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THE STRUCTURE OF C^*-CONVEX SETS

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ABSTRACT. Compact C^*-convex subsets of M_n correspond exactly to n-th matrix ranges of operators. The main result of this paper is to discover the "right" analog of linear extreme points, called *structural elements*, and then to prove a generalised Krein-Milman theorem for C^*-convex subsets of M_n. The relationship between structural elements and an earlier attempted generalisation, called C^**-extreme points*, is examined, solving affirmatively a conjecture of Loebl and Paulsen [8]. An improved bound for a C^*-convex version of the Caratheodory theorem for convex sets is also given.

0. **Introduction.** For T a bounded linear operator on a Hilbert space, Arveson [1] introduced a generalisation of the familiar numerical range, called the *n-th matrix range* of T, and defined by $W^n(T) = \{\varphi(T) : \varphi: \mathcal{B}(\mathcal{H}) \longrightarrow M_n \text{ unital, completely positive}\}$. Among other things, he observed that $W^n(T)$ has a particularly strong convexity property. Loebl and Paulsen [8] named this property C^*-convexity and defined it as follows:

DEFINITION 0.1. Let $\mathcal{A}$ be a C^*-algebra. A C^**-convex combination* of $x_1 \cdots x_m \in \mathcal{A}$ is a sum of the form $\sum_{i=1}^m t_i^* x_i t_i$ where the $t_i \in \mathcal{A}$ satisfy $\sum_{i=1}^m t_i^* t_i = \mathbf{1}$. A set $\mathcal{S} \subset \mathcal{A}$ is C^*-convex iff it is closed under C^*-convex combinations of elements of $\mathcal{S}$.

The analogy with linear convexity is obvious, and the paper [8] is a good introduction to the basic facts about C^*-convexity. The motivation for studying C^*-convex sets is their connection with n-th matrix ranges of operators: n-th matrix ranges are compact C^*-convex sets. Furthermore, by reinterpreting some earlier work of Salinas [10], Loebl and Paulsen observed that the converse also holds: for any compact, C^*-convex subset $\mathcal{S} \subset M_n$, there exists a separable Hilbert space $\mathcal{H}$, and some $T \in \mathcal{B}(\mathcal{H})$ satisfying $\mathcal{S} = W^n(T)$. Thus compact C^*-convex subsets of M_n correspond exactly to n-th matrix ranges of operators. The importance of the structure of C^*-convex sets is an immediate corollary to the widespread interest in n-th matrix ranges. (The survey paper [6] is a good introduction to the literature on n-th matrix ranges. An example of the usefulness of n-th matrix ranges is Arveson's result [1] that an irreducible compact operator is characterised up to unitary equivalence by the set of its n-th matrix ranges.) The main goal of a structure theory of C^*-convex sets is to prove a generalised version of the Krein-Milman theorem for ordinary convex sets. That is, we seek to identify the "right" analog of extreme points, and to prove that these are necessary and sufficient to reconstruct the original set (using C^*-convex combinations).

The paper is organised as follows. Section 1 presents some elementary preliminaries (although, for the most part, the reader is assumed to be already familiar with the basic

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facts about C^*-convexity, see [8]), along with references to some more substantial results already in the literature which will be needed later in the paper. Section 2 introduces structural elements and uncovers some elementary facts about them. The C^*-summands, pieces, and weights, and the C^*-faces defined in Section 3 are the technical tools used to prove a generalised Krein-Milman theorem and its converse in Section 4. Section 5 examines the relationship between structural elements and C^*-extreme points, and the paper ends with an improved bound for the Caratheodory type theorem for C^*-convex sets.

1. **Preliminaries and previous results.** The definition of a C^*-convex set was given in the introduction. Three elementary examples of C^*-convex sets are the following:

i) $\{T \in \mathcal{B}(\mathcal{H}) : 0 \leq T \leq \mathbf{1}\}$,

ii) the unit ball of $\mathcal{A}$, and

iii) $W_1 = \{T \in \mathcal{B}(\mathcal{H}) : w(T) \leq 1\}$ (where $w(T)$ is the numerical radius of T).

The proofs that these sets are C^*-convex are elementary, and can be found in [8]. We will reuse these three examples throughout to illustrate new concepts as they are introduced. It is a trivial consequence of the definition that a C^*-convex set $\mathcal{S}$ is closed under unitary equivalence. That is, if $u \in \mathcal{A}$ is unitary, and $x \in \mathcal{S}$, then $y = u^*xu \in \mathcal{S}$. This is a recurrent theme, and many of the concepts introduced (*e.g.* C^*-extreme points, structural elements) are "up to unitary equivalence". We write $y \sim x$ for y is unitarily equivalent to x, and we write $\mathcal{U}(x)$ for the unitary orbit of x.

Loebl and Paulsen [8] first proposed the search for a generalised Krein-Milman theorem, and suggested the following definitions as the appropriate analogue to the definition of an extreme point.

DEFINITION 1.1. Let $\mathcal{A}$ be a C^*-algebra. A *proper C^*-convex combination of* $x_1, \ldots, x_m \in \mathcal{A}$ is a sum of the form $\sum_{i=1}^m t_i^* x_i t_i$ where, in addition to the condition $\sum_i t_i^* t_i = \mathbf{1}$ (for a C^*-convex combination) each $t_i \in \mathcal{A}$ is invertible.

DEFINITION 1.2. Let $\mathcal{S} \subset \mathcal{A}$ be C^*-convex. $x \in \mathcal{S}$ is a *C^*-extreme point of $\mathcal{S}$*, written $x \in \partial_e^* \mathcal{S}$, provided that if $x = \sum_i t_i^* x_i t_i$ is a proper C^*-convex combination of $x_i \in \mathcal{S}$, then $x \sim x_i \; \forall i$.

Once again the analogy with linear extreme points is clear. In fact, C^*-extreme points are linearly extreme [8], but not conversely ([7]—see iii) below). Elementary examples of C^*-extreme points include:

i) for $\{x \in M_n : 0 \leq x \leq \mathbf{1}\}$ the C^*-extreme points are exactly the orthogonal projections (including 0 and $\mathbf{1}$);

ii) for the unit ball the C^*-extreme points are exactly the isometries [7]; and

iii) for $W_1 \subset M_2$, the C^*-extreme points are $\{\lambda \mathbf{1} : |\lambda| = 1\} \cup \mathcal{U}\left\{\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\right\}$ [7].

It should be noted that the restriction to *proper* C^*-convex combinations is often quite significant, more so than in the case of linear convexity. The definition of C^*-extreme points is one example; we shall encounter others (see, for instance, Remark 3.5.8).

Earlier efforts to prove a generalised Krein-Milman theorem have focused on Loebl and Paulsen's definition of C^*-extreme points, although it wasn't until 1990 that Farenick [3] proved that every compact C^*-convex set must have C^*-extreme points. At the same time it is clear that although C^*-extreme points might be sufficient to recover the original set, they certainly cannot be necessary. A simple example is

$$\mathcal{S} = \{x \in M_2 : 0 \le x \le \mathbf{1}\} = C^* - \operatorname{conv}\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}\right\}$$

and these two elements of $\mathcal{S}$ are not even linearly extreme let alone C^*-extreme. The main result of this paper is to overcome this difficulty by defining the structural elements of $\mathcal{S}$, $\operatorname{str}(\mathcal{S})$, ($\mathcal{S} \subset M_n$ compact and C^*-convex), and proving that this set is the right analog of linear extreme points in the sense that $\operatorname{str}(\mathcal{S})$ is both necessary and sufficient to reconstruct the original compact C^*-convex set using C^*-convex combinations.

The next two results are elementary, but they will be used later, and perhaps also serve to give the newcomer to C^*-convexity some of the flavour of the proofs.

PROPOSITION 1.3. *Let $\mathcal{S} \subset M_n$ be C^*-convex, $x = \sum_{i=1}^m t_i^* x_i t_i$ a C^*-convex combination of $x_i \in \mathcal{S}$. Then by combining the terms with $i \ge 2$ we can find $y \in \mathcal{S}$ and $r \in M_n$ so that we can rewrite x as a C^*-convex combination of elements of $\mathcal{S}$ with only two terms $x = t_1^* x_1 t_1 + r^* y r$, ($x_1, y \in \mathcal{S}, t_1^* t_1 + r^* r = \mathbf{1}$).*

PROOF. It is clear that essentially what we want to do is the following: let $r = (\sum_{i\ge 2} t_i^* t_i)^{1/2}$ (notice $r = r^*$), and let $y = r^{-1}(\sum_{i\ge 2} t_i^* x_i t_i) r^{-1}$. It is obvious that $y \in C^* - \operatorname{conv}\{x_2 \cdots x_m\} \subset \mathcal{S}$ because $\sum_{i\ge 2} r^{-1} t_i^* t_i r^{-1} = \mathbf{1}$. Then $x = t_1^* x_1 t_1 + r^* y r \in \mathcal{S}$ because $t_1^* t_1 + r^* r = \mathbf{1}$. The difficulty is, of course, that r need not be invertible. Choose unitary $u \in M_n$ so that $r' = u^* r u = \begin{pmatrix} r'' & \\ & 0 \end{pmatrix}$ with $r'' \ge 0$ invertible. The idea is that it is enough to be able to invert r''. The rest is merely details.

Let $x' = u^* x u = \sum_i (u^* t_i^* u)(u^* x_i u)(u^* t_i u) = \sum_i t_i'^* x_i' t_i'$ a C^*-convex combination. Without loss of generality we may assume $0 \in \mathcal{S}$. Let

$$s' = \begin{pmatrix} (r'')^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix},$$

and $y' = s'(\sum_{i\ge 2} t_i'^* x_i' t_i') s' + s\mathbf{0}s$. $y' \in \mathcal{S}$ because $\mathbf{0}, x \in \mathcal{S}$ and $\sum_{i\ge 2} s' t_i'^* t_i' s' + s^2 = \mathbf{1}_n$. Notice $t_i'^* t_i' \le r'^2$ so $t_i' s' r' = t_i'$. Thus

$$\begin{aligned} t_1'^* x_1' t_1' + r' y' r' &= t_1'^* x_1' t_1' + r'\Big(s'\Big(\sum_{i\ge 2} t_i'^* x_i' t_i'\Big) s' + s\mathbf{0}s\Big) r' \\ &= t_1'^* x_1' t_1' + \sum_{i\ge 2} t_i'^* x_i' t_i' \\ &= x' \\ &= u^* x u, \end{aligned}$$

so

$$x = ux'u^* = ut_1'^* x_1' t_1' u^* + ur'y'r'u^*$$
$$= (ut_1'^* u^*)(ux_1'u^*)(ut_1'u^*) + u\Big(\sum_{i\geq 2} t_i'^* x_i' t_i'\Big)u^*$$
$$= t_1^* x_1 t_1 + \sum_{i\geq 2} t_i^* x_i t_i.$$

Thus $x = t_1^* x_1 t_1 + r^* y r$ where $r = ur'u^*$ and $y = uy'u^* \in \mathcal{S}$. ∎

Notice that, unlike ordinary convexity, if $x_i = x_j$ it is not usually possible to combine the terms $t_i^* x_i t_i + t_j^* x_j t_j$ into a single term $s^* x_i s$. As an example

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

but

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq u^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u$$

for any (unitary) u.

