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LOCAL MINIMA OF THE GAUSS CURVATURE OF A MINIMAL SURFACE

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Let D be a domain in the complex w-plane and let $x\colon D \to \mathbb{R}^3$ be a regular minimal surface. Let $M(K)$ be the set of points $w_0 \in D$ where the Gauss curvature K attains local minima: $K(w_0) \leqslant K(w)$ for $|w - w_0| < \delta(w_0)$, $\delta(w_0) > 0$. The components of $M(K)$ are of three types: isolated points; simple analytic arcs terminating nowhere in D; analytic Jordan curves in D. Components of the third type are related to the Gauss map.

1. Introduction and results

Our purpose is to study the set of parameter points where the Gauss curvature of a minimal surface in the Euclidean space $\mathbb{R}^3$ attains local minima. A nonconstant map x from a domain D in the complex w-plane $\mathbb{C}(w = u + iv)$ into the Euclidean space $\mathbb{R}^3$, in notation, $x\colon D \to \mathbb{R}^3$, is said to determine a regular minimal surface, or, simply, x is a regular minimal surface defined in D, if the following three conditions hold:

(HA) Each component x_k of $x = (x_1, x_2, x_3)$ is harmonic in D.

(IS) The real parameters u and v are isothermal in the sense that

$$\sum_{k=1}^{3} \phi_k^2 \equiv 0$$

in D, where

$$\phi_k = \frac{\partial x_k}{\partial u} - i\frac{\partial x_k}{\partial v} = 2\frac{\partial x_k}{\partial w}, \qquad k = 1, 2, 3.$$

(RE) The function $\sum_{k=1}^{3} |\phi_k|^2$ never vanishes in D.

Suppose that the surface x is not contained in any plane in $\mathbb{R}^3$. Then $f = \phi_1 - i\phi_2$ is analytic and $g = \phi_3/f$ is meromorphic in D. The Gauss map Γ of x is the map from x into the unit sphere S in $\mathbb{R}^3$ defined by

$$\Gamma(w) \equiv \Gamma(x(w)) = \left(\frac{2\operatorname{Re} g(w)}{|g(w)|^2 + 1}, \frac{2\operatorname{Im} g(w)}{|g(w)|^2 + 1}, \frac{|g(w)|^2 - 1}{|g(w)|^2 + 1}\right), \ w \in D;$$

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this is the unit normal at $x(w)$ with the standard orientation, together with $\Gamma(w) = (0, 0, 1)$, if w is a pole of g. Then Γ is identified with g via the stereographic projection from S onto $\mathbb{C} \cup \{\infty\}$. The Gauss curvature at the point $x(w)$ is then

$$K(w) = -\left[\frac{4g^{\#}(w)}{|f(w)|\left(1+|g(w)|^2\right)}\right]^2,$$

where the spherical derivative $h^{\#}(w)$ at w of h meromorphic in D is defined by

$$h^{\#}(w) = \begin{cases} |h'(w)|/\left(1+|h(w)|^2\right) & \text{if } h(w) \neq \infty, \\ |(1/h)'(w)| & \text{if } h(w) = \infty. \end{cases}$$

Condition (RE) is valid if and only if the function

$$|f|\left(1+|g|^2\right) = \sqrt{2\sum_{k=1}^{3}|\phi_k|^2}$$

never vanishes in D. Thus, if x is not contained in any plane, then $K(w) \neq 0$ if and only if $g^{\#}(w) \neq 0$. This is the case if and only if w is a simple pole of g or $g(w) \neq \infty$ and $g'(w) \neq 0$. Therefore, $-\infty < K \leqslant 0$ everywhere in D. For the basic properties of minimal surfaces, see [**1**, **2**].

Let $M(K)$ be the set of points $w_0 \in D$ where K has local minima: $K(w_0) \leqslant K(w)$ for w in a disk $\{|w - w_0| < \delta\}$ with δ depending on K and w_0.

