IMMERSIONS OF METRIC SPACES INTO EUCLIDEAN SPACES

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1. Introduction. In a recent paper on isotopy invariants **(1)**, S. T. Hu defined the enveloping space $E_m(X)$ of any given topological space X for each integer $m > 1$. By an application of the Smith theory to the singular cohomology of the enveloping space $E_m(X)$, he obtained his immersion classes $\Psi_m{}^n(X)$ for every $n = 1, 2, 3, \ldots$ and proved **(3)** the main theorem that a necessary condition for a compact metric space X to be immersible into the n-dimensional Euclidean space R^n is $\Psi_2{}^n(X) = 0$. This theorem was proved earlier by W. T. Wu **(4)** for finitely triangulable spaces X using purely combinatorial methods.

The objective of the present paper is to prove the above-mentioned theorem for arbitrary metric spaces. Our treatment follows that of S. T. Hu **(3)** in which he considers a homotopically equivalent subspace of $E_m(X)$. By a further localization of the situation, we obtain a homotopically equivalent subspace of $E_m(X)$ for locally finite open coverings of X. This enables us to remove the compactness condition.

The reader is referred to **(1)** and **(3)** for definitions and notation.

2. The map δ and the subspace $E_m(X, \delta)$. Let $\mathfrak{F}$ be a given locally finite open covering of an arbitrary metric space X with a distance function $d\colon X^m x X^m \to R$ in the topological power X^m. Define a positive real-valued function δ on X as follows. Let x be an arbitrary point of X. Since $\mathfrak{F}$ is a locally finite open covering of X, the point x meets only a finite number of members of $\mathfrak{F}$, say $V_1, V_2, \ldots V_q$. Then define

$$\delta(x) = \max_{1 \leqslant i \leqslant q} [d(x, X \backslash V_i)].$$

Continuity of δ is obvious. Call δ the *canonical map* of the given covering $\mathfrak{F}$. Next, for any path $\sigma \in E_m(x)$, $\sigma(0)$ is a point of the diagonal X of the mth power X^m and thus $\delta[\sigma(0)]$ is a well-defined positive real number. Let $E_m(X, \delta)$ denote the subspace of the mth enveloping space $E_m(x)$ which consists of all paths $\sigma \in E_m(X)$ satisfying the condition

$$d[\sigma(0), \sigma(t)] < \tfrac{1}{2}\delta[\sigma(0)]$$

for every $t \in I$. Since

$$d[\sigma(0), \sigma(t)] = d\{\xi[\sigma(0)], \xi[\sigma(t)]\},$$

Received July 20, 1964. This research was supported in part by the Air Force Office of Scientific Research.

ξ sends $E_m(X, \delta)$ onto itself. Therefore, we have the orbit space

$$E_m^*(X, \delta) = E_m(X, \delta)/G.$$

Since the canonical map δ is continuous and positive for all points of X, the following theorem holds as a result of the proof of **(3**, 4.1**)**.

THEOREM 2.1. *There exists a homotopy*

$$h_t: E_m(X) \to E_m(X) \qquad (t \in I)$$

satisfying the following conditions:

(2.1A) *h_0 is the identity map on $E_m(X)$.*
(2.1B) *h_1 sends $E_m(X)$ into $E_m(X, \delta)$.*
(2.1C) *For every $t \in I$, h_t sends $E_m(X, \delta)$ into itself.*
(2.1D) *For every $t \in I$, $h_t \circ \xi = \xi \circ h_t$.*

COROLLARY 2.2. *The inclusion map*

$$i^*: E_m^*(X, \delta) \subset E_m^*(X)$$

is a homotopy equivalence.

3. The main theorem. We are concerned here with an arbitrarily given immersion $j: X \to Y$ of a metric space X into any topological space Y.

For each point x of X, choose an open neighbourhood U_x of x in X such that $j|U_x$ is an imbedding. Since every metric space is paracompact, the open cover $\mathfrak{C} = \{U_x | x \in X\}$ has a locally finite open refinement $\mathfrak{F} = \{V_\mu | \mu \in M\}$ (M an index set) which covers X. Let δ denote the canonical map of the covering $\mathfrak{F}$, and consider the subspace $E_m(X, \delta)$ of the mth enveloping space $E_m(X)$ of the metric space X as defined in §2.