In order to define the compression of a C^*-convex set from M_n to M_k ($k < n$), we define

$$P_{nk} = \begin{pmatrix} \mathbf{1}_k \\ 0 \end{pmatrix} \in M_{n,k}.$$

Thus for $x \in M_n$, $P_{nk}^* x P_{nk}$ is the compression of x to M_k.

PROPOSITION 1.4. *Let $\mathcal{S} \subset M_n$ be C^*-convex. Let $\mathcal{S}_k = P_{nk}^* \mathcal{S} P_{nk}$ be its compression to M_k. Then $\mathcal{S}_k$ is C^*-convex, and if $\mathcal{S}$ is compact then so is $\mathcal{S}_k$. Furthermore, if $\mathcal{S} = C^* - \mathrm{conv}(\mathcal{G})$ for some set $\mathcal{G} \subset M_n$, then $\mathcal{S}_k = C^* - \mathrm{conv}\Big(P_{nk}^*\big(\mathcal{U}(\mathcal{G})\big)P_{nk}\Big)$.*

PROOF. Suppose $x = \sum_i t_i^* x_i t_i$ is a C^*-convex combination of $x_i \in \mathcal{S}_k$ ($i = 1, \ldots, m$). To show $x \in \mathcal{S}_k$ we must find $\tilde{x} \in \mathcal{S}$ such that $x = P_{nk}^* \tilde{x} P_{nk}$. Since $x_i \in \mathcal{S}_k$, $\exists \tilde{x}_i \in \mathcal{S}$ such that $x_i = P_{nk}^* \tilde{x}_i P_{nk}$. Let $\tilde{t}_i = t_i \oplus (\frac{1}{\sqrt{m}} \mathbf{1}_{n-k})$, so $\sum_i \tilde{t}_i^* \tilde{t}_i = \mathbf{1}_n$, and let $\tilde{x} = \sum_i \tilde{t}_i^* \tilde{x}_i \tilde{t}_i \in \mathcal{S}$. Clearly $P_{nk}^* \tilde{x} P_{nk} = x$. The statement about compactness is trivial because compression is continuous.

Let $\mathcal{S} = C^* - \mathrm{conv}(\mathcal{G})$. Let $x \in \mathcal{S}_k$ where $x = P_{nk}^* \tilde{x} P_{nk}$ for some $\tilde{x} \in \mathcal{S}$. Thus $\tilde{x} = \sum_i t_i^* g_i t_i$ C^*-convex combination of $g_i \in \mathcal{G}$. Choose unitaries $u_i \in M_n$ such that $u_i t_i P_{nk} = u_i t_i \begin{pmatrix} \mathbf{1}_k \\ 0 \end{pmatrix} = \begin{pmatrix} s_i \\ 0 \end{pmatrix}$ for some $s_i \in M_k$. Now $\sum_i s_i^* s_i = \sum_i P_{nk}^* t_i^* u_i^* u_i t_i P_{nk} = \mathbf{1}_k$ so

$$x = P_{nk}^* \tilde{x} P_{nk} = \sum_i P_{nk}^* t_i^* u_i^* (u_i g_i u_i^*) u_i t_i P_{nk}$$
$$= \sum_i \begin{pmatrix} s_i \\ 0 \end{pmatrix}^* u_i g_i u_i^* \begin{pmatrix} s_i \\ 0 \end{pmatrix}$$
$$= \sum_i s_i^* P_{nk}^* u_i g_i u_i^* P_{nk} s_i$$
$$\in C^* - \mathrm{conv}\Big(P_{nk}^* \mathcal{U}(\mathcal{G}) P_{nk}\Big).$$
∎

We will also require the following more substantial results, already in the literature.

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DEFINITION 1.5. Given $B, M \in M_n$, $r \geq 0$, ($r \in \mathbf{R}$), define the matrix-valued disk $D(B,M;r) = \{\Lambda \in M_n : \|\Lambda \otimes B + \mathbf{1} \otimes M\| \leq r\}$.

These sets were introduced by Farenick [4] where he shows that they are compact and C^*-convex. They are called *matrix-valued disks* because if $n = 1$ they give rise to a classical disk and the next result reduces to a well known result in linear convexity theory. Although little is known of their structure (*e.g.* C^*-extreme points), and for $n > 1$ they are not disks in the usual sense, their importance is the following separation theorem, proved by Farenick.

THEOREM 1.6. *Let $\mathcal{S} \subset M_n$ be compact and C^*-convex. If $T \notin \mathcal{S}$ then $\exists$ matrix-valued disk $D(B,M;r)$ separating T from $\mathcal{S}$, i.e., $D(B,M;r) \supset \mathcal{S}$ but $T \notin D(B,M;r)$.*

This separation theorem is central to the later proof of the generalised Krein-Milman theorem in Section 4, somewhat analogously to the use of the Hahn-Banach separation theorem in the proof of the original Krein- Milman theorem.

Much of what is already known about C^*-extreme points can be found in [3] and [6]. We will require the following theorem from [6] and its corollary from [9].

THEOREM 1.7. *Suppose that $\mathcal{S} = C^* - \mathrm{conv}\{x_\alpha : \alpha \in I\} \subset M_n$, where I is any index set (finite or infinite). If x is C^*-extreme in $\mathcal{S}$ then x is unitarily equivalent to some x_α or x is reducible. Moreover, there exist projections q_i such that $\sum_i q_i = \mathbf{1}, x = \sum_i q_i x'_{\alpha_i} q_i$, and each $x'_{\alpha_i} \sim x_{\alpha_i}$.*

COROLLARY 1.8. *Let $\mathcal{S} = C^* - \mathrm{conv}\{x_\alpha : \alpha \in I\} \subset M_n$, and let $x \in \mathcal{S}$ be irreducible and C^*-extreme. If $x = \sum_{i=1}^m s_i^* z_i s_i$ ($z_i \in \mathcal{S}, s_i \neq 0$), then $\exists$ unitaries $u_i \in M_n$ and $\lambda_i \in \mathbf{R}$, ($0 \leq \lambda_i \leq 1$, $\sum_i \lambda_i{}^2 = 1$), such that $s_i = \lambda_i u_i$, and $z_i = u_i x u_i^*$. (i.e. any C^*-convex combination $x = \sum_i s_i^* z_i s_i = \sum_i (\lambda_i u_i^*)(u_i x u_i^*)(u_i \lambda_i) = \sum_i \lambda_i{}^2 x$ is essentially trivial.)*

PROOF. First we will show that $z_i \sim x \, \forall i$. Suppose, on the contrary, that $z_1 \not\sim x$. From Proposition 1.3 we can write $x = s_1^* z_1 s_1 + r^* y r$, C^*-convex combination with $y \in \mathcal{S}$. Using the same procedure as in the proof of the previous theorem (polar decomposition of s_1, r, absorbing unitaries, etc.) and the same application of Technique B (for all $\lambda \in (0,1)$), we reach the conclusion that x is reducible, which is once again a contradiction. Thus we must have $x \sim z_1$. Similarly, $x \sim z_i \, \forall i$.

Thus $\exists u_i$ unitary such that $z_i = u_i x u_i^*$ and $x = \sum_i s_i^* z_i s_i = \sum_i (s_i^* u_i) x (u_i^* s_i)$. Next we show that $u_i^* s_i = \lambda_i \mathbf{1}$.

Let $\phi(z) = \sum_i (s_i^* u_i) z (u_i^* s_i)$ so $\phi : M_n \longrightarrow M_n$ is (unital) completely positive. Further $\phi(x) = x$ and x is irreducible so $\phi(z) = z \, \forall z$ [1]. Now it follows from Choi's result [2] on the uniqueness of the decomposition of ϕ that $u_i^* s_i = \lambda_i \mathbf{1}$ ($\lambda_i \in \mathbf{C}$) and $\sum_i \overline{\lambda_i} \lambda_i = 1$. To get $\lambda_i \in \mathbf{R}$, (instead of $\lambda_i \in \mathbf{C}$), we simply absorb the argument of λ_i into u_i. That is, if $\lambda_i = e^{i\theta_i} |\lambda_i|$, replace λ_i by $|\lambda_i|$, and u_i by $e^{i\theta_i} u_i$. ■

2. **Structural elements.** In this section we change the focus of our attention from C^*-extreme points as defined by Loebl and Paulsen to structural elements, as defined shortly. The existence of the generalised Krein- Milman Theorem in terms of structural elements (Theorem 4.5) shows that they are a better analog of linear extreme points than C^*-extreme points are. Some of the relation between C^*-extreme points and structural elements is apparent in Propositions 2.2, 5.1, 5.2, and 5.3.

DEFINITION 2.1. Let $\mathcal{S} \subset M_n$ be a compact C^*-convex set. For $x \in \mathcal{S}$, call x a *structural element (of $\mathcal{S}$) of size n* if whenever $x = \sum_i t_i^* x_i t_i$ is a C^*-convex combination of elements of $\mathcal{S}$, then there exist unitaries $u_i \in M_n$, and scalars $\lambda_i \in [0,1]$, such that $x_i = u_i x u_i^*$, $t_i = u_i \lambda_i$, and $\sum_i \lambda_i^2 = 1$. We write $x \in \mathrm{str}(\mathcal{S}, n)$.

A necessary consequence of the definition of $\mathrm{str}(\mathcal{S}, n)$ is that if $x = \sum_i t_i^* x_i t_i$ then each $t_i^* x_i t_i = \lambda_i^2 x$. Thus structural elements of size n have a certain similarity to linear extreme points in that the only ways to write them as a C^*-convex combination are essentially trivial. The following gives the first relation between structural elements and C^*-extreme points.

