TRIPLE ORTHOGONAL SERIES

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In a number of recent papers, we have developed an abstract approach to dual orthogonal series (see [**1**], [**2**], [**3**], and [**4**]). Such series arise in crack theory, heat transfer, etc. In this paper, we generalize these results to triple orthogonal series. We also show, via a counterexample, that, surprisingly, the results in the dual case are not generalizable as completely as expected.

NOTATION. In this paper we shall always have:

(1) H is a real separable Hilbert space.

(2) $\mathbf{P}$ and $\mathbf{Q}$ are orthogonal subspaces of H (i.e., $H = \mathbf{P} \oplus \mathbf{Q}$).

(3) $\mathbf{P}_1, \mathbf{P}_2$, and $\mathbf{P}_3$ are mutually orthogonal subspaces of H (i.e. $H = \mathbf{P}_1 \oplus \mathbf{P}_2 \oplus \mathbf{P}_3$).

(4) P and Q are projection operators from H onto $\mathbf{P}$ and $\mathbf{Q}$ respectively, so that $P + Q = I$, the identity operator.

(5) P_1, P_2, and P_3 are projection operators from H onto $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$ respectively, so that $P_1 + P_2 + P_3 = I$.

(6) $\{\varphi_n\}_{n=1}^{\infty}$ is a complete orthonormal sequence in H.

The main results of [**1**] and [**3**] can be written as:

THEOREM A. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers and let $\psi_n = P\varphi_n + a_n Q\varphi_n$. Then $\{\psi_n\}_{n=1}^{\infty}$ is complete in H.*

THEOREM B. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that $0 < m \le a_n \le M$. If $\psi_n = P\varphi_n + a_n Q\varphi_n$, then $\{\psi_n\}_{n=1}^{\infty}$ is an ℓ^2 basis in H, i.e. given $F \in H$ there is a unique sequence $\{K_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} K_n \psi_n = F$ (in the norm of H) and, moreover, $\{K_n\}_{n=1}^{\infty}$ is square summable .*

It is these two theorems for which we shall try to find analogues in the case when H, instead of being split into two subspaces, is split into three subspaces. As we said earlier, a surprising factor arises in the analogues of both theorems.

THEOREM 1. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences such that*

$$1 - \frac{1}{\sqrt{2}} \le a_n \le 1 + \frac{1}{\sqrt{2}} \quad \textit{and} \quad 1 - \frac{1}{\sqrt{2}} \le b_n \le 1 + \frac{1}{\sqrt{2}}.$$

If $\psi_n = P_1\varphi_n + a_n P_2 \varphi_n + b_n P_3 \varphi_n$, then $\{\psi_n\}_{n=1}^{\infty}$ is complete in H.

Proof. Let $a_n = 1 - \epsilon_n$ and $b_n = 1 - \delta_n$. Then $|\epsilon_n| \le \frac{1}{\sqrt{2}}$, $|\delta_n| \le \frac{1}{\sqrt{2}}$ and $\psi_n = \varphi_n - \epsilon_n P_2 \varphi_n - \delta_n P_3 \varphi_n$. Assume $F \in H$ and $(F, \psi_n) = 0$, $(n = 1, 2, 3, \ldots)$. Then

$$(F, \varphi_n) = \epsilon_n (F, P_2 \varphi_n) + \delta_n (F, P_3 \varphi_n) = \epsilon_n (P_2 F, \varphi_n) + \delta_n (P_3 F, \varphi_n),$$

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Therefore, using $(a+b)^2 \leq 2(a^2+b^2)$, we have

$$\begin{aligned}(F, \varphi_n)^2 &= (\epsilon_n(P_2F, \varphi_n) + \delta_n(P_3F, \varphi_n))^2 \\ &\leq 2(\epsilon_n^2(P_2F, \varphi_n)^2 + \delta_n^2(P_3F, \varphi_n)^2) \quad (1) \\ &\leq (P_2F, \varphi_n)^2 + (P_3F, \varphi_n)^2 \quad (2)\end{aligned}$$