THEOREM 1. *Let $x\colon D \to \mathbb{R}^3$ be a regular minimal surface contained in no plane and with nonempty $M(K)$. Then the connected components of $M(K)$ are at most countable and each component is one of the following:*

(1) *An isolated point.*
(2) *A simple analytic arc terminating nowhere in D.*
(3) *A simple closed analytic curve.*

All the cases of (1), (2) and (3) actually happen; see the next section. We let $M_1(K)$, $M_2(K)$, and $M_3(K)$ be the set of components of $M(K)$ of type (1), (2), and (3), respectively.

Let D_1 be a subdomain of D. The total curvature of the subsurface $x\colon D_1 \to \mathbb{R}^3$ is defined by

$$T(D_1) = \frac{1}{2}\iint_{D_1} K \cdot \sum_{k=1}^{3}|\phi_k|^2\, du dv.$$

Then

$$-T(D_1) = 4\iint_{D_1} g^{\#2} du dv, g$$

the area of the image of D_1 by g covering over S.

THEOREM 2. *Let $x\colon D \to \mathbb{R}^3$ be a regular minimal surface contained in no plane. Suppose that $c \in M_3(K)$ exists and suppose further that the Jordan domain Δ bounded by c is contained in D. Then,*

$$-T(\Delta) = \pi(Z'_\Delta + P_\Delta - n), \tag{4}$$

where Z'_Δ is the sum of all orders of all the distinct zeros of g' in Δ, while P_Δ is the sum of all orders of all the distinct n poles of g in Δ.

In particular, if $g^{\#}$ never vanishes in D, then the right-hand side of (4) is zero. Thus, either $M_3(K)$ is empty or else each Jordan domain bounded by $c \in M_3(K)$ is not contained in D.

There does exist x for which $\Delta \subset D$ actually happens as described in Theorem 2; see TYPE 3 in the next section.

2. EXAMPLES

Suppose that $D \subset \mathbb{C}$ is simply connected and g is nonconstant and analytic in D. With the aid of g we can construct a minimal surface $x\colon D \to \mathbb{R}^3$ as follows:

$$x_1(w) = \frac{1}{2}\operatorname{Re}\int_a^w \{1 - g(\zeta)^2\}d\zeta,$$
$$x_2(w) = \frac{1}{2}\operatorname{Re}\int_a^w i\{1 + g(\zeta)^2\}d\zeta,$$
$$x_3(w) = \operatorname{Re}\int_a^w g(\zeta)d\zeta,$$

where a is a fixed point of D. The Gauss map is just g; see [**2**, p.64]. Therefore,

$$\frac{\sqrt{-K(w)}}{4} = \frac{|g'(w)|}{\left(1 + |g(w)|^2\right)^2}, \; w \in D.$$

TYPE 1. Let $D = \mathbb{C}$ and $g(w) = w$. Then $M(K) = \{0\}$. (Enneper's surface)

TYPE 2. Let $D = \mathbb{C}$ and $g(w) = e^w$. Then $M(K) = \{\operatorname{Re} w = -(1/2)\log 3\}$.

TYPE 3. Let $D = \mathbb{C}$ and $g(w) = w^2$. Then $M(K) = \{|w| = 7^{-1/4}\}$.

The restriction of the above surfaces to $\{|w| > 1\}$ yields $M(K) = \emptyset$. A problem is to find $x\colon D \to \mathbb{R}^3$ for which two or three types appear at the same time for $M(K)$.

It would be interesting to consider the typical minimal surfaces given in the non-parametric form, namely:

The helicoid:

$$x_3 = \tan^{-1}\left(\frac{x_2}{x_1}\right), \qquad (x_1, x_2) \in \mathbb{R}^2.$$

The catenoid:

$$x_3 = \cosh^{-1}\sqrt{x_1^2 + x_2^2}, \qquad x_1^2 + x_2^2 \geqslant 1.$$

See [**2**, pp.17–18] and [**3**, pp.34 and 47]. When $(x_1, x_2) = (0, 0)$ in the helicoid we interpret this to express the x_3-axis.