PROPOSITION 2.2. *Let $\mathcal{S} \subset M_n$ be compact and C^*-convex. Then the structural elements of $\mathcal{S}$ of size n correspond exactly to the irreducible C^*-extreme points.*

PROOF. It is immediate from the definition that if x is structural of size n then x is C^*-extreme. Furthermore if x is reducible then choose any scalar $\lambda \mathbf{1} \in \mathcal{S}$ and observe

$$x \sim \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}$$

where the coefficient $\begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}$ cannot be of the form $\lambda_i u_i$ for any scalar λ_i and unitary u_i. Thus $x \in \mathrm{str}(\mathcal{S}, n)$ implies x is irreducible and C^*-extreme.

The converse is Corollary 1.8. ■

Unfortunately a C^*-convex set $\mathcal{S}$ may not have any structural elements of size n, let alone enough to reconstruct $\mathcal{S}$, so now we extend the definition of structural elements to sizes other than n.

DEFINITION 2.3. For $\mathcal{S} \subset M_n$ compact and C^*-convex let $\mathcal{S}_k$ be its compression to M_k ($\mathcal{S}_k$ is still compact, C^*-convex; see Proposition 1.4). Define $x \in M_k$ to be a *structural element (of $\mathcal{S}$) of size k* provided

i) x is a structural element of $\mathcal{S}_k$ of size k, and

ii) $x \notin \{$compressions to M_k of structural elements (of $\mathcal{S}$) of size j, $k < j \leq n\}$. (The definition is inductive, starting from structural elements of size n as defined previously.) We write $x \in \mathrm{str}(\mathcal{S}, k)$. We also define the structure set of $\mathcal{S}$ to be $\mathrm{str}(\mathcal{S}) = \bigcup_{k=1}^{n} \mathrm{str}(\mathcal{S}, k)$.

The following examples of structural elements should be compared with the C^*-extreme points of the same sets (as given in a previous example):

i) for $\mathcal{S} =$ the unit ball of M_n: $\mathrm{str}(\mathcal{S}) = \{\lambda \in \mathbf{C} : |\lambda| = 1\}$,

ii) for $\mathcal{S} = \{x \in M_n : \mathbf{0} \leq x \leq \mathbf{1}\}$: $\mathrm{str}(\mathcal{S}) = \{0, 1\} \subset \mathbf{C}$,

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iii) for $\mathcal{S} = \{x \in M_2 : w(x) \leq 1\}$: $\operatorname{str}(\mathcal{S}) = \mathcal{U}\left\{\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\right\}$.

Because the elements of $\operatorname{str}(\mathcal{S})$ are not necessarily all the same size we need the following notation. For $x \in \operatorname{str}(\mathcal{S}, k)$ and $m > k$ define $x(m) \in M_m$ by $x(m) = x \oplus (\lambda \mathbf{1}_{m-k})$ for some $\lambda \in W(x)$. (It is not important which $\lambda \in W(x)$ is chosen since any such choice will ensure $x(m) \in \mathcal{S}_m$. It is important that $\lambda \in W(x)$ in order that $C^* - \operatorname{conv}\{x\} = P^*_{mk}\Big(C^* - \operatorname{conv}\big(x(m)\big)\Big)P_{mk}$, see the next Lemma.) Further define $C^* - \operatorname{conv}_m\big(\operatorname{str}(\mathcal{S})\big) = C^* - \operatorname{conv}\{x(m) : x \in \operatorname{str}(\mathcal{S})\} \subset M_m$; *i.e.*, extend all the structural elements to $m \times m$ matrices and take the C^*-convex hull in M_m. Of course this makes sense only when all elements of $\operatorname{str}(\mathcal{S})$ are of size $\leq m$. We will continue to use the notation $C^* - \operatorname{conv}(\mathcal{G})$, with no subscript, provided all elements of $\mathcal{G}$ are the same size (*i.e.* $\mathcal{G} \subset M_n$).

In order to prove the generalised Krein-Milman theorem we need some elementary facts about structural elements. The following lemma, useful for this purpose, can also be used to show that $C^* - \operatorname{conv}_n(s)$ (for $s \in M_k$) doesn't depend on the choice of $\lambda \in W(s)$.

LEMMA 2.4. *Let $s \in M_k$ be irreducible, $\lambda \in W(s)$, $s(n) = s \oplus \lambda \mathbf{1}_{n-k}$. Let $\mathcal{S} = C^* - \operatorname{conv}\big(s(n)\big)$ $\big(= C^* - \operatorname{conv}_n(s)\big)$. Then $\mathcal{S}_k = C^* - \operatorname{conv}(s)$ (independent of λ).*

PROOF. Consider

$$x = \sum_i t_i^* \begin{pmatrix} s & \\ & \lambda \mathbf{1}_{n-k} \end{pmatrix} t_i \in \mathcal{S}$$

($t_i \in M_n, \sum_i t_i^* t_i = \mathbf{1}$). We must show that $P^*_{nk} x P_{nk} \in C^* - \operatorname{conv}(s)$. Let

$$t_i = \begin{pmatrix} t_{i1} & t_{i2} \\ t_{i3} & t_{i4} \end{pmatrix}.$$

The condition $\sum_i t_i^* t_i = \mathbf{1}_n$ implies $\sum_i (t_{i1}^* t_{i1} + t_{i3}^* t_{i3}) = \mathbf{1}_k$ where $t_{i1} \in M_k, t_{i3} \in M_{n-k,k}$. Now $P^*_{nk} x P_{nk} = \sum_i (t_{i1}^* s t_{i1} + t_{i3}^* \lambda \mathbf{1}_{n-k} t_{i3})$.

If $n - k \leq k$ let

$$\tilde{t}_{i3} = \begin{pmatrix} t_{i3} \\ 0 \end{pmatrix} \in M_k$$

so $\tilde{t}_{i3}^* \tilde{t}_{i3} = t_{i3}^* t_{i3}$ and $P^*_{nk} x P_{nk} = \sum_i t_{i1}^* s t_{i1} + \tilde{t}_{i3}^* \lambda \mathbf{1}_k \tilde{t}_{i3} \in C^* - \operatorname{conv}(s)$. (Recall $\lambda \in W(s) \Rightarrow \lambda \mathbf{1}_k \in C^* - \operatorname{conv}(s)$.)

If $n - k > k$ then there exist unitaries $u_i \in M_{n-k}$ such that

$$u_i t_{i3} = \begin{pmatrix} t'_{i3} \\ 0 \end{pmatrix} \quad \text{for some } t'_{i3} \in M_k.$$

Hence $t_{i3}^* \lambda \mathbf{1}_{n-k} t_{i3} = t_{i3}^* u_i^* \lambda \mathbf{1}_{n-k} u_i t_{i3} = t'_{i3} \lambda \mathbf{1}_k t'_{i3}$. Also $t_{i3}^* t_{i3} = t_{i3}^* u_i^* u_i t_{i3} = t'^*_{i3} t'_{i3}$ so $P^*_{nk} x P_{nk} = \sum_i (t_{i1}^* s t_{i1} + t'^*_{i3} \lambda \mathbf{1}_k t'_{i3}) \in C^* - \operatorname{conv}(s)$. ∎

PROPOSITION 2.5. *Let $s \in M_k$ be irreducible and let $\mathcal{S} = C^* - \operatorname{conv}_n(s)$ $(n \geq k)$. Then $\operatorname{str}(\mathcal{S}) = \mathcal{U}(s)$.*

PROOF. In order for $x \in M_j$ to be in $\operatorname{str}(\mathcal{S})$, x must be irreducible and C^*-extreme in $\mathcal{S}_j$. If $j > k$ then it is easy to see that $\mathcal{S}_j = C^* - \operatorname{conv}_j(s) = C^* - \operatorname{conv}\big(s(j)\big)$; and since $s(j)$

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is reducible, so are the C^*-extreme points of $\mathcal{S}_j$ (by Theorem 1.7), and so $\text{str}(\mathcal{S}, j) = \phi$ $(j > k)$. If $j = k$, $\mathcal{S}_k = C^* - \text{conv}_k(s)$, and s is irreducible so $\text{str}(\mathcal{S}, k) = \mathcal{U}(s)$. Finally for $j < k$, $\mathcal{S}_j = P^*_{kj}\mathcal{S}_k P_{kj} = P^*_{kj}C^* - \text{conv}_k(s)P_{kj}$, so $\text{str}(\mathcal{S}, j) = \phi$ $(j < k)$. ∎

The next two propositions concern the behaviour of the set of structural elements under compression.

PROPOSITION 2.6. *Let $\mathcal{S} \subset M_n$ be compact and C^*-convex, $\mathcal{S}_k$ its compression to M_k $(1 \leq k < n)$. Then* $\text{str}(\mathcal{S}, k) \subset \text{str}(\mathcal{S}_k, k) \subset \text{str}(\mathcal{S}, k) \cup \{$*compressions of* $\text{str}(\mathcal{S}, j)$ *to* $M_k, k < j \leq n\}$.

PROOF. Let $x \in \text{str}(\mathcal{S}, k)$. Then x is irreducible and C^*-extreme in $\mathcal{S}_k$ so $x \in \text{str}(\mathcal{S}_k, k)$.

Next, suppose $x \in \text{str}(\mathcal{S}_k, k)$. Then x is irreducible and C^*-extreme in $\mathcal{S}_k$, so either $x \in \text{str}(\mathcal{S}, k)$ or $x \in \{$compressions of $\text{str}(\mathcal{S}, j)$ to $M_k, k < j \leq n\}$. ∎

PROPOSITION 2.7. *Same hypotheses as in the previous proposition, plus* $1 \leq m < k$. *Then* $\text{str}(\mathcal{S}, m) \subset \text{str}(\mathcal{S}_k, m) \subset \text{str}(\mathcal{S}, m) \cup \{$*compressions of* $\text{str}(\mathcal{S}, j)$ *to* $M_m, m < j \leq n\}$.

PROOF. Notice $\mathcal{S}_m = (\mathcal{S}_k)_m$ (the compression of $\mathcal{S}_k$ to M_m). We have proven the case $m = k$ above. Let $m = k - 1$. Let $x \in \mathcal{S}_m$ be irreducible and C^*-extreme. If $x \notin \text{str}(\mathcal{S}_k, m)$ then x belongs to the compression of $\text{str}(\mathcal{S}_k, k)$ to $M_k \subset \{$compressions of $\text{str}(\mathcal{S}, j)$ to M_m, $m < j \leq n\} \Rightarrow x \notin \text{str}(\mathcal{S}, m)$. Thus $\text{str}(\mathcal{S}, m) \subset \text{str}(\mathcal{S}_k, m)$.

