-1)_+\overline{S}\chi_{\{I_0>0\}},\quad \forall\, x\in\bar\Omega,\end{aligned} \tag{4.28}$$

where $R:=\beta/\gamma$.

(ii) If $1/(\beta-\gamma)\in L^1(H^+\cap\{I_0>0\})$, then

$$\lim_{t\to\infty}(S(x,t),I(x,t)=(S^*,I^*)),\quad \text{uniformly for } x\in\bar\Omega,$$

where

$$I^* = \frac{(\beta-\gamma)_+S^*}{\gamma}\chi_{H^+\cap\{I_0>0\}}$$

and S^* is a positive constant given by

$$S^* = \frac{N}{|\Omega|+\int_{H^+\cap\{I_0>0\}}\frac{(\beta-\gamma)_+}{\gamma}dx}. \tag{4.29}$$

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Proof. (i) Note that (4.26) follows from Proposition 4.5. Next, observe that

$$\begin{aligned}\frac{d}{dt}\int_\Omega Sdx=&\int_\Omega \gamma Idx-\int_\Omega \beta\frac{I}{I+S}Sdx\\ \geq&\gamma_m\int_\Omega Idx-\beta_M\int_\Omega Sdx\\ =&\gamma_m\left(N-\int_\Omega Sdx\right)-\beta_M\int_\Omega Sdx\\ =&\gamma_mN-(\beta_M+\gamma_m)\int_\Omega Sdx.\end{aligned}$$

Therefore, by $\int_\Omega S_0dx\geq 0$ and the comparison principle, we have that

$$\int_\Omega S(x,t)dx\geq\frac{\gamma_mN}{\beta_M+\gamma_m}\left(1-e^{-t(\beta_M+\gamma_m)}\right),\quad \forall\, t\geq 0. \tag{4.30}$$

Next by (4.19), we see that

$$\begin{cases}\partial_tS\geq d_S\Delta S-\beta_MS, & x\in\Omega, t>0,\\ \partial_\nu S=0, & x\in\partial\Omega,\ t>0.\end{cases}$$

For any $t_0>0$, it follows from the comparison principle for parabolic equations that

$$S(\cdot,t+t_0)\geq e^{-t\beta_M}e^{td_S\Delta}S(\cdot,t_0),\quad \forall\, t>0. \tag{4.31}$$

Thanks to the Harnack's inequality (see Lemma 2.3), there is a positive constant c_0 such that

$$e^{td_S\Delta}S(\cdot,t_0)(x)\geq c_0e^{td_S\Delta}S(\cdot,t_0)(y),\quad \forall\, td_S\geq 1,\ x,y\in\bar{\Omega},\ t_0>0. \tag{4.32}$$

This in turn together with (4.30) implies that

$$\begin{aligned}\min_{x\in\overline{\Omega}}e^{d_St\Delta}S(\cdot,t_0)(x)\geq&\frac{c_0}{|\Omega|}\int_\Omega e^{td_S\Delta}S(\cdot,t_0)(y)dy\\ =&\frac{c_0}{|\Omega|}\int_\Omega S(y,t_0)dy\\ \geq&\frac{\gamma_mNc_0}{|\Omega|(\beta_M+\gamma_m)}\left(1-e^{-t_0(\beta_M+\gamma_m)}\right),\qquad t\geq\frac{1}{d_S},\ t_0>0.\end{aligned}$$

As a result, it follows from (4.31) that

$$S\left(x,\frac{1}{d_S}+t_0\right)\geq\frac{\gamma_mNc_0e^{-\frac{\beta_M}{d_S}}}{|\Omega|(\beta_M+\gamma_m)}\left(1-e^{-t_0(\beta_M+\gamma_m)}\right),\qquad x\in\bar{\Omega}, t_0>0.$$

Therefore, taking $t_1=\frac{1}{d_S}+\frac{\ln(2)}{\beta_M+\gamma_m}$, (4.27) holds with $\underline{S}:=\frac{\gamma_mNc_0e^{-\frac{\beta_M}{d_S}}}{2|\Omega|(\beta_M+N\gamma_m)}>0$, where $\underline{S}$ and t_1 are independent of the initial data.