A parametric form of the helicoid is then given by $x\colon \mathbb{C} \to \mathbb{R}^3$, where,

$$x_1(w) = \sinh u \cos v,$$
$$x_2(w) = \sinh u \sin v,$$
$$x_3(w) = v.$$

Thus, $f(w) = e^{-w}$ and $g(w) = -ie^{w}$, so that a calculation shows that $M(K)$ is the imaginary axis in $\mathbb{C}$, which corresponds to the x_3-axis lying on the surface.

A parametric form of the catenoid is given by $x\colon \mathbb{C} \setminus \{0\} \to \mathbb{R}^3$, where

$$x_1(w) = -\frac{u}{2}\left(1 + \frac{1}{|w|^2}\right),$$
$$x_2(w) = \frac{v}{2}\left(1 + \frac{1}{|w|^2}\right),$$
$$x_3(w) = \log|w|.$$

Thus, $f(w) = -1$ and $g(w) = -1/w$, so that a calculation shows that $M(K)$ is the unit circle, which corresponds to the unit circle on the surface.

Note that, in all examples in this section, K actually attains the global minimum at each point of $M(K)$.

3. Proof of Theorem 1

It suffices to prove the following proposition:

(I). *Let $a \in M(K)$ be an accumulation point of $M(K)$. Then there exists $\delta > 0$ such that $M(K) \cap \{|w - a| < \delta\}$ is a simple analytic arc with both terminal points on the circle $\{|w - a| = \delta\}$.*

Lemma 1. *Let G be analytic and H be meromorphic in a domain $D_1 \subset \mathbb{C}$. Suppose that*

$$L(G, H) = \{w \in D_1; \overline{G(w)} = H(w)\}$$

has an accumulation point $a \in D_1$ *and* $G'(a) \neq 0$. *Then there exists an open disk* $U(a)$ *of centre* a *such that* $U(a) \cap L(G, H)$ *is a simple, analytic arc passing through* a *with both terminal points on the circle* $\partial U(a)$.

The proof of this lemma is the same as that of [3, Lemma 1] (see also [4]) in case $G(w) \equiv w$. In the general case, let $V(a)$ be an open disk with centre a where G is univalent. Regarding $G(V(a))$ as D_1, $G(w)$ as w, and H as $H \circ G^{-1}$, we can reduce this case to the case specified in the above.

We are ready to prove (I). Set

$$\Phi(w) = \frac{\sqrt{-K(w)}}{4}, \qquad w \in D.$$

We first note that $g^{\#}(a) \neq 0$ for $a \in M(K)$.

Suppose the case where $g(a) \neq \infty$ and $g'(a) \neq 0$. Then there exists $\delta_1 > 0$ such that g is analytic and univalent in $\Delta_1 = \{|w-a| < \delta_1\}$ and $\Phi(w) \leqslant \Phi(a)$ for each $w \in \Delta_1$. Hence at each $w \in \Delta_1 \cap M(K)$ we have

$$\text{(3.1)} \qquad \frac{\partial \Phi(w)}{\partial w} \Big/ \Phi(w) = \frac{1}{2}\left(\frac{g''(w)}{g'(w)} - \frac{f'(w)}{f(w)}\right) - \frac{2\overline{g(w)}g'(w)}{1+|g(w)|^2} = 0.$$

Consequently

$$\Delta_1 \cap M(K) \subset L(g, H),$$

where $L(g, H)$ is considered in Δ_1 with

$$H = \frac{Q}{2g' - Qg}, \qquad Q = \frac{1}{2}\left(\frac{g''}{g'} - \frac{f'}{f}\right).$$

It follows from Lemma 1 that there exists $U(a)$ such that $L_1 = U(a) \cap L(g, H)$ is a simple analytic arc described there. Let $L_1 : w = w(t)$ be an analytic expression with a real parameter t. Then,

$$\frac{d}{dt}\Phi(w(t)) = 2\,\mathrm{Re}\left[\left\{\frac{\partial \Phi(w)}{\partial w}\right\}_{w=w(t)} w'(t)\right] = 0$$

on L_1. Hence Φ is constant on L_1. Furthermore, $L_1 = U(a) \cap M(K)$. This proves (I) for the present case.