Thus

$$\begin{aligned}\|F\|^2 &= \sum_{n=1}^{\infty} (F, \varphi_n)^2 \\ &\leq \sum_{n=1}^{\infty} (P_2F, \varphi_n)^2 + \sum_{n=1}^{\infty} (P_3F, \varphi_n)^2 \\ &= \|P_2F\|^2 + \|P_3F\|^2\end{aligned}$$

or

$$\|P_1F\|^2 + \|P_2F\|^2 + \|P_3F\|^2 \leq \|P_2F\|^2 + \|P_3F\|^2$$

Thus $P_1F = 0$ and all the inequalities must be equalities. Since $(a+b)^2 = 2(a^2+b^2)$ if and only if $a = b$, (1) gives us

$$\epsilon_n(P_2F, \varphi_n) = \delta_n(P_3F, \varphi_n) \quad \text{for all } n. \quad (3)$$

(2) gives us

$$2\epsilon_n^2(P_2F, \varphi_n)^2 = (P_2F, \varphi_n)^2 \quad (4)$$

and

$$2\delta_n^2(P_3F, \varphi_n)^2 = (P_3F, \varphi_n)^2 \quad \text{for all } n. \quad (5)$$

Finally $(F, \varphi_n) = \epsilon_n(P_2F, \varphi_n) + \delta_n(P_3F, \varphi_n)$ and $P_1F = 0$ give us

$$(1-\epsilon_n)(P_2F, \varphi_n) = (\delta_n - 1)(P_3F, \varphi_n) \quad \text{for all } n, \quad (6)$$

and we note that neither $(1-\epsilon_n)$ nor $(\delta_n - 1)$ can be zero. Thus, for each n, either (P_2F, φ_n) and (P_3F, φ_n) are both zero or neither is zero.

If, for some n, neither (P_2F, φ_n) nor (P_3F, φ_n) are zero (4) and (5) give us $|\epsilon_n| = |\delta_n| = 1/\sqrt{2}$, (3) gives us $|(P_2F, \varphi_n)| = |(P_3F, \varphi_n)|$ and (6) gives us $|1-\epsilon_n| = |\delta_n - 1|$. Thus ϵ_n and δ_n must have the same sign. Then (3) tells us (P_2F, φ_n) and (P_3F, φ_n) have the same sign while (6) tells us they must have opposite signs. We thus conclude that, for each n, (P_2F, φ_n) and (P_3F, φ_n) are zero and the theorem is proven.

COROLLARY. *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be sequences such that* $0 < m_1 \leq a_n \leq M_1$ *and* $0 < m_2 \leq b_n \leq M_2$ *where* $M_i \leq (\sqrt{2}+1)/(\sqrt{2}-1)m_i$ $(i = 1, 2)$. *If* $\psi_n = P_1\varphi_n + a_nP_2\varphi_n + b_nP_3\varphi_n$, *then* $\{\psi_n\}_{n=1}^{\infty}$ *is complete in H.*

Proof. Assume F is orthogonal to $\{\psi_n\}_{n=1}^{\infty}$. Then

$$P_1F + \frac{M_1\sqrt{2}}{1+\sqrt{2}} P_2F + \frac{M_2\sqrt{2}}{1+\sqrt{2}} P_3F$$

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is orthogonal to

$$P_1\varphi_n+\frac{1+\sqrt{2}}{M_1\sqrt{2}}a_nP_2\varphi_n+\frac{1+\sqrt{2}}{M_2\sqrt{2}}b_nP_3\varphi_n.$$

However, by Theorem 1,

$$P_1\varphi_n+\frac{1+\sqrt{2}}{M_1\sqrt{2}}a_nP_2\varphi_n+\frac{1+\sqrt{2}}{M_2\sqrt{2}}b_nP_3\varphi_n$$

is complete in H and hence

$$P_1F+\frac{M_1\sqrt{2}}{1+\sqrt{2}}P_2F+\frac{M_2\sqrt{2}}{1+\sqrt{2}}P_3F$$

is zero which says that F is zero.

We now turn to a generalization of Theorem B. As before an unexpected factor of $(\sqrt{2}+1)/(\sqrt{2}-1)$ arises.