Next, we show that (4.28) holds. To this end, we first note that

$$\begin{aligned}I_t=\beta\Big((1-r)-\frac{I}{I+S}\Big)I\leq&\beta\Big((1-r)-\frac{I}{I+\overline{S}}\Big)I\\ =&\gamma\Big((R-1)\overline{S}-I\Big)\frac{I}{I+\overline{S}},\qquad t>0,\end{aligned}$$

which in view of the comparison principle for ordinary differential equations implies that $\limsup_{t\to\infty}I(x,t)\leq(R-1)_+\overline{S}$. Recalling that $I(x,t)=0$ for all $t>0$ whenever $I_0(x)=0$, we then conclude that $\limsup_{t\to\infty}I(x,t)\leq(R-1)_+\overline{S}\chi_{\{I_0>0\}}$.

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Similarly, observing that

$$I_t = \beta\left(\left(1-\frac{\gamma}{\beta}\right)-\frac{I}{I+S}\right)I \geq \beta\left((1-r)-\frac{I}{I+\underline{S}}\right)I$$
$$=\gamma\left((R-1)\underline{S}-I\right)\frac{I}{I+\underline{S}}, \quad t>t_1,$$

we can proceed as in the previous case to establish that $\liminf_{t\to\infty} I(x,t) \geq (R-1)_+\underline{S}\chi_{\{I_0>0\}}$, which completes the proof of (i).

(ii) By the equation of I, we have

$$\int_0^t \int_{H^-\cup H^0\cup\{I_0=0\}} \left(\beta\frac{SI}{S+I}-\gamma I\right)dxds = \int_{H^-\cup H^0\cup\{I_0=0\}} I(x,t)dx - \int_{H^-\cup H^0\cup\{I_0=0\}} I_0 dx.$$

Note that

$$\beta\frac{SI}{S+I}-\gamma I = \frac{((\beta-\gamma)S-\gamma I)I}{S+I} \leq 0, \quad \forall x\in H^-\cup H^0\cup\{I_0=0\}.$$

We have

$$0\leq \int_0^\infty \int_{H^-\cup H^0\cup\{I_0=0\}} \left(-\beta\frac{SI}{S+I}+\gamma I\right)dxds < \infty,$$

which together with (4.26) yields that

$$0\leq \int_0^\infty \int_{H^-\cup H^0\cup\{I_0=0\}} S\left(-\beta\frac{SI}{S+I}+\gamma I\right)dxdt < \infty.$$

Hence, integrating (4.25) over $(0,\infty)$, we obtain that

$$\int_0^\infty \int_\Omega |\nabla S|^2 dxdt < \infty \tag{4.33}$$

and

$$\int_0^\infty \int_{H^+\cap\{I_0>0\}} (\beta-\gamma)_+ I\frac{(S-\kappa I)^2}{S+I}dxdt < \infty. \tag{4.34}$$

By (4.26), we have

$$\sup_{t\geq 0}\left\|\frac{\beta SI}{S+I}-\gamma I\right\|_{L^\infty(\Omega)} < \infty.$$

So by the regularity theory for parabolic equations and Sobolev embedding theorem, the mappings $\nabla S(x,t)$ and S are Hölder continuous on $\overline{\Omega}\times[1,\infty)$, and $\{S(\cdot,t)\}_{t\geq 1}$ is precompact in $C^{1+\alpha}(\Omega)$, $0<\alpha<1$. Then by (4.33) and Lemma 2.2, $\int_\Omega |\nabla S|^2 dx \to 0$ as $t\to\infty$. Hence, the set $w_S := \cap_{t\geq 1}\overline{\cup_{s\geq t}\{S(\cdot,s)\}}$ consists of positive constant functions. Furthermore, since $\sup_{t\geq 0}\|\partial_t I(\cdot,t)\|_{L^\infty(\Omega)} < \infty$ and S is Hölder continuous on $\overline{\Omega}\times[1,\infty)$, the mapping

$$t\mapsto \int_{H^+\cap\{I_0>0\}} (\beta-\gamma)_+\frac{(S-\kappa I)^2}{S+I}Idx$$

is Hölder continuous on $[1,\infty)$. Therefore, by Lemma 2.2 and (4.34), we have

$$\lim_{t\to\infty}\int_{H^+\cap\{I_0>0\}} (\beta-\gamma)_+\frac{(S-\kappa I)^2}{S+I}Idx = 0. \tag{4.35}$$