Let $\sigma \in E_m(X, \delta)$ be arbitrarily given. Since $\sigma: I \to X^m$ is a path in the mth topological power X^m of X, we may compose σ with the mth topological power $j^m: X^m \to Y^m$ of the given immersion $j: X \to Y$ and obtain a path $j^m \circ \sigma: I \to Y^m$.

LEMMA 3.1. *For every $\sigma \in E_m(X, \delta)$, we have*

$$j^m \circ \sigma \in E_m(Y).$$

Proof. Let σ be an arbitrary path from $E_m(X, \delta)$. We must show that $j^m[\sigma(t)]$ is a point on the diagonal Y of Y^m if and only if $t = 0$. If $t = 0$, the result follows immediately. On the other hand, suppose that $j^m[\sigma(t)]$ is a point on the diagonal Y for some $t \in I$. In order to conclude that $t = 0$, it suffices to show that $\sigma(t)$ is a point on the diagonal X of X^m. Let

$$\sigma(t) = (x_1, x_2, \ldots, x_m) \in X^m \quad \text{and} \quad j^m[\sigma(t)] = (y, y, \ldots, y)$$

where y is a point of the space Y. Then

$$j(x_1) = j(x_2) = \ldots = j(x_m) = y.$$

Let $V_1, V_2, \ldots, V_q$ be the members of $\mathfrak{F}$ which contain the point $\sigma(0)$. Since $\sigma \in E_m(X, \delta)$, it follows that

$$d[\sigma(0), \sigma(t)] < \tfrac{1}{2}\delta[\sigma(0)] = \tfrac{1}{2} \max_{1 \leqslant i \leqslant q} [d(\sigma(0), X \backslash V_i)]$$
$$= \tfrac{1}{2} d[\sigma(0), X \backslash V_k]$$

for some $k = 1, 2, \ldots, q$, and thus the set of points $\{x_1, x_2, \ldots, x_m\}$ is in V_k. Since $\mathfrak{F}$ is a refinement of $\mathfrak{C}$, there is an open neighbourhood U of $\mathfrak{C}$ containing V_k. But the restriction $j|U$ is an imbedding, and hence $x_1 = x_2 = \ldots = x_m$. This completes the proof.

According to 3.1, j^m defines an imbedding

$$E_m(j) : E_m(X, \delta) \to E_m(Y).$$

By means of the induced isomorphism

$$i^{**} : H^n[E_m{}^*(X); G] \to H^n[E_m{}^*(X, \delta); G]$$

of the homotopy equivalence i^* in (2.2) and the map $E_m(j)$, one can define a homomorphism

$$E_m{}^{**}(j) : H^n[E_m{}^*(Y); G] \to H^n[E_m{}^*(X); G]$$

for each dimension n and every abelian coefficient group G using methods analogous to **(3)**. Routine verification shows that $E_m{}^{**}(j)$ is independent of the choice of the locally finite open refinement $\mathfrak{F}$ of $\mathfrak{C}$; that is to say, if $\mathfrak{F}'$ is another locally finite open refinement of $\mathfrak{C}$ and δ' is the canonical map of $\mathfrak{F}'$, then the diagram

$$\begin{array}{ccc}
H^n[E_m{}^*(Y); G] & \xrightarrow{E_m{}^{**}(j, \delta)} & H^n[E_m{}^*(X, \delta); G] \\
\Big\downarrow {\scriptstyle E_m{}^{**}(j, \delta')} & & \Big\downarrow {\scriptstyle (i^{**})^{-1}} \\
H^n[E_m{}^*(X, \delta'); G] & \xrightarrow{(i'^{**})^{-1}} & H^n[E_m{}^*(X); G]
\end{array}$$

is commutative. Furthermore, one obtains the following proposition.

PROPOSITION 3.2. *For each* $n = 1, 2, \ldots,$ *we have*

$$E_m{}^{**}(j)[\Psi_m{}^n(Y)] = \Psi_m{}^n(X).$$

Because of **(3**, 5.1**)**, (3.2), and the fact that $\Phi_2{}^n(R^n) = 0$ **(3**; **4)**, we are able to state the main theorem.

THEOREM 4.3. *If a metric space* X *can be immersed in the n-dimensional Euclidean space* R^n, *then* $\Psi_2{}^n(X) = 0$

https://doi.org/10.4153/CJM-1965-095-6 Published online by Cambridge University Press

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