Next, $x \in \text{str}(\mathcal{S}_k, m) \Rightarrow x$ irreducible, C^*-extreme in $\mathcal{S}_m \Rightarrow x \in \text{str}(\mathcal{S}, m)$ or $x \in \{$compressions of $\text{str}(\mathcal{S}, j)$ to $M_m, m < j \leq n\}$. Continuing inductively gets the result for all $m < k$.

3. **Technical tools.** We will have more to say about structural elements, including their relation to C^*-extreme points, after we have proved the main theorem. Now we introduce some concepts whose main interest is their usefulness in proving the Krein-Milman theorem. They are generalisations of similar concepts in the linear convexity case, and it may help the reader to translate the definitions and the results immediately following into the more familiar setting of linear convexity where the geometry is more transparent. The idea behind the next set of definitions is as follows: Suppose $x = a^*ya + b^*zb$ is a C^*-convex combination (*i.e.* $a^*a + b^*b = \mathbf{1}$) with $a \neq 0$. Then y (also z) is called a C^*-*summand* of x, a^*ya is called a C^*-*piece* of x, and a^*a is called the *weight* of a^*ya as a piece of x. These three sets are useful in proving the generalised Krein-Milman theorem. The precise definitions are given next.

DEFINITIONS 3.1. Let $\mathcal{S} \subset M_n$ be compact and C^*-convex, $x \in \mathcal{S}$.

i) The set of C^*-*summands of* x, $C^* - \text{summ}(x) = \{y \in \mathcal{S} : \exists z \in \mathcal{S}, a, b \in M_n, a \neq 0$, such that $x = a^*ya + b^*zb$ (C^*-convex combination)$\}$.
ii) The set of C^*-*pieces of* x, $\text{pcs}(x) = \{a^*ya : \exists y, z \in \mathcal{S}, a, b \in M_n$ such that $x = a^*ya + b^*zb$(C^*-convex combination)$\}$.
iii) Given $r \in \text{pcs}(x)$, the set of *weights of* r *as a piece of* x, $\mathcal{W}(r; x) = \{a^*a : \exists a, b \in M_n, y_1, y_2 \in \mathcal{S}$ with $x = a^*y_1a + b^*y_2b$ and $r = a^*y_1a\}$.

REMARKS 3.2. 1) In i) above the restriction $a \neq 0$ keeps the set of C^*-summands proper. In ii) taking $a = 0$ implies $0 \in \text{pcs}(x)\ \forall x$ which is convenient—see 3) below.

2) $C^* - \text{summ}(x)$ need not be convex. Let $\mathcal{S} = \{t \in M_2 : 0 \leq t \leq \mathbf{1}\}$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ are } C^*\text{-summands of } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ but}$$
$$\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} = 1/2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 1/2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

is not. It is easy to see that $x \in C^* - \text{summ}(\mathbf{1})$ iff x is unitarily equivalent to a matrix of the form $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$, $0 \leq a \leq 1$.

3) Unlike ordinary convexity, $y \in C^*-\text{summ}(x)$ and $z \in C^*-\text{summ}(y)$ does not imply $z \in C^* - \text{summ}(x)$. As an example, let $\mathcal{S}$ be as in the previous remark, $x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$, $z = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/4 \end{pmatrix}$.

4) $\text{pcs}(x)$ is compact and convex.

5) $\mathcal{W}(r; x)$ is compact and convex.

6) Let $r, r_1, r_2 \in \text{pcs}(x)$, $\alpha \in (0, 1)$, with $r = \alpha r_1 + (1 - \alpha) r_2$. Then $\mathcal{W}(r; x) \supset \alpha \mathcal{W}(r_1; x) + (1 - \alpha) \mathcal{W}(r_2; x)$.

7) If $0 \in \mathcal{S}$ then $\text{pcs}(x) \subset \mathcal{S}$.

We illustrate the above definitions with some examples taken from $\mathcal{S} = W_1 \subset M_2$. Let $x = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \in \mathcal{S}$. x is extreme but not C^*-extreme in $\mathcal{S}$ [7]. In fact

$$x = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -i/2 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & i/2 \end{pmatrix} + \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & i/2 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & -i/2 \end{pmatrix}.$$

PROPOSITION 3.3. *$y \in C^* - \text{summ}(x)$ iff at least one of $1, i \in W(y)$.*

PROOF. Suppose $1 \in W(y)$. Then $y \sim \begin{pmatrix} 1 & * \\ * & * \end{pmatrix}$ so

$$x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} y \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and so $y \in C^* - \text{summ}(x)$. Similarly for $i \in W(y)$.

On the other hand, suppose $y \in C^* - \text{summ}(x)$, so $x = a^* y a + b^* z b$, a C^*-convex combination with $a \neq 0$. Let $\{e_1, e_2\}$ be the usual orthonormal basis for $\mathbf{C}^2$. If $ae_1 \neq 0$ then

$$\begin{aligned} 1 &= (xe_1, e_1) \\ &= (yae_1, ae_1) + (zbe_1, be_1) \\ &= \|ae_1\|^2 \Big(y\Big(\frac{ae_1}{\|ae_1\|}\Big), \Big(\frac{ae_1}{\|ae_1\|}\Big) \Big) + \|be_1\|^2 \Big(z\Big(\frac{be_1}{\|be_1\|}\Big), \Big(\frac{be_1}{\|be_1\|}\Big) \Big) \\ &= \|ae_1\|^2 \alpha + \|be_1\|^2 \beta. \end{aligned}$$

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This represents 1 as a convex combination (since $\|ae_1\|^2 + \|be_1\|^2 = 1$) of $\alpha \in W(y) \subset \Delta = \{\lambda \in \mathbf{C} : |\lambda| \leq 1\}$, and $\beta \in W(z) \subset \Delta$. But 1 is extreme in Δ so $\alpha = 1 \in W(y)$. If $ae_1 = 0$, then $ae_2 \neq 0$ and a similar argument shows $i \in W(y)$. ■

The set pcs(x) is slightly harder to describe, and we will begin by giving some examples. Obviously $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} \in \text{pcs}(x)$. Also, from the C^*-convex combination given above,

$$\begin{pmatrix} 1/2 & i/2 \\ 1/2 & i/2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -i/2 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & i/2 \end{pmatrix}$$

and,

$$\begin{pmatrix} 1/2 & -i/2 \\ -1/2 & i/2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & i/2 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & -i/2 \end{pmatrix} \in \text{pcs}(x).$$

The set pcs(x) can be described completely as follows. For any $k \in \mathbf{N}$ and $j < k$, choose orthogonal unit vectors $(\alpha_1, \ldots, \alpha_k)$, $(\beta_1, \ldots, \beta_k) \in \mathbf{C}^k$. Let $p = \sum_{i=1}^{j} \bar{\alpha}_i \alpha_i$, $q = \sum_{i=1}^{j} \bar{\beta}_i \beta_i$, $\lambda = \sum_{i=1}^{j} \bar{\alpha}_i \beta_i$. Then $y = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} p & \lambda \\ \bar{\lambda} & q \end{pmatrix} \in \text{pcs}(x)$, and every $y \in \text{pcs}(x)$ is of this form. The conditions on p, q, and λ are, of course, equivalent to requiring $\begin{pmatrix} p & \lambda \\ \bar{\lambda} & q \end{pmatrix} \geq 0$.

For $r = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\mathcal{W}(r;x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. For $r = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}$, $\mathcal{W}(r;x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. To see an easy example where $\mathcal{W}(r;x)$ is not a singleton, take $x = \begin{pmatrix} 1/2 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{S}$, $r = \begin{pmatrix} 1/4 & 1/2 \\ 0 & 0 \end{pmatrix}$. $x = (1/\sqrt{2})x(1/\sqrt{2}) + (1/\sqrt{2})x(1/\sqrt{2})$ so $r = (1/2)x \in \text{pcs}(x)$ and $1/2 \in \mathcal{W}(r;x)$. But we can also write

$$x = \left\{(1/4)\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} + (1/4)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix}\right\}$$
$$+ \left\{\begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix} + (1/4)\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\right\}.$$

This is a C^*-convex combination, and the sum of the first three terms is $r = x/2$ so $\begin{pmatrix} 1/2 & 0 \\ 0 & 3/4 \end{pmatrix} \in \mathcal{W}(r;x)$. The sum of the last two terms is also r, so $\begin{pmatrix} 1/2 & 0 \\ 0 & 1/4 \end{pmatrix} \in \mathcal{W}(r;x)$. This shows that $\mathcal{W}(r;x)$ need not be a singleton. It is easy to construct a similar C^*-convex combination to show $\begin{pmatrix} 3/4 & 0 \\ 0 & 1/2 \end{pmatrix} \in \mathcal{W}(r;x)$, which shows that not all elements of $\mathcal{W}(r;x)$ need be comparable. If we had examined the analog of these definitions in the linear convexity case, there too it would be true that $\mathcal{W}(r;x)$ need not be a singleton. However in that case of course, every element of $\mathcal{W}(r;x)$ would be comparable.

Another useful concept that we can borrow from linear convexity theory is that of a face, generalised below to a C^*-face.

DEFINITION 3.4. A *C^*-face* $\mathcal{F}$ of a C^*-convex set $\mathcal{S}$ is a non-empty subset of $\mathcal{S}$ such that if $x \in \mathcal{F}$ and $x = \sum_i t_i^* x_i t_i$ is a proper C^*-convex combination of elements $x_i \in \mathcal{S}$ (*i.e.* $\sum_i t_i^* t_i = \mathbf{1}$, and t_i invertible $\forall i$), then necessarily $x_i \in \mathcal{F}\, \forall i$.

REMARKS 3.5. 1) C^*-faces exist; $\mathcal{S}$ is a C^*-face of $\mathcal{S}$. Also if $\mathcal{S} \subset M_n$ is compact and C^*-convex and $B, M \in M_n$ then $\{x \in \mathcal{S} : \|x \otimes B + \mathbf{1} \otimes M\| = \sup_{s \in \mathcal{S}}\{\|s \otimes B + \mathbf{1} \otimes M\|\}\}$ is a compact C^*-face of $\mathcal{S}$ [9].