THEOREM 2. *Let* $\psi_n=P_1\varphi_n+a_nP_2\varphi_n+b_nP_3\varphi_n$ *where* $1-1/\sqrt{2}<m_1\le a_n\le M_1<1+1/\sqrt{2}$ *and* $1-1/\sqrt{2}<m_2\le b_n\le M_2<1+1/\sqrt{2}$. *Then* $\{\psi_n\}_{n=1}^{\infty}$ *is an* ℓ^2 *basis in H.*

Proof. Let $a_n=1+\epsilon_n$ and $b_n=1+\delta_n$. Then $|\epsilon_n|\le\epsilon<1/\sqrt{2}$ and $|\delta_n|\le\delta<1/\sqrt{2}$ where $\epsilon=\max(|m_1-1|,|M_1-1|)$ and $\delta=\max(|m_2-1|,|M_2-1|)$. Also $\psi_n=\varphi_n+\epsilon_nP_2\varphi_n+\delta_nP_3\varphi_n$. We consider the linear map T taking H into H defined by $T\varphi_n=\psi_n$ $(n=1,2,\ldots)$. For any $F\in H$ we have $F=\sum_{n=1}^{\infty}F_n\varphi_n$. Therefore

$$\begin{aligned}\|TF\|&=\left\|T\sum_{n=1}^{\infty}F_n\varphi_n\right\|\\&=\left\|\sum_{n=1}^{\infty}F_n\varphi_n+\sum_{n=1}^{\infty}\epsilon_nF_nP_2\varphi_n+\sum_{n=1}^{\infty}\delta_nF_nP_3\varphi_n\right\|\\&\le\left\|\sum_{n=1}^{\infty}F_n\varphi_n\right\|+\left\|P_2\sum_{n=1}^{\infty}\epsilon_nF_n\varphi_n\right\|+\left\|P_3\sum_{n=1}^{\infty}\delta_nF_n\varphi_n\right\|\\&\le\|F\|+\left\|\sum_{n=1}^{\infty}\epsilon_nF_n\varphi_n\right\|+\left\|\sum_{n=1}^{\infty}\delta_nF_n\varphi_n\right\|\\&=\|F\|+\sqrt{\left(\sum_{n=1}^{\infty}\epsilon_n^2F_n^2\right)}+\sqrt{\left(\sum_{n=1}^{\infty}\delta_n^2F_n^2\right)}\\&\le\|F\|+\epsilon\|F\|+\delta\|F\|\\&\le(1+\sqrt{2})\|F\|\end{aligned}$$

Thus $\|T\|\le1+\sqrt{2}$.

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We also have, for each $F \in H$,

$$\begin{aligned}\|TF\| &= \left\| \sum_{n=1}^{\infty} F_n\varphi_n + \sum_{n=1}^{\infty} \epsilon_n F_n P_2 \varphi_n + \sum_{n=1}^{\infty} \delta_n F_n P_3 \varphi_n \right\| \\ &\geq \left\| \sum_{n=1}^{\infty} F_n \varphi_n \right\| - \left\| P_2 \sum_{n=1}^{\infty} \epsilon_n F_n \varphi_n + P_3 \sum_{n=1}^{\infty} \delta_n F_n \varphi_n \right\| \\ &\geq \|F\| - \sqrt{(\epsilon^2+\delta^2)} \left\| \sum_{n=1}^{\infty} F_n \varphi_n \right\| \\ &= (1\sqrt{(\epsilon^2+\delta^2)})\|F\|\end{aligned}$$

(and we note that $1-\sqrt{(\epsilon^2+\delta^2)}>0$). By standard theory of linear operators (see [5]) T^{-1} exists and is bounded on $T(H)$. However, since, by Theorem 1, $T(H)$ is dense in H, we can extend T^{-1} to be a bounded operator on H. Consequently, given $F \in H$ we can write $F = Tg$ for a unique $g \in H$. Since g is uniquely of the form $\sum_{n=1}^{\infty} g_n \varphi_n$ (with $\{g_n\} \in \ell^2$) we have $F = Tg = T\left(\sum_{n=1}^{\infty} g_n\varphi_n\right) = \sum_{n=1}^{\infty} g_n \psi_n$ and the theorem is proved.