Suppose the case where a is a simple pole of g. Then there exists $\delta_2 > 0$ such that $G = 1/g$ is analytic and univalent in $\Delta_2 = \{|w-a| < \delta_2\}$ and $\Phi(w) \leqslant \Phi(a)$ in

Δ_2. Consequently, at each $w \in (\Delta_2 \setminus \{a\}) \cap M(K)$, we have

$$\text{(3.2)} \qquad \frac{\partial \Phi(w)}{\partial w} \Big/ \Phi(w) = \frac{1}{2}\left(\frac{G''(w)}{G'(w)} - \frac{f'(w)}{f(w)}\right) - \frac{2\overline{G(w)}G'(w)}{1+|G(w)|^2} + \frac{G'(w)}{G(w)} = 0$$

because

$$\Phi(w) = \frac{|G(w)|^2 G^{\#}(w)}{|f(w)|\left(1+|G(w)|^2\right)}.$$

Hence

$$(\Delta_2 \setminus \{a\}) \cap M(K) \subset L(G, H_1),$$

where $L(G, H_1)$ is considered in Δ_2 with

$$H_1 = \frac{Q_1}{2G' - Q_1 G}, \qquad Q_1 = \frac{1}{2}\left(\frac{G''}{G'} - \frac{f'}{f}\right) + \frac{G'}{G}.$$

Thus, a is an accumulation point of $L(G, H_1)$ and $G'(a) \neq 0$. It follows from Lemma 1 that there exists $U(a)$ such that

$$L_2 = U(a) \cap L(G, H_1)$$

is a simple analytic arc described there. On the other hand, Φ is constant on $L_2 \setminus \{a\}$, so that $\Phi(w) \equiv \Phi(a)$, $w \in L_2$, by the continuity of Φ at a. Accordingly

$$L_2 = U(a) \cap M(K)$$

and this completes the proof of (I). □

REMARK. We let $M^*(K)$ be the set of points $w_0 \in D$ where K has the (global) minimum: $K(w_0) \leqslant K(w)$, $w \in D$. Suppose that $a \in D$ is an accumulation point of $M^*(K)(\subset M(K))$. Then there exists $c \in M_2(K) \cup M_3(K)$ such that $a \in c$. Since K is constant on c, it follows that $c \subset M^*(K)$. Hence we have the analogous classification: $M_k^*(K)$, $k = 1, 2, 3$, of components of $M^*(K)$.

4. Proof of Theorem 2

First of all $g^{\#}$ never vanishes on $c = \partial\Delta$ because this is the case at each point of $M(K)$. Let α_k, $1 \leqslant k \leqslant p$, be all the simple poles of g on c, and let γ_k be all the distinct poles of g in Δ of orders ν_k, $1 \leqslant k \leqslant n$, so that

$$P_\Delta = \sum_{k=1}^{n} \nu_k.$$

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Set $A = \{\alpha_1, \dots, \alpha_p, \gamma_1, \dots, \gamma_n\}$. For $\varepsilon > 0$ and $\alpha \in A$ we set

$$\delta(\alpha, \varepsilon) = \{z; |z-\alpha| \leqslant \varepsilon\},$$
$$c(\alpha, \varepsilon) = \{z \in \Delta; |z-\alpha| = \varepsilon\}.$$

Then, from sufficiently small ε on,

$$\Delta(\varepsilon) = \Delta \setminus \bigcup_{\alpha \in A} \delta(\alpha, \varepsilon)$$

is a domain bounded by Jordan curves. Set

$$\lambda = \frac{\overline{g}g'}{1+|g|^2} \quad \text{and} \quad \mu = i\lambda.$$

Then the Green formula

$$\iint_{\Delta(\varepsilon)} (\mu_u - \lambda_v)\,du\,dv = \int_{\partial\Delta(\varepsilon)} (\lambda\,du + \mu\,dv)$$

can be rewritten as

$$4\iint_{\Delta(\varepsilon)} g^{\#}(w)^2\,du\,dv = -2i\int_{\partial\Delta(\varepsilon)} \lambda(w)\,dw, \tag{4.1}$$

where the line integral is in the positive sense with respect to $\Delta(\varepsilon)$.