Now, let $S^*\in w_S$. Then there is a sequence $t_k\to\infty$ such that $S(\cdot,t_k)\to S^*$ uniformly on $\bar{\Omega}$ as $k\to\infty$. Since (4.35) holds, after passing to a subsequence if necessary, we may suppose that

$$\lim_{k\to\infty} I(x,t_k)\frac{(S^*-\kappa(x)I(x,t_k))^2}{S^*+I(x,t_k)} = 0 \quad \text{a.e. on } H^+\cap\{I_0>0\}. \tag{4.36}$$

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However, when $x \in H^+ \cap \{I_0 > 0\}$, we have from (4.28) that $\liminf_{t\to\infty} I(x,t) \geq \underline{S}(\beta(x) - \gamma(x)) > 0$. Therefore, we conclude from (4.36) that

$$\lim_{k\to\infty} I(x,t_k) = \frac{S^*}{\kappa(x)} = \frac{(\beta(x)-\gamma(x))S^*}{\gamma(x)} \quad \text{a.e. on } H^+ \cap \{I_0 > 0\}.$$

By (4.28) again, $\lim_{t\to\infty} I(x,t) = 0$ almost everywhere for $x \in \Omega \setminus (H^+ \cap \{I_0 > 0\})$. So by the dominated convergence theorem, we have

$$\begin{aligned} N &= \lim_{k\to\infty} \int_\Omega (S(\cdot,t_k) + I(\cdot,t_k))dx \\ &= |\Omega|S^* + S^* \int_{H^+\cap\{I_0>0\}} \frac{1}{\kappa}dx = \left(|\Omega| + \int_{H^+\cap\{I_0>0\}} \frac{\beta-\gamma}{\gamma}dx\right) S^*. \end{aligned}$$

This yields that

$$S^* = \frac{N}{|\Omega| + \int_{H^+\cap\{I_0>0\}} \frac{\beta-\gamma}{\gamma}dx}.$$

Since S^* is independent of the chosen subsequence, we have

$$w_S = \left\{ \frac{N}{|\Omega| + \int_{H^+\cap\{I_0>0\}} \frac{\beta-\gamma}{\gamma}dx} \right\},$$

and therefore (4.29) holds. Since $S(\cdot,t) \to S^*$ uniformly on Ω as $t\to\infty$, we obtain from (4.24) that $I(\cdot,t) \to \frac{S^*(\beta-\gamma)^+}{\gamma}\chi_{H^+\cap\{I_0>0\}}$ uniformly on $\bar{\Omega}$ as $t\to\infty$. □

5. Simulations

In this section, we run numerical simulations to illustrate the results. Let $\Omega = [0,1]$, $S_0 = 2 + \cos(\pi x)$ and $I_0 = 1.5 + \cos(\pi x)$. Then the total population is $N = \int_\Omega (S_0 + I_0)dx = 3.5$.

5.1. Mass action mechanism

We first simulate the models with mass action mechanism.

5.1.1. Simulation 1: control the movement of susceptible people

Let $d_S = 0$, $d_I = 1$, and $\gamma = 4 - \pi \sin(\pi x)$. First, choose $\beta = 0.5$, and so $N < \int_\Omega \gamma/\beta dx = 4$, i.e. the total population is small. By Theorem 3.3, we have $I(\cdot,t) \to 0$ as $t\to\infty$ and the infected population will be eliminated, which is confirmed by Figure 1a. Then, choose $\beta = 2$, and so $N > \int_\Omega \gamma/\beta dx = 1$, i.e. the total population is large. Now Corollary 3.5 predicts that $S(\cdot,t) \to \gamma/\beta$ and $I \to I^* = (N - \int_\Omega r dx)/|\Omega| = 2.5$, which is confirmed by Figure 1b. Finally, choose $\gamma = 0.5(1+x)$ such that $\int_\Omega r dx = \int_\Omega \gamma/\beta dx \approx 2.82$ and $S_0 - r$ changes sign on Ω (By Theorem 3.3 and Corollary 3.5, we have already known that $N = \int_\Omega r dx$ is a threshold value for the two alternatives in Theorem 3.3 if β is a constant or $S_0 - r$ does not change sign on Ω). Replace the initial condition by $I_0 = a + \cos(\pi x)$ and then $N = \int_\Omega (S_0 + I_0)dx = a + 2$. Figure 1c shows $\int_\Omega I(x,40)dx$ as a function of N, which indicates that a bifurcation appears at $N \approx 2.82$, i.e. $N \approx \int_\Omega r dx$ is a threshold value for alternative (i) vs (ii) in Theorem 3.3. It is still an open problem to rigorously show whether $N = \int_\Omega r dx$ is the threshold value for the two alternatives in Theorem 3.3 without the additional assumptions in Corollary 3.5. From the simulations, we can see that the disease can be eliminated only when the total population is small by controlling the movement of susceptible people.