COROLLARY. *Let $\psi_n = P_1\varphi_n + a_n P_2 \varphi_n + b_n P_3 \varphi_n$ where $0 < m_1 \leq a_n \leq M_1$, $0 < m_2 \leq b_n \leq M_2$ and $M_1 < (\sqrt{2}+1)/(\sqrt{2}-1)m_i$ $(i = 1, 2)$. Then $\{\psi_n\}_{n=1}^{\infty}$ is an ℓ^2 basis in H.*

Proof. For $i = 1, 2$, choose C_i such that

$$\frac{\sqrt{2}-1}{\sqrt{2}m_i} < C_i < \frac{\sqrt{2}+1}{\sqrt{2}M_i}.$$

Then by Theorem 2, $\{P_1\varphi_n + C_1 a_n P_2 \varphi_n + C_2 b_n P_3 \varphi_n\}$ is an ℓ^2 basis in H. Thus given any $F \in H$ there is a unique sequence $\{F_n\}_{n=1}^{\infty} \in \ell^2$ such that

$$P_1 F + C_1 P_2 F + C_2 P_3 F = \sum_{n=1}^{\infty} F_n (P_1 \varphi_n + C_1 a_n P_2 \varphi_n + C_2 b_n P_3 \varphi_n)$$

or equivalently, $F = \sum_{n=1}^{\infty} F_n \psi_n$ which was to be proved.

COUNTEREXAMPLE. Upon comparing Theorems 1 and 2 (and their corollaries) with Theorems A and B, we notice that Theorems A and B are not generalized as completely as would be expected. For example, in Theorem B we have $0 < m \leq a_n \leq M$ which, in the case of triple series, we would expect to become

$$0 < m_1 \leq a_n \leq M_1 \quad \text{and} \quad 0 < m_2 \leq b_n \leq M_2. \tag{*}$$

Instead, as in the corollary to Theorem 2, we have an additional factor of $M_i < (\sqrt{2}+1)/(\sqrt{2}-1)m_i$ $(i = 1, 2)$. As a means of proving that some additional condition is needed, we have the following counterexample. For $n = 1, 2, \ldots$ let $\varphi_n(x) = \sin nx$. Let

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$H=L^2[0,\pi]$, $\mathbf{P}_1=L^2[0,5\pi/16]$, $\mathbf{P}_2=L^2[5\pi/16,\pi/2]$, and $\mathbf{P}_3=L^2[\pi/2,\pi]$. For $n=4,5,6,\ldots$ let $a_n=b_n=1$; i.e. $\psi_n(x)=\varphi_n(x)=\sin nx$. Let t be a real number, $a_1=b_1=b_2=a_3=t$ and let $a_2=b_3=2-t$. Thus for all $t\in(0,1)$ we have $0<t\le a_n\le 2-t$ and $0<t\le b_n\le 2-t$. Thus for all $t\in(0,1)$ $\{a_n\}$ and $\{b_n\}$ would satisfy condition (*). However, we shall show that for some $t\in(0,1)$ not only is $\{\psi_n\}$ not an ℓ^2 basis but it is not even complete in H.

In searching for a function $F(x)\in L^2[0,\pi]$ orthogonal to $\{\psi_n\}_{n=1}^{\infty}$ we write $F(x)=\sum_{n=1}^{\infty} F_n \sin nx$. Then since, for $n=4,5,\ldots$ F is orthogonal to $\sin nx$ we have $F_n=0$ $(n=4,5,\ldots)$. Thus we merely need find non-zero F_1, F_2, and F_3 such that $\sum_{n=1}^{3} F_n \sin nx$ is orthogonal to $\{\psi_n\}_{n=1}^{3}$. We can find such $\{F_i\}_{i=1}^{3}$ if and only if the determinant $|(\psi_i(x),\sin jx)|$ is zero. Keeping in mind that the coefficients of $\{\psi_i\}_{i=1}^{3}$ depend on $t\in(0,1)$ we set $D(t)=|(\psi_i(x),\sin jx)|$ $(i,j=1,2,3)$. $D(t)$ is a continuous function of t and $8D(0{\cdot}07)=-0{\cdot}071942$ while $8D(0{\cdot}08)=0{\cdot}318591$. Thus for some $t\in(0{\cdot}07,0{\cdot}08)$, $D(t)$ is zero and hence $\{\psi_n(x)\}_{n=1}^{\infty}$ is incomplete (and certainly not a basis) in $L^2[0,\pi]$ even though it satisfies (*). We note that for $t>(\sqrt{2}-1)/\sqrt{2}=0.29289$ we have all the hypotheses of the corollary to Theorem 2 satisfied and hence $\{\psi_n(x)\}_{n=1}^{\infty}$ would be an ℓ^2 basis in $L^2[0,\pi]$. Thus while there is some gap between the t of our counterexample and $t>(\sqrt{2}-1)/\sqrt{2}$ we have established that something more than (*) is necessary and that Theorem B is not generalized in the obvious manner.

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