2) Unlike faces of ordinary convex sets, C^*-faces are not usually C^*-convex, or even convex. In view of Remark 8 below, our previous example that $C^* - \mathrm{summ}(x)$ need not be convex also serves as an example that C^*-faces need not be convex.

The fact that C^*-faces are not C^*-convex seems unavoidable. The difficulty is that almost all C^*-convex subsets of M_n have non-empty interior. Specifically, if $\mathcal{S}$ is C^*-convex, let $W(\mathcal{S}) = \bigcup\{W(x)$ (the numerical range of x): $x \in \mathcal{S}\}$. $\mathcal{S}$ has non-empty interior iff $W(\mathcal{S})$ does [11]. Furthermore, $W(\mathcal{S})$ has empty interior iff it is the line segment $[\alpha, \beta]$ (in $\mathbf{C}$), and so $\mathcal{S} = (\beta - \alpha)\mathcal{P} + \alpha\mathbf{1}$ (where $\mathcal{P} = \{x \in M_n : 0 \leq x \leq \mathbf{1}\}$ is a particularly simple C^*-convex set). If we included in the definition of a C^*-face the requirement that it be C^*-convex (and hence convex, and hence a face in the usual sense), then most C^*-faces would have non-empty interior, and so would be the trivial face, all of $\mathcal{S}$. The only other possibilities would be of the form $\mathcal{F} = (\beta - \alpha)\mathcal{P} + \alpha\mathbf{1}$ where $[\alpha, \beta]$ was a face of $W(\mathcal{S})$. This would clearly give too little structural information to be useful. See also Corollary 5.4 which is the C^*-version of the usual result that a minimal compact face is an extreme point.

3) If $\mathcal{F}$ is a C^*-face and $x \in \mathcal{F}$ then $\mathcal{U}(x) \subset \mathcal{F}$. $\mathcal{F} = \mathcal{U}(x)$ if and only if $x \in \partial_e^* \mathcal{S}$.

4) If $\mathcal{F}$ is a C^*-face of $\mathcal{S}$ and $\mathcal{F}_1 \subset \mathcal{F}$ is a C^*-face of $C^* - \mathrm{conv}\, \mathcal{F}$ then $\mathcal{F}_1$ is a C^*-face of $\mathcal{S}$.

5) Let $\mathcal{F}$ be a C^*-face of $\mathcal{S}$ and let $x \in \mathcal{F}$ be C^*-extreme in $C^* - \mathrm{conv}\, \mathcal{F}$. Then $x \in \partial_e^* \mathcal{S}$.

6) The intersection of C^*-faces is a C^*-face, provided it is non-empty.

7) Every compact C^*-face contains a minimal compact C^*-face (by Zorn's lemma). Furthermore, a minimal compact C^*-face "is" a C^*-extreme point—see Proposition 5.4.

8) $C^* - \mathrm{summ}(x)$, as defined previously, is obviously a C^*-face, but because the definition of a C^*-face requires the coefficients to be invertible, and the definition of $C^* - \mathrm{summ}(x)$ does not, $C^* - \mathrm{summ}(x)$ need not be the minimal C^*-face containing x. We will see (Lemma 4.1) that in M_n, $C^* - \mathrm{summ}(x)$ is closed.

We will also need the following elementary properties of minimal compact C^*-faces.

PROPOSITION 3.6. *Let $\mathcal{F}$ be a minimal compact C^*-face of $\mathcal{S}$. Then $\forall x, y \in \mathcal{F}$, $C^* - \mathrm{conv}\{x\} = C^* - \mathrm{conv}\{y\}$.*

PROOF. Suppose $y \notin C^* - \mathrm{conv}\{x\}$. Then by [4] $\exists B, M \in M_n$, $r > 0$, so that $D(B, M; r)$ separates y from $C^* - \mathrm{conv}\{x\}$. Let $r' = \sup_{z \in \mathcal{F}}\{\|z \otimes B + \mathbf{1} \otimes M\|\}$. Then $\mathcal{F}' = \{z \in \mathcal{F} : \|z \otimes B + \mathbf{1} \otimes M\| = r'\}$ is a compact C^*-face of $\mathcal{S}$: if $z \in \mathcal{F}' \subset \mathcal{F}$ and $z = \sum_i t_i^* x_i t_i$ is a proper C^*-convex combination, then $x_i \in \mathcal{F}$ (because $\mathcal{F}$ is a C^*-face) so $\|x_i\| \leq \|z\| = r'\, \forall i$. Using the invertibility of the t_i it is clear that $\|x_i\| = \|z_i\| = r'\, \forall i$, so $x_i \in \mathcal{F}' \, \forall i$. Now $\mathcal{F}' \subset_{\neq} \mathcal{F}$ (since $x \in \mathcal{F} \setminus \mathcal{F}'$) contradicting minimality of $\mathcal{F}$. Thus $y \in C^* - \mathrm{conv}\{x\}$ and similarly $x \in C^* - \mathrm{conv}\{y\}$ so $C^* - \mathrm{conv}\{x\} = C^* - \mathrm{conv}\{y\}$. ∎

COROLLARY 3.7. *Let $\mathcal{F}$ be a minimal compact C^*-face of $\mathcal{S}$. If $\mathcal{F} = \mathcal{U}(x)$ for some $x \in \mathcal{S}$, then $x \in \partial_e^* \mathcal{S}$. In particular, if any $x \in \mathcal{F}$ is irreducible then $\mathcal{F} = \mathcal{U}(x)$ and $x \in \partial_e^* \mathcal{S}$.*

PROOF. Suppose $\mathcal{F} = \mathcal{U}(x)$ and $x = \sum_i t_i^* x_i t_i$ is a proper C^*-convex combination of $x_i \in \mathcal{S}$. Then $x_i \in \mathcal{F} \, \forall i$ (by definition of a C^*-face), so $x_i \sim x \, \forall i$ and $x \in \partial_e^* \mathcal{S}$.

Suppose some $x \in \mathcal{F}$ is irreducible. By Proposition 3.6, $C^* - \mathrm{conv}\{x\} = C^* - \mathrm{conv}\{y\} \, \forall y \in \mathcal{F}$. By [6], because x is irreducible, it follows that $x \sim y \, \forall y \in \mathcal{F}$, so $\mathcal{F} = \mathcal{U}(x)$. Thus by the first paragraph above, $x \in \partial_e^* \mathcal{S}$. ∎

4. **Generalised Krein-Milman theorem.** Our first step is to prove that $C^* - \mathrm{summ}(x)$ is closed. It would be nice if we could use the following argument. Let $y \in \overline{C^* - \mathrm{summ}(x)}$. That is, $\exists y_i, z_i \in C^* - \mathrm{summ}(x)$ ($x = a_i^* y_i a_i + b_i^* z_i b_i$ C^*-convex combinations) with $y_i \longrightarrow y$. Passing to subsequences if necessary, we may assume $a_i \longrightarrow a, b_i \longrightarrow b, z_i \longrightarrow z$ and so $x = a^* y a + b^* z b$, and we could conclude $y \in C^* - \mathrm{summ}(x)$, except that a might be 0. The following result essentially ensures that this can be avoided.

LEMMA 4.1. *Let $\mathcal{S}$ be compact and C^*-convex, $x \in \mathcal{S}$. Then $\exists \epsilon > 0$ such that if $y \in C^* - \mathrm{summ}(x)$ and a^*a is maximal in $\{a^*a : x = a^*ya + b^*zb \ (C^*\text{-convex combination})\}$, then $\|a^*a\| \geq \epsilon$.*

PROOF. Without loss of generality, we will assume $0 \in \mathcal{S}$. Consider the set $P_W = \{(r, a^*a) : r \in \mathrm{pcs}(x), a^*a \in \mathcal{W}(r; x)\} \subset M_n \times M_n^+$. This set is convex by 3, 4, and 5 of Remarks 3.2, and $(0,0) \in P_W$ (because $0 \in \mathcal{S}$) so it is contained in a k-dimensional subspace $F_k \subset M_n \times M_n^+$. Furthermore P_W is symmetric with respect to the point $(x/2, \mathbf{1}/2)$. To see this, if $(r, a^*a) \in P_W$ then $\exists y, z \in \mathcal{S}, b \in M_n$ such that $x = a^*ya + b^*zb$ a C^*-convex combination with $r = a^*ya$. Thus $(b^*zb, b^*b) \in P_W$ and

$$(x/2, \mathbf{1}/2) = \frac{1}{2}(a^*ya, a^*a) + \frac{1}{2}(b^*zb, b^*b).$$

From this symmetry it follows that $\exists \epsilon > 0$ so that $B_\epsilon(x/2, \mathbf{1}/2) \cap F_k \subset P_W$. (Define the open ball $B_\epsilon(x/2, \mathbf{1}/2) = \{(r, w) \in M_n \times M_n^+ : \|r - x/2\| + \|w - \mathbf{1}/2\| < \epsilon\}$.)

Now suppose $x \in \mathcal{S}$ and $x = a^*ya + b^*zb$ (a C^*-convex combination of $y, z \in \mathcal{S}$) with $a^*a < \epsilon \mathbf{1}$ (*i.e.* $\|a^*a\| < \epsilon$). Without loss of generality we will assume that $\forall s \in \mathcal{S}$, $\|s\| \leq 1$, and so $\|a^*ya\| < \epsilon \, \forall y \in \mathcal{S}$. Let

$$r_1 = \frac{b^*zb}{2}, \quad r_2 = \frac{b^*zb}{2} + a^*ya \in \mathrm{pcs}(x)$$

with

$$\frac{b^*b}{2} \in \mathcal{W}(r_1; x), \quad \frac{b^*b}{2} + a^*a \in \mathcal{W}(r_2; x).$$

Notice

$$\left(r_1, \frac{b^*b}{2}\right), \quad \left(r_2, \frac{b^*b}{2} + a^*a\right) \in B_\epsilon(x/2, \mathbf{1}/2) \cap F_k \subset P_W$$

because

$$(x/2, \mathbf{1}/2) = \Big(\frac{a^*ya + b^*zb}{2}, \frac{a^*a + b^*b}{2}\Big)$$

and $\|a^*ya\| < \epsilon$. Thus there exists $\delta > 0$ such that

$$(1+\delta)\Big(r_2, \frac{b^*b}{2} + a^*a\Big), \quad \Big(r_1 - \delta r_2, (1-\delta)\frac{b^*b}{2} - \delta a^*a\Big) \in B_\epsilon(x/2, \mathbf{1}/2) \cap F_k \subset P_W.$$

Thus there exists $z' \in \mathcal{S}$, $b' \in M_n$ such that $r_1 - \delta r_2 = b'^*z'b'$ and $b'^*b' = (1-\delta)b^*b/2 - \delta a^*a$, and so

$$\begin{aligned} x &= (1+\delta)r_2 + (r_1 - \delta r_2) \\ &= (1+\delta)\Big(\frac{b^*zb}{2} + a^*ya\Big) + b'^*z'b' \\ &= (1+\delta)a^*ya + \Big(\frac{1+\delta}{2}\Big)b^*zb + b'^*z'b' \\ &= (1+\delta)a^*ya + b''^*z''b'' \qquad \text{(using Proposition 1.2.1)} \end{aligned}$$

a C^*-convex combination . But $(1+\delta)a^*a > a^*a$, so (if $\|a^*a\| < \epsilon$) a^*a is not maximal. This completes the proof. ∎

Now it is a simple matter to prove the following.