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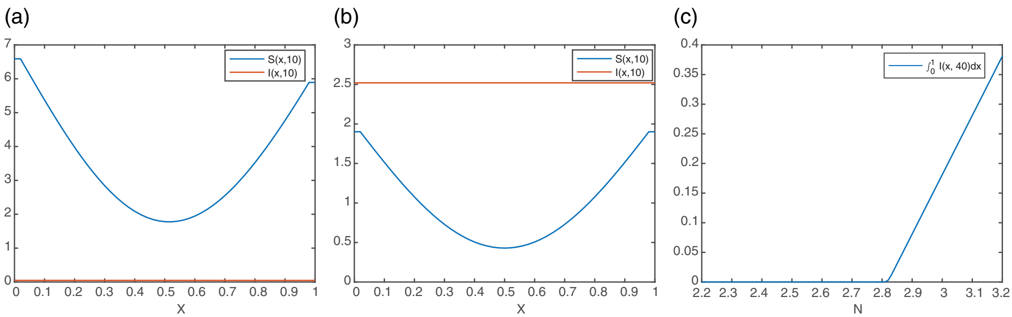

Figure 1. *Simulations of the model with mass action mechanism and $d_S = 0$. Parameters: $d_I = 1$, $\gamma = 4 - \pi \sin(\pi x)$. Left figure: $\beta = 0.5$ and $N < \int_\Omega \gamma/\beta dx$; middle figure: $\beta = 2$ and $N > \int_\Omega \gamma/\beta dx$; right figure: $\beta = 0.5(1 + x)$ and I_0 is replaced by $a + cos(\pi x)$ with $a \in [0.2, 1.2]$.*

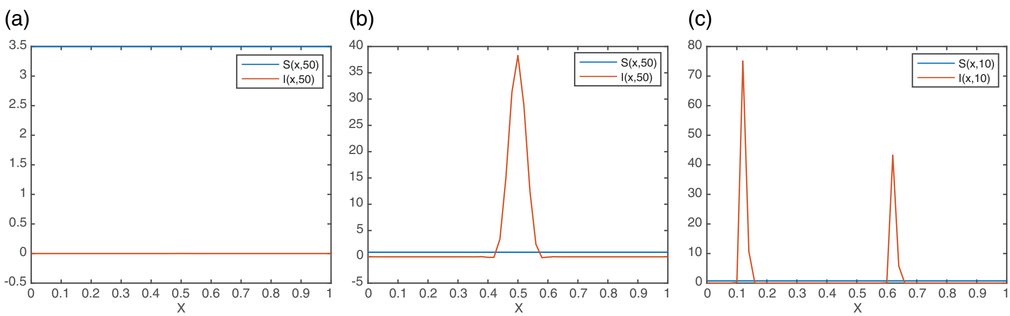

Figure 2. *Simulations of the model with mass action mechanism and $d_S = 1, d_I = 0$. Left figure: $\beta = 0.2$, $\gamma = 4 - \pi \sin(\pi x)$, and $H^+ = \emptyset$; middle figure: $\beta = 1$, $\gamma = 4 - \pi \sin(\pi x)$, $H^+ \neq \emptyset$, and the minimum of γ/β is attached at $x = 0.5$; right figure: $\beta = 2$, $\gamma = 14 - 4\pi \sin(4\pi x)$, $H^+ \neq \emptyset$, and the minimum of γ/β is attached at $x = 1/8, 5/8$.*

5.1.2. Simulation 2: control the movement of infected people

Let $d_S = 1$ and $d_I = 0$. First, choose $\beta = 0.2$ and $\gamma = 4 - \pi \sin(\pi x)$ such that $H^+ = \emptyset$. By Theorem 3.11, $S(\cdot, t)$ converges to $N/|\Omega| = 3.5$ and $I(\cdot, t)$ converges to 0, which is confirmed by Figure 2a. Then, choose $\beta = 1$ and $\gamma = 4 - \pi \sin(\pi x)$ such that $H^+ \neq \emptyset$ and the minimum of γ/β is attached at $x = 0.5$. By Theorem 3.11, the infected people will concentrate at $x = 0.5$, which is confirmed by Figure 2b. Finally, we choose $\beta = 2$ and $\gamma = 14 - 4\pi \sin(4\pi x)$ such that $H^+ \neq \emptyset$. The minimum of γ/β is attached at $x = 1/8, 5/8$, and the infected people concentrate at these two points as shown in Figure 2c. From the simulations, we can see that the infected people may not be eliminated by controlling the movement of infected people if $H^+ \neq \emptyset$. Instead, the infected people will concentrate at certain points that are of the highest risk.

