A BANACH LATTICE NOT WEAKLY PROJECTABLE

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In [4] a concept of a weakly projectable vector lattice has been introduced. Stone vector lattices [3] and thus all special types of them, like Riesz [5], σ-complete and complete vector lattices are weakly projectable. Moreover $C[0,1]$ is weakly projectable but not Stone [4]. As we see the collection W of weakly projectable vector lattices is quite large. This explains to some extent the difficulty in producing examples of vector lattices which do not belong to W. In this note an example of a Banach lattice [1] which is not weakly projectable is described.

DEFINITION. A vector lattice E is said to be *weakly projectable* if for any $x, y \in E$ there exists $z \in x^{\perp}$ such that $y \in (|x| + |z|)^{\perp\perp}$.

(For definitions of symbols used above we refer e.g. to [2]).

EXAMPLE. Let $F[0,1]$ denote the space of bounded real valued functions defined on $[0,1]$ and discontinuous at most at a countable set of points. Addition and multiplication by scalars are introduced in the usual way. The order is defined by: $x \geqq 0$ if and only if $x(t) \geqq 0$ for each $t \in [0,1]$. Define also $\| x \| = \sup_{0 \leqq t \leqq 1} | x(t) |$.

Using standard methods it is easy to prove that $F[0,1]$ is a Banach lattice and thus an Archimedean vector lattice [1]. We shall prove that $F[0,1] \notin W$.

Let $\{r_n\}$ be a sequence dense in the interval $[0,1]$. Denote by A the set

$$A = [0,1] \cap \left(\bigcup_{n=1}^{\infty} (r_n - 4^{-n}, r_n + 4^{-n}) \right).$$

A is open in $[0,1]$ and mes $(A) \leqq \frac{2}{3}$, thus $A' = [0,1] \setminus A$ is a closed uncountable set. We have also $\bar{A} = [0,1]$. Define $x: [0,1] \to R$ by

$$x(t) = \text{distance from } t \text{ to } A'.$$

x is continuous on $[0,1]$, and so $x \in F[0,1]$. To show that $F[0,1] \notin W$ it is sufficient to prove that for any $z \in x^{\perp}$, the identity function e: $e(t) = 1$ for all $t \in [0,1]$ does not belong to $(|x| + |z|)^{\perp\perp}$. Take any $z \in x^{\perp}$. Then $z(t) = 0$ for all $t \in A$.

Moreover, since A is dense in $[0,1]$, any point t_0 of $[0,1]$ is a limit of a sequence $\{t_n\}$ of points in A. Therefore if z is continuous at t_0 then $z(t_0) = \lim_{n\to\infty} z(t_n) = 0$. Since $z \in F[0,1]$, it is discontinuous at most at a countable set. On the other hand A' is not countable. Consequently, there exists a point $\tau \in A'$ such that $z(\tau) = 0$. Since $\tau \in A'$, we have also $x(\tau) = 0$. Thus $w = |z| + |x|$ vanishes at τ. Hence the function u defined by

$$u(t) = \begin{cases} 1 & \text{if } t = \tau, \\ 0 & \text{if } t \neq \tau \end{cases}$$

belongs to $F[0,1]$ and $u \perp w$.

Let $v \in w^{\perp\perp}$. Since $u \in w^{\perp}$ and $u(\tau) \neq 0$, it follows that $v(\tau) = 0$. On the other hand $e(\tau) = 1$ and thus $e \notin w^{\perp\perp}$. This concludes the proof that $F[0,1]$ is not weakly projectable.

References

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