PROPOSITION 4.2. *Let $\mathcal{S} \subset M_n$ be compact and C^*-convex. Then* $C^* - \text{summ}(x)$ *is closed.*

PROOF. Let $y \in \overline{C^* - \text{summ}}(x)$. Then $\exists a_k, b_k \in M_n$, $y_k, z_k \in \mathcal{S}$ such that $x = a_k^*(y_k)a_k + b_k^*z_kb_k$ with $a_k^*a_k + b_k^*b_k = \mathbf{1}$ and $y_k \longrightarrow y$. Without loss of generality we can assume the $a_k^*a_k$ are maximal in the same sense as the previous lemma. Passing to subsequences if necessary we may assume $a_k \longrightarrow a$, $b_k \longrightarrow b$, $z_k \longrightarrow z$, and $x = a^*ya + b^*zb$ (C^*-convex combination). By maximality of a_k^*a and the previous lemma, it follows that $\|a^*a\| \geq \epsilon$, (in particular $a \neq 0$), so $y \in C^* - \text{summ}(x)$. ∎

The proof of the generalised Krein-Milman theorem involves induction, and the following Lemma is an essential part of the inductive step. In proving the following Lemma we make use of two facts about convex sets in a finite dimensional space. First, if C and C' are convex with C' a dense subset of C, and if $x \in \text{int}\, C$, then $x \in C'$. (If C is contained in a hyperplane H, $x \in \text{int}\, C$ is understood to mean $x \in \text{int}(C \cap H)$). Second, let C be a ray $\{kx : k \geq 0\}$ ($x \neq 0$), and let C' be a closed pointed cone with $C \cap C' = \{0\}$. Then there is a linear functional ρ separating C' from C in the sense that $\rho(C) \geq 0$, $\rho(C') \leq 0$, and for $x \in C'$, $\rho(x) = 0$ iff $x = 0$. The proofs of these two facts are elementary and are omitted. We write $\mathcal{S}_{\text{red}}$ for $\Big\{x \in \mathcal{S} : x \text{ is reducible, } \textit{i.e.}, x \sim \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}\Big\}$.

LEMMA 4.3. *Let $\mathcal{S} \subset M_n$ be compact and C^*-convex. Then $\mathcal{S} = C^* - \text{conv}\big(\mathcal{S}_{\text{red}} \cup \text{str}(\mathcal{S}, n)\big)$.*

PROOF. Suppose $x \in \mathcal{S} \setminus C^* - \text{conv}\big(\mathcal{S}_{\text{red}} \cup \text{str}(\mathcal{S}, n)\big)$. Clearly x is irreducible. In fact we can restrict ourselves to those x which are maximal in the sense that

$$C^* - \text{conv}(\{x\} \cup \mathcal{S}_{\text{red}}) \not\subset C^* - \text{conv}(\{y\} \cup \mathcal{S}_{\text{red}})$$

for any $y \in \mathcal{S}, y \not\succ x$. (If $\exists x \in \mathcal{S} \backslash C^*\text{-conv}\big(\mathcal{S}_{\text{red}} \cup \text{str}(\mathcal{S}, n)\big)$ the existence of such maximal x is easily shown using Zorn's Lemma and Lemma 4.4 following.) Such an x cannot have any C^*-summands in $\mathcal{S}_{\text{red}}$ without violating maximality so $C^*-\text{summ}(x)$ consists entirely of irreducibles. This fact ensures that if $x = a^*ya + b^*zb$ is a C^*-convex combination then both a and b must be invertible, and so we can conclude that if $y \in C^* - \text{summ}(x)$ then $C^* - \text{summ}(y) \subset C^* - \text{summ}(x)$. Thus any $y \in C^* - \text{summ}(x)$ also has no C^*-summands in $\mathcal{S}_{\text{red}}$, and furthermore, for any such y, there must be $z \in \text{str}(\mathcal{S}, n)$ with $z \in C^* - \text{summ}(y)$. ($C^* - \text{summ}(y)$ is a compact C^*-face consisting only of irreducibles, thus it contains a minimal compact C^*-face containing only irreducibles, that is $\mathcal{U}(z)$ for some $z \in \text{str}(\mathcal{S}, n)$.)

Consider the set $P_W = \{(r, w) : r \in \text{pcs}(x), w \in \mathcal{W}(r; x)\}$ and its subset $P'_W = \{(r, w) : (r, w) \in P_W, r = \sum_i a_i^* y_i a_i, w = \sum_i a_i^* a_i \text{ and } y_i \in \text{str}(\mathcal{S}, n)\ \forall i\}$. ($P'_W$ is those pieces of x, and their weights, which can be generated by structural elements of size n.) P'_W is clearly convex, and since x must have a C^*-summand in $\text{str}(\mathcal{S}, n)$, non-empty. Furthermore, for any $k \geq 0$, if $(r', w') \in P'_W$ and $k(r', w') \in P_W$, then $k(r', w') \in P'_W$.

Next we show that P'_W is dense in P_W. Suppose not. Then there exists $(r, w) \in P_W \backslash \overline{P'_W}$. Let C be the ray $\{k(r, w) : k \geq 0\}$ and let C' be the closed, pointed cone generated by $\overline{P'_W}$. Then by the second fact preceding the lemma, there is a linear functional ρ such that $\rho(r, w) \geq 0$, and $\rho(r', w') < 0\ \forall (r', w') \in P'_W \setminus \{(0, 0)\}$. Choose $(r_1, w_1) \in P_W$ such that $\rho(r_1, w_1)$ is maximal. Now $(r_1, w_1) = (a^*ya, a^*a)$ for some $y \in C^* - \text{summ}(x)$. By the first paragraph above, there exists $z \in \text{str}(\mathcal{S}, n) \cap C^* - \text{summ}(y)$ so $y = b^*zb + c^*z'c$ and

$$(r_1, w_1) = (a^*b^*zba, a^*b^*ba) + (a^*c^*z'ca, a^*c^*ca)$$

with

$$(a^*b^*zba, a^*b^*ba) \in P'_W \setminus \{(0, 0)\},$$

and

$$(a^*c^*z'ca, a^*c^*ca) \in P_W.$$

But $\rho(a^*b^*zba, a^*b^*ba) < 0 \Rightarrow \rho(a^*c^*z'ca, a^*c^*ca) > \rho(r_1, w_1)$ which is a contradiction. Thus P'_W is dense in P_W.

Now by the first fact preceding the Lemma, applied to P_W, P'_W and $(x/2, \mathbf{1}/2) \in \text{int}\, P_W$, we conclude that $(x/2, \mathbf{1}/2) \in P'_W$. But then $2(x/2, \mathbf{1}/2) = (x, \mathbf{1}) \in P'_W$, that is $x \in C^* - \text{conv}\big(\text{str}(\mathcal{S}, n)\big)$. ∎

LEMMA 4.4. *Let $\mathcal{S} \subset M_n$ be compact and C^*-convex. Let $\{x_k : k \in \mathbf{N}\}$ be a sequence in M_n, converging to x, satisfying $C^* - \text{conv}(\mathcal{S} \cup \{x_k\}) \supset C^* - \text{conv}(\mathcal{S} \cup \{x_j\})\ \forall k > j$. Then $C^* - \text{conv}(\mathcal{S} \cup \{x\}) \supset C^* - \text{conv}(\mathcal{S} \cup \{x_k\})\ \forall k$.*

PROOF. It suffices to show $x_k \in \overline{C^* - \text{conv}}\,(\mathcal{S} \cup \{x\})\ \forall k$. Fix k. Given $\epsilon > 0$ choose j so that $\|x_j - x\| < \epsilon$. Write x_k as a C^*-convex combination $x_k = \sum_i v_{ij}^* y_{ij} v_{ij} + \sum_i w_{ij}^* x_j w_{ij}$ of $y_{ij} \in \mathcal{S}$, and x_j. Thus

$$\begin{aligned} \Big\| x_k - \sum_i v_{ij}^* y_{ij} v_{ij} + \sum_i w_{ij}^* x w_{ij} \Big\| &= \Big\| \sum_i w_{ij}^* (x_j - x) w_{ij} \Big\| \\ &\leq \|x_j - x\| \\ &< \epsilon \end{aligned}$$

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and so $x_k \in \overline{C^* - \text{conv}}(\mathcal{S} \cup \{x\})$. ■

Now we are in a position to state and prove the generalised Krein-Milman theorem and its converse.

THEOREM 4.5. *Let $\mathcal{S} \subset M_n$ be compact and C^*-convex. Then $\mathcal{S} = C^* - \text{conv}_n(\text{str}\,\mathcal{S})$. Conversely, if $\mathcal{S} = C^* - \text{conv}(\mathcal{G})$ then $\forall s \in \text{str}(\mathcal{S}), \exists$ unitary $u \in M_n$, and $g \in \mathcal{G}$ such that $g' = u^*gu = s \oplus s'$ for some appropriate s'. That is, every structural element of $\mathcal{S}$ must appear as a direct summand (irreducible block) of some g' unitarily equivalent to $g \in \mathcal{G}$.*

PROOF. We begin by proving the first statement using induction on n. $n = 1$ is the usual Krein-Milman theorem because C^*-convexity and structural elements (C^*-extreme points) correspond to ordinary convexity and extreme points.