5.2. Standard incidence mechanism

We then simulate the models with standard incidence mechanism.

5.2.1. Simulation 3: control the movement of susceptible people.

Let $d_S = 0$ and $d_I = 1$. First, choose $\beta = 1 + \sin(\pi x)$ and $\gamma = 1.5$ such that $\int_\Omega (\beta - \gamma)dx = 2/\pi - 0.5 > 0$, the high-risk sites are $H^+ = (1/6, 5/6)$, the low-risk sites are $H^- = [0, 1/6) \cup (5/6, 1]$ and

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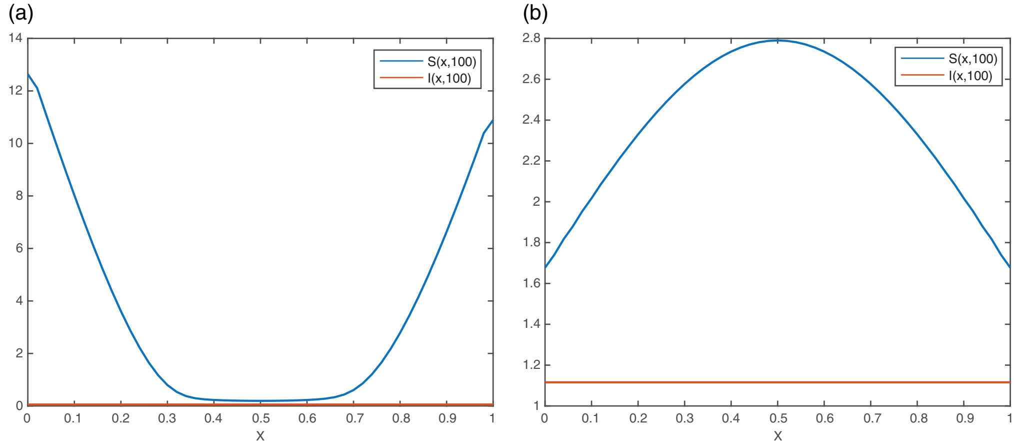

Figure 3. *Simulations of the model with standard incidence mechanism and $d_S = 0, d_I = 1$. Left figure: $\beta = 1 + \sin(\pi x)$ and $\gamma = 1.5$ such that $H^+ = (1/6, 5/6)$, $H^- = [0, 1/6) \cup (5/6, 1]$ and $H^0 = \{1/6, 5/6\}$; right figure: $\beta = 2.5 + \sin(\pi x)$ and $\gamma = 1.5 + \sin(\pi x)$ such that $\beta > \gamma$ and $I^* = N/\int_\Omega (\beta/(\beta - \gamma))dx \approx 1.1159$.*

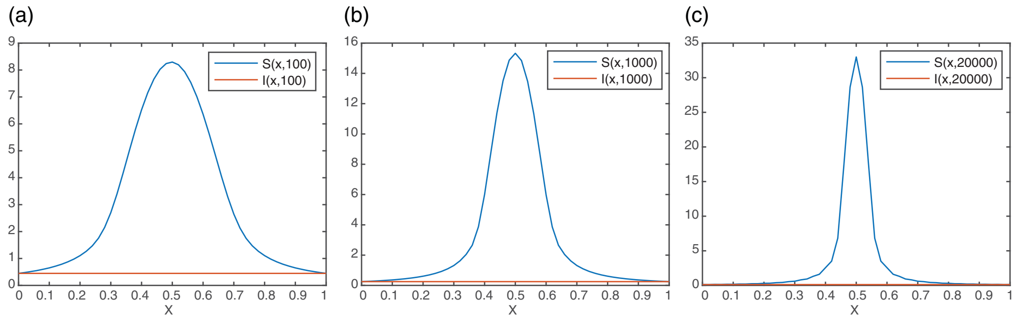

Figure 4. *Simulations of the model with standard incidence mechanism and $d_S = 0, d_I = 1$. Parameters: $\beta = 2 - \sin(\pi x)$ and $\gamma = 1$ such that $\int_\Omega 1/(\beta - \gamma)dx = \infty$.*