Now, the Laurent expansion of g about $\alpha \in A$ yields

$$g(w) = (w-\alpha)^{-N} h(w) \text{ in } \delta(\alpha, \varepsilon) \setminus \{\alpha\},$$

where h is analytic and zero-free in $\delta(\alpha, \varepsilon)$ and $N = 1$ if $\alpha = \alpha_k$, while $N = \nu_k$ if $\alpha = \gamma_k$. The differentiation yields that

$$g'(w) = (w-\alpha)^{-N-1}\Psi(w) \text{ in } \delta(\alpha, \varepsilon) \setminus \{\alpha\}, \tag{4.2}$$

where

$$\Psi(w) = -Nh(w) + (w-\alpha)h'(w).$$

Since

$$\varepsilon e^{it}\lambda(\varepsilon e^{it} + \alpha) = \frac{\overline{h(\varepsilon e^{it}+\alpha)}\Psi(\varepsilon e^{it}+\alpha)}{\varepsilon^{2N} + |h(\varepsilon e^{it}+\alpha)|^2} \to -N \text{ as } \varepsilon \to 0$$

uniformly for real t, it follows that

$$\int_{c(\alpha,\varepsilon)} \lambda(w)\,dw \to \begin{cases} \pi i & \text{if } \alpha = \alpha_k, \\ 2\pi\nu_k i & \text{if } \alpha = \gamma_k, \end{cases}$$

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as $\varepsilon \to 0$, where the integral is in the clockwise sense. Letting $\varepsilon \to 0$ in (4.1), we now have

$$(4.3) \qquad 4\iint_\Delta g^\#(w)^2 du dv = -2i\int_c \lambda(w)dw + 2\pi p + 4\pi P_\Delta = \frac{1}{2i}\int_c \left(\frac{g''(w)}{g'(w)} - \frac{f'(w)}{f(w)}\right) dw + 2\pi p + 4\pi P_\Delta$$

by $\partial\Phi(w)/\partial w = 0$ on c.

We remember that f vanishes precisely at the poles of g. Thus, γ is a zero of order 2ν of f if and only if γ is a pole of order ν of g. Hence,

$$(4.4) \qquad \frac{1}{2\pi i}\int_{\partial\Delta_0(\varepsilon)} \left(\frac{g''(w)}{g'(w)} - \frac{f'(w)}{f(w)}\right) dw = Z'_\Delta - (3P_\Delta + n),$$

where

$$\Delta_0(\varepsilon) = \Delta \setminus \bigcup_{k=1}^{p} \delta(\alpha_k, \varepsilon).$$

We have in $\delta(\alpha, \varepsilon) \setminus \{\alpha\}$, $\alpha = \alpha_k$,

$$\frac{g''(w)}{g'(w)} - \frac{f'(w)}{f(w)} = \frac{-4}{w-\alpha} + \frac{X'(w)}{X(w)},$$

where X is analytic and zero-free in $\delta(\alpha, \varepsilon)$ for small $\varepsilon > 0$. Consequently, letting $\varepsilon \to 0$ in the left-hand side of (4.4) we have the identity

$$(4.5) \qquad \frac{1}{2\pi i}\int_c \left(\frac{g''(w)}{g'(w)} - \frac{f'(w)}{f(w)}\right) dw + 2p = Z'_\Delta - 3P_\Delta - n.$$

Combining (4.3) and (4.5) we have (4).

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