We begin the inductive step by showing that $C^* - \text{conv}_n(\text{str}\,\mathcal{S}) \supset \mathcal{S}_{\text{red}}$. For $\mathcal{S}$ as in the statement of the theorem, let $\mathcal{S}_{n-1}$ be its compression to M_{n-1}. $\mathcal{S}_{n-1}$ is compact and C^*-convex so by induction $\mathcal{S}_{n-1} = C^* - \text{conv}_{n-1}\big(\text{str}(\mathcal{S}_{n-1})\big)$. For any $x \in \mathcal{S}_{\text{red}}, x \sim t_1 \oplus \cdots \oplus t_m$, where $t_i \in M_{k_i}$ are irreducible. It is easy to see that if we can show each $t_i(n) \in C^* - \text{conv}_n(\text{str}\,\mathcal{S})$ then $x \in C^* - \text{conv}_n(\text{str}\,\mathcal{S})$. Now by Propositions 2.6 and 2.7

$$\text{str}(\mathcal{S}_{n-1}) \subset \bigcup_{k=1}^{n-1} \Big[\text{str}(\mathcal{S}, k) \cup \Big(\bigcup_{j=k+1}^{n} P_{jk}^* \,\text{str}(\mathcal{S}, j) P_{jk}\Big)\Big]$$

and notice that if $t \in P_{jk}^*\,\text{str}(\mathcal{S}, j)P_{jk}$ then $t(j) \in C^* - \text{conv}_j\big(\text{str}(\mathcal{S}, j)\big)$; in fact $t(n) \in C^* - \text{conv}_n\big(\text{str}(\mathcal{S}, j)\big)$. Now for any $x \sim t_1 \oplus \cdots \oplus t_m \in \mathcal{S}_{\text{red}}$, it is clear that $t_i(n-1) \in \mathcal{S}_{n-1}$ and so

$$\begin{aligned} t_i(n) \in C^* - \text{conv}_n\big(\text{str}(\mathcal{S}_{n-1})\big) &= C^* - \text{conv}_n\Big\{\bigcup_{k=1}^{n-1}\Big[\text{str}(\mathcal{S}, k) \cup \Big(\bigcup_{j=k+1}^{n} P_{jk}^*\,\text{str}(\mathcal{S}, j)P_{jk}\Big)\Big]\Big\} \\ &\subset C^* - \text{conv}_n\Big\{\bigcup_{k=1}^{n} \text{str}(\mathcal{S}, k)\Big\} \\ &= C^* - \text{conv}_n\big(\text{str}(\mathcal{S})\big). \end{aligned}$$

Thus $C^* - \text{conv}_n(\text{str}\,\mathcal{S}) \supset \mathcal{S}_{\text{red}}$, and by the previous lemma, $\mathcal{S} = C^* - \text{conv}_n(\text{str}\,\mathcal{S})$.

To prove the converse, we begin by assuming $\mathcal{G}$ is closed under unitary equivalence, thus making the unitary u in the statement of the theorem redundant. Let $\mathcal{S}_k$, $\mathcal{G}_k$ be the compressions of $\mathcal{S}$, $\mathcal{G}$ to M_k, and recall that $\mathcal{S}_k = C^* - \text{conv}(\mathcal{G}_k)$ (see Proposition 1.4). For $t \in \text{str}(\mathcal{S})$, t is irreducible and C^*-extreme in $\mathcal{S}_k$ for some k, so $t = P_{nk}^* g P_{nk}$ for some $g \in \mathcal{G}$. We wish to show that in fact $g = t \oplus t'$ (some $t' \in M_{n-k}$). The proof is completed by the following Lemma 4.6. ■

LEMMA 4.6. *Let $\mathcal{S} \subset M_n$ be compact and C^*-convex, and let $t \in \text{str}(\mathcal{S}, k)$. If $t = P_{nk}^* g P_{nk}$ for some $g \in \mathcal{S}$ then, in fact, $g = t \oplus t'$ for some $t' \in M_{n-k}$.*

PROOF. It follows from the first half of Theorem 4.5 that $g \in C^* - \text{conv}_n\big(\text{str}(\mathcal{S})\big)$, so $g = \sum_i a_i^* s_i(n) a_i$ where $s_i \in \text{str}(\mathcal{S})$. Now there exist unitaries u_i such that

$$u_i a_i P_{nk} = \begin{pmatrix} a_i' \\ 0 \end{pmatrix}$$

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with $a_i' \in M_k$ so

$$\begin{aligned} t &= P_{nk}^* g P_{nk} \\ &= \sum_i P_{nk}^* a_i^* s_i(n) a_i P_{nk} \\ &= \sum_i \begin{pmatrix} a_i'^* & 0 \end{pmatrix} u_i s_i(n) u_i^* \begin{pmatrix} a_i' \\ 0 \end{pmatrix} \\ &= \sum_i a_i'^* \Big(P_{nk}^* u_i s_i(n) u_i^* P_{nk} \Big) a_i' \end{aligned}$$

a C^*-convex combination in M_k. Now t is C^*-extreme in $\mathcal{S}_k$ so $\forall i$, $P_{nk}^* u_i s_i(n) u_i^* P_{nk} = v_i^* t v_i \sim t$ and $a_i' = \lambda_i v_i^*$ for some $\lambda \in [0,1]$ and $v_i \in M_k$ unitary. Without loss of generality we may absorb the v_i and write $P_{nk}^* u_i s_i(n) u_i^* P_{nk} = t$, $a_i' = \lambda_i$. It follows that $t(n) \in C^* - \operatorname{conv}\big(s_i(n)\big)$. But $t \in \operatorname{str}(\mathcal{S})$ implies $t \in \operatorname{str}\big\{C^* - \operatorname{conv}\big(s_i(n)\big)\big\} = \mathcal{U}(s_i)$, so $t \sim s_i$, and $t(n) \sim s_i(n)$. Without loss of generality, $g = \sum_i b_i^* t(n) b_i$, a C^*-convex combination where

$$t(n) = \begin{pmatrix} t & \\ & \mu \mathbf{1} \end{pmatrix},$$

and

$$b_i = \begin{pmatrix} \lambda_i \mathbf{1} & c_i \\ 0 & d_i \end{pmatrix},$$

with $\lambda_i \in [0,1]$, $c_i \in M_{k,n-k}$, and $d_i \in M_{n-k}$.

$$\begin{aligned} \mathbf{1} &= \sum_i b_i^* b_i \\ &= \begin{pmatrix} \sum_i |\lambda_i|^2 & \sum_i \lambda_i^* c_i \\ \sum_i c_i^* \lambda_i & \sum_i (c_i^* c_i + d_i^* d_i) \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}. \end{aligned}$$

Equating entries in the above matrices and applying the results to the following we see that,

$$\begin{aligned} g &= \sum_i b_i^* t(n) b_i \\ &= \sum_i \begin{pmatrix} |\lambda_i|^2 t & \lambda_i^* t c_i \\ c_i^* t \lambda_i & c_i^* t c_i + d_i^* \mu d_i \end{pmatrix} \\ &= \begin{pmatrix} t & 0 \\ 0 & t' \end{pmatrix}. \end{aligned}$$

Thus t is in fact a direct summand of g and we're done. ∎

5. **Loose ends.** With the proof of the generalised Krein-Milman theorem in hand we can easily show the relationship between structural elements and C^*-extreme points as defined in [8].

Using Lemma 4.6 we can prove the following.

PROPOSITION 5.1. *Let $\mathcal{S} \subset M_n$ be compact and C^*-convex. Let $m + k = n$ and let $\mathcal{S}_k$ be the compression of $\mathcal{S}$ to M_k. If $x = s \oplus z$ with $s \in \operatorname{str}(\mathcal{S}, m)$ and $z \in \partial_e^*(\mathcal{S}_k)$ then $x \in \partial_e^* \mathcal{S}$.*

PROOF. Let $x = \sum_i t_i^* y_i t_i$ be a proper C^*-convex combination (*i.e.* t_i invertible, $\sum_i t_i^* t_i = \mathbf{1}$) of $y_i \in \mathcal{S}$. Since $x = s \oplus z$, it follows that $s = P_{nm}^* x P_{nm} = \sum_i P_{nm}^* t_i^* y_i t_i P_{nm}$. As usual, there exist unitaries $u_i \in M_n$ so that $u_i t_i P_{nm} = \begin{pmatrix} t_i' \\ 0 \end{pmatrix}$ and $t_i' \neq 0$ because t_i is invertible. Letting $\tilde{y}_i = u_i y_i u_i^*$ we can write $s = \sum_i t_i' P_{nm}^* \tilde{y}_i P_{nm} t_i'$.

Since $s \in \operatorname{str}(\mathcal{S}, m)$ it follows that there exist unitaries $v_i \in M_m$ and scalars $\lambda_i \in [0, 1]$ with $\sum_i \lambda_i^2 = 1$, such that $P_{nm}^* \tilde{y}_i P_{nm} = v_i s v_i^* \sim s$ and $t_i' = \lambda_i v_i \ \forall i$. (No $\lambda_i = 0$ since $t_i' \neq 0$). By Lemma 4.6 it follows that $\tilde{y}_i = (v_i s v_i^*) \oplus z_i$ for some $z_i \in M_k$. Let $v_i' = v_i \oplus \mathbf{1}_k$, which is unitary in M_n, to get $\tilde{y}_i = v_i'(s \oplus z_i) v_i'^*$. Let $w_i = v_i'^* u_i$ to get $y_i = w_i^*(s \oplus z_i) w_i$. Thus we can write

$$x = \sum_i t_i^* y_i t_i = \sum_i (t_i^* w_i^*)(s \oplus z_i)(w_i t_i).$$

At the same time, notice

$$\begin{aligned} w_i t_i = v_i'^* u_i t_i &= \begin{pmatrix} v_i^* & \\ & \mathbf{1}_k \end{pmatrix} \begin{pmatrix} t_i' & * \\ 0 & * \end{pmatrix} \\ &= \begin{pmatrix} \lambda_i \mathbf{1}_m & \rho_i \\ 0 & \eta_i \end{pmatrix} \end{aligned}$$

for some $\rho_i \in M_{m,k}$, $\eta_i \in M_{k,k}$, and η_i must be invertible (because t_i was). Thus

$$\begin{aligned} x &= \sum_i \begin{pmatrix} \lambda_i \mathbf{1}_m & 0 \\ \rho_i^* & \eta_i^* \end{pmatrix} \begin{pmatrix} s & \\ & z_i \end{pmatrix} \begin{pmatrix} \lambda_i \mathbf{1}_m & \rho_i \\ 0 & \eta_i \end{pmatrix} \\ &= \sum_i \begin{pmatrix} \lambda_i s \lambda_i & \lambda_i s \rho_i \\ \rho_i^* s \lambda_i & \rho_i^* s \rho_i + \eta_i^* z_i \eta_i \end{pmatrix} \\ &= \begin{pmatrix} s & 0 \\ 0 & z \end{pmatrix}. \end{aligned}$$

Equating lower right hand corners we find $z = \sum_i (\rho_i^* s \rho_i + \eta_i^* z_i \eta_i)$. We can consider this to be a C^*-convex combination of elements of $\mathcal{S}_k$ by suitable modifications of the s and the ρ_i, as the following paragraph shows.