moderate-risk sites are $H^0 = \{1/6, 5/6\}$. Since $H^- \neq \emptyset$, Theorem 4.2-(i) predicts that infected population will be eliminated and susceptible people occupy $H^- \cup H^0$. Moreover, since $\int_\Omega (\beta - \gamma) > 0$, then $\mathcal{R}_0 > 1$ and Remark 4.3 suggests that the local size of the susceptible population maybe significantly low on some portion of the high-risk area, which is confirmed by Figure 3a. Then choose $\beta = 2.5 + \sin(\pi x)$ and $\gamma = 1.5 + \sin(\pi x)$ such that $\beta > \gamma$ and $I^* = N/\int_\Omega (\beta/(\beta - \gamma))dx \approx 1.1159$. Theorem 4.2-(ii) predicts that $I(\cdot, t) \to I^*$ as $t \to \infty$, which is confirmed by Figure 3b. Finally, choose $\beta = 2 - \sin(\pi x)$ and $\gamma = 1$ such that $\beta \geq \gamma$, $H^0 = \{0.5\}$, and $\int_\Omega 1/(\beta - \gamma)dx = \infty$. As shown in Figure 4, infected people are eliminated which agrees with Theorem 4.4. Moreover, susceptible people seem to concentrate near H^0, which we cannot prove. Our theoretical results and simulations show that the disease may be controlled by limiting the movement of susceptible people if there exist low-risk or moderate-risk sites.

5.2.2. Simulation 4: control the movement of infected people.

Let $d_S = 1$ and $d_I = 0$. First, choose $\beta = 2 - |x - 0.5|^{0.5}$ and $\gamma = 1.5$ such that the high-risk sites are $H^+ = (0.25, 0.75)$ and $\int_\Omega 1/|\beta - \gamma|dx < \infty$. As shown in Figure 5a, $S(\cdot, t)$ converges to a positive constant and $I(x, t)$ is positive when $x \in H^+$. Then we choose $\beta = 2 - \sin(\pi x)$ and $\gamma = 1.5$ such that the high-risk sites are $H^+ = (0, 1/6) \cup (5/6, 1)$ and $\int_\Omega 1/|\beta - \gamma|dx = \infty$. The infected people will live in

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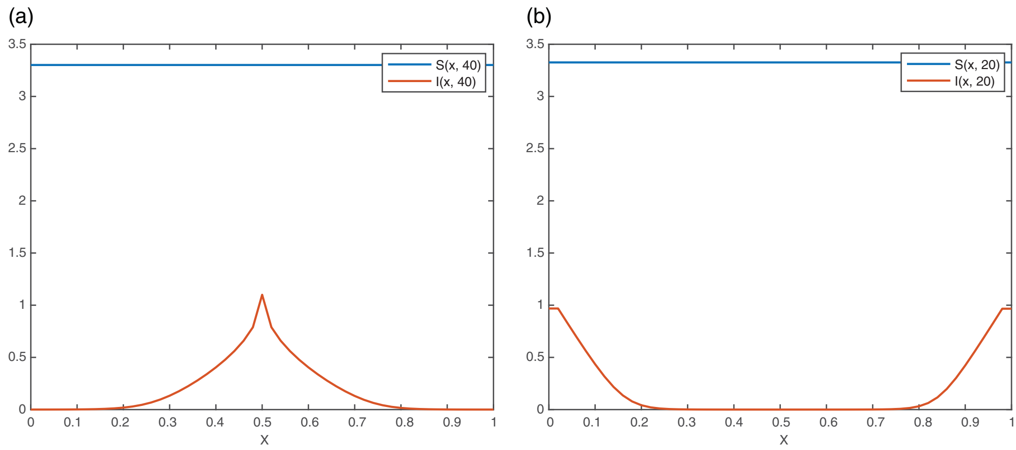

Figure 5. *Simulations of the model with standard incidence mechanism and* $d_S = 1, d_I = 0$. *Left figure:* $\beta = 2 - |x - 0.5|^{0.5}$ *and* $\gamma = 1.5$ *such that the high-risk sites are* $H^+ = (0.25, 0.75)$ *and* $\int_\Omega 1/|\beta - \gamma|dx < \infty$; *right figure:* $\beta = 2 - \sin(\pi x)$ *and* $\gamma = 1.5$ *such that the high-risk sites are* $H^+ = (0, 1/6) \cup (5/6, 1)$ *and* $\int_\Omega 1/|\beta - \gamma|dx = \infty$.