If $m > k$, then ρ_i has rank at most k, so choose unitaries r_i so that $r_i \rho_i = \begin{pmatrix} \rho_i' \\ 0 \end{pmatrix}$, $\rho_i' \in M_k$. Let $s_i' = P_{nk} r_i s r_i^* P_{nk} \in \mathcal{S}_k$, and we see that $\sum_i \rho_i^* s \rho_i = \sum_i \rho_i'^* s_i' \rho_i'$. Also $\sum_i \rho_i'^* \rho_i' + \eta_i^* \eta_i = \sum_i \rho_i^* \rho_i + \eta_i^* \eta_i = \mathbf{1}_k$. Thus $z = \sum_i (\rho_i'^* s_i' \rho_i' + \eta_i^* z_i \eta_i)$ is a C^*-convex combination of the s_i' and $z_i \in \mathcal{S}_k$. The case $m \leq k$ requires only replacing s by $s' = \begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix} \in M_k$, and ρ_i by $\rho_i' = \begin{pmatrix} \rho_i \\ 0 \end{pmatrix} \in M_k$.

Now the η_i are invertible (even though the ρ_i' may not be), and it is an easy application of Proposition 1.3 and the definition of a C^*-extreme point (in $\mathcal{S}_k$) to show that $z \sim z_i \ \forall i$.

Thus if $x = \sum_i t_i^* y_i t_i$ is a proper C^*-convex combination, then $y_i \sim \begin{pmatrix} s & \\ & z_i \end{pmatrix} \sim \begin{pmatrix} s & \\ & z \end{pmatrix} = x \ \forall i$, and so $x \in \partial_e^* \mathcal{S}$. ∎

COROLLARY 5.2. *Let $\mathcal{S} \subset M_n$ be compact and C^*-convex, $x = x_1 \oplus \cdots \oplus x_m \in \mathcal{S}$ with $x_i \in \mathrm{str}(\mathcal{S}, k_i) \subset M_{k_i}$. Then $x \in \partial_e^* \mathcal{S}$.*

PROOF. The proof is by induction on m (the number of summands). The case $m = 1$ is Proposition 2.2. The induction step is Proposition 5.1. ∎

The converse to the above corollary is false: it is not true that if $x = x_1 \oplus \cdots \oplus x_m \in \partial_e^* \mathcal{S}$ then each $x_i \in \mathrm{str}(\mathcal{S}, k_i)$, as can easily be shown by example [9].

Another relationship between structural elements and C^*-extreme points is the following, which says essentially that any structural element can be extended to a C^*-extreme point, and so we can substitute C^*-extreme points for structural elements in the forward direction of the generalised Krein-Milman theorem (Theorem 4.5).

COROLLARY 5.3. *Let $\mathcal{S} \subset M_n$ be compact and C^*-convex, $x \in \mathrm{str}(\mathcal{S}, k)$, $\lambda \in \partial_e^* W(\mathcal{S})$. Then $x \oplus (\lambda \mathbf{1}_{n-k}) \in \partial_e^* \mathcal{S}$, and $\mathcal{S} = C^* - \mathrm{conv}\, \partial_e^* \mathcal{S}$.*

PROOF. Notice that $\lambda \mathbf{1}_{n-k} \in \partial_e^*(\mathcal{S}_{n-k})$ by [3]. Thus $x \oplus (\lambda \mathbf{1}_{n-k}) \in \partial_e^* \mathcal{S}$ by Proposition 5.1. Since $\mathcal{S} = C^* - \mathrm{conv}\big(\mathrm{str}(\mathcal{S})\big)$ (by Theorem 4.5), and every element of $\mathrm{str}(\mathcal{S})$ can be extended to a C^*-extreme point of $\mathcal{S}$ as above, $\mathcal{S} = C^* - \mathrm{conv}(\partial_e^* \mathcal{S})$. ∎

This settles in the affirmative a conjecture of [8]. At this point, we can also prove the following C^*-analog to the theorem that a minimal compact face is an extreme point.

PROPOSITION 5.4. *Let $\mathcal{S} \subset M_n$ be compact and C^*-convex. Let $\mathcal{F} \subset \mathcal{S}$ be a minimal compact C^*-face. Then $\exists x \in \partial_e^* \mathcal{S}$ such that $\mathcal{F} = \mathcal{U}(x)$, i.e., a minimal compact C^*-face "is" a C^*-extreme point.*

PROOF. The proof is by induction on n, with the case $n = 1$ being trivial. Let $x \in \mathcal{F} \subset M_n$. We know (Proposition 3.6) that for any $y \in \mathcal{F}$, $C^* - \mathrm{conv}\{x\} = C^* - \mathrm{conv}\{y\}$. If x is irreducible, or if $x \sim y \; \forall y \in \mathcal{F}$, we're done by Corollary 3.7.

Suppose now that x is reducible. Write $x \sim x_1 \oplus \cdots \oplus x_m$ where each $x_i \in M_{k_i}$ is irreducible. The (converse to the) generalised Krein-Milman theorem tells us that the structural elements of $C^* - \mathrm{conv}\{x\}$ are some subset of the x_i. So without loss of generality we can assume that $x \sim \begin{pmatrix} s & \\ & r \end{pmatrix}$ with $s \in \mathrm{str}(C^* - \mathrm{conv}\{x\})$, $r \in M_k$. Similarly, for any $y \in \mathcal{F}$ we may assume $y \sim \begin{pmatrix} s & \\ & t \end{pmatrix}$. (Since $C^* - \mathrm{conv}\{x\} = C^* - \mathrm{conv}\{y\}$, they have the same structural elements.) Define $\mathcal{F}' = \left\{ t : y \sim \begin{pmatrix} s & \\ & t \end{pmatrix} \text{ for some } y \in \mathcal{F} \right\}$. $\mathcal{F}'$ is a compact C^*-face of $\mathcal{S}_k$. In fact $\mathcal{F}'$ is a minimal compact C^*-face (else $\mathcal{F}$ is not minimal). By induction, $\mathcal{F}'$ is a C^*-extreme point, *i.e.*, $\mathcal{F}' = \mathcal{U}(r)$. Thus $\mathcal{F} = \mathcal{U}\begin{pmatrix} s & \\ & r \end{pmatrix} = \mathcal{U}(x)$ and $x \sim \begin{pmatrix} s & \\ & r \end{pmatrix} \in \partial_e^* \mathcal{S}$ by Proposition 5.1 or Corollary 3.7. ∎

An obvious question to ask at this point is whether there is a C^*-analog to the Caratheodory theorem for convex sets in finite dimensions. That theorem says that every point in a convex set $\mathcal{S}$ contained in an n-dimensional (real) linear space is a convex combination of at most $n + 1$ extreme points of $\mathcal{S}$. Farenick [3] showed that for a C^*-convex set $\mathcal{S} \subset M_n$, with $\mathcal{S} = C^* - \operatorname{conv} \mathcal{G}$, then any $x \in \mathcal{S}$ can be written as a C^*-convex combination of at most $n^3(n^2+1)$ elements of $\mathcal{G}$. The following result improves this upper bound to $3n^2$, but it is still unclear whether or not this bound is the best possible. In particular, if $n = 1$, it is (Notice $n = 1$ corresponds to $\mathbf{C}$ which is 2 dimensional as a real linear space.).

PROPOSITION 5.5. *Let $\mathcal{S} \subset M_n$ be compact and C^*-convex, and let $x \in \mathcal{S}$. Then x is a C^*-convex combination of at most $3n^2$ elements of* $\operatorname{str}(\mathcal{S})$.

PROOF. Without loss of generality assume $0 \in \mathcal{S}$. Let $C = \left\{\left(a^*s(n)a, a^*a\right) : s \in \operatorname{str}\mathcal{S}, a^*a \leq \mathbf{1}\right\} \subset M_n \times M_n^+$. Let C' be the convex cone $C' = \{\sum_{i=1} k\alpha_i c_i : k \in \mathbf{N}, \alpha_i > 0, c_i \in C\}$. $0 \in C'$ since $0 \in \mathcal{S}$, and C' is a proper cone because $\mathcal{S}$ is compact. C' is clearly a pointed cone. To see that C' is closed, consider the following similar construction. Let $B = \{(a^*xa, a^*a) : x \in \mathcal{S}, a^*a \leq \mathbf{1}\}$, and $B' = \{\sum_{i=1} k\beta_i b_i : k \in \mathbf{N}, \beta_i > 0, b_i \in B\}$. B is closed because $\mathcal{S}$ is compact, hence B' is closed. But $B' = C'$ because $\mathcal{S} = C^* - \operatorname{conv}(\operatorname{str}\mathcal{S})$: if $b = (a^*xa, a^*a) \in B$ and $x = \sum_i t_i^* s_i(n) t_i$ then $b = \sum_i \left(a^* t_i^* s_i(n) t_i a, a^* t_i^* t_i a\right) \in C'$.

The space $M_n \times M_n^+$ has dimension $2n^2 + n^2 = 3n^2$ as a real linear space, and any $c' \in C'$ can be expressed as a convex combination of the extreme rays of C'. Also, any extreme ray of C' must contain some $c = \left(a^*s(n)a, a^*a\right) \in c$.

Finally, if $x \in \mathcal{S}$ then $(x, \mathbf{1}) \in C'$ so $(x, \mathbf{1}) = \sum_i \lambda_i\left(a_i^* s_i(n) a_i, a_i^* a_i\right)$ a convex combination of at most $3n^2$ terms (by the usual Caratheodory theorem applied to a pointed cone). Thus $x = \sum_i \sqrt{\lambda_i} a_i^* s_i(n) a_i \sqrt{\lambda_i}$ is a C^*-convex combination of at most $3n^2$ terms (since $\sum_i \sqrt{\lambda_i} a_i^* a_i \sqrt{\lambda_i} = \mathbf{1}$). ■

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