H^+ as shown in Figure 5b. From the simulations, we can see that the infected people may be eliminated exactly at the low-risk sites by controlling the movement of infected people, which is predicted by Theorem 4.6.

6. Conclusions

We studied the impact of limiting population movement on disease outbreak by examining the large time behaviour of classical solutions of a class of epidemic models with mass action or standard incidence transmission mechanism. To this end, we set the diffusion rate of the population subgroups (susceptible or infected) to zero separately and presented detailed mathematical analysis of the corresponding degenerate epidemic models. First, we established the existence and uniqueness of global classical solutions (see Propositions 3.1, 3.7, 4.1 and 4.5). Next, we discussed the global dynamics of the solutions (Theorems 3.3, 3.11, 4.4, and 4.6). Finally, we conducted some numerical simulations to complement and illustrate our theoretical results (see Figures 1–5). Our results revealed the intricate effects of restricting population movement on disease dynamics and how predictions may depend on the choices of transmission mechanisms and spatially heterogeneous parameters.

First, we considered the global dynamics of the model with mass action transmission mechanism. We defined a risk function $r := \gamma/\beta$ as the ratio of the recovery and transmission rates. When the susceptible population movement is restricted, Theorem 3.3 indicates that the average population size, $N/|\Omega|$, compared to that of the risk function, $\int_\Omega r dx/|\Omega|$, largely determines the disease outcome. In particular, regardless of the magnitude of the movement rate of the infected population, the disease may be eradicated only if the average population size is smaller than the average risk function. Moreover, if the disease persists, the infected population would eventually be uniformly distributed across the whole habitat. On the other hand, when the movement of the infected population is restricted, Theorem 3.11 suggests that the disease could be eradicated only if the habitat does not have a high-risk area. Moreover, whether the magnitude of the movement rate of the susceptible population affects the results depends on whether the average size of the population is larger than the risk function in at least one location. Furthermore when the disease persists, the infected population would concentrate on the highest-risk area of the habitat. Our simulations in Figures 1 and 2 illustrated the above theoretical results.

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Next, we studied the global dynamics of the model with standard incidence mechanism. Theorems 4.2 and 4.4 indicate that restricting the susceptible population's movement could completely eradicate the disease only if the habitat accommodates either a low-risk or moderate-risk area. These requirements are achieved if the local distribution of the risk function is less than or equal to one. However when the habitat consists of only high-risk areas, the disease would persist with the infected population being uniformly distributed across the whole habitat. On the other hand, when the movement of the infected population is restricted, Theorem 4.6 indicates that the disease may persist if the habitat has a nonempty high-risk area. Moreover, the infected population precisely occupy the high-risk sites, and the magnitude of the movement of the susceptible subgroup does not influence the disease persistence. Our simulations in Figures 3–5 illustrated these theoretical results.

Our above theoretical results suggest that the transmission mechanisms play an important role when the population movement of one subgroup is restricted. Indeed, when mass action mechanism is used, the persistence of the disease depends on the average population size compared to either the average of the risk function (when $d_S = 0$) or its local distribution (when $d_I = 0$). However, when standard incidence mechanism is adopted, the disease persistence prediction depends on whether the habitat has only high-risk areas (when $d_S = 0$) or has a nonempty high-risk area (when $d_I = 0$). Interestingly, our results suggest that for either mass action or standard incidence mechanism disease control strategies which focus on limiting the movement of the susceptible population should be preferred over those focusing on restricting the movement of the infected population.

Finally, we point out that most of the previous works (e.g. [2, 9, 13, 45, 62]) considered the asymptotic profiles of the EE solutions to understand the effects of limiting population movements on the dynamics of infectious diseases, under the assumptions that the evolution of the disease happens on a fast scale compared to control strategies and populations ultimately stabilise at the EE solutions. In this work, we study the effects of limiting population movement by focusing on the global dynamics of the degenerate systems with either $d_S = 0$ or $d_I = 0$, assuming instead that the control strategies happen on much faster scale than the evolution of the disease. We remark that some of our results depend on the initial value. Other than that, the biological implications from these two approaches seem to align well (e.g. both approaches predict that the effectiveness of controlling the movement of susceptible people with mass action mechanism depends on the size of the total population).

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