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FROBENIUS–SCHUR INDICATORS FOR PROJECTIVE CHARACTERS WITH APPLICATIONS

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Abstract

Let α be a complex valued 2-cocycle of finite order of a finite group G. The nth Frobenius–Schur indicator of an irreducible α-character of G is defined and its properties are investigated. The indicator is interpreted in general for $n = 2$ and it is shown that it can be used to determine whether an irreducible α-character is real-valued under the assumption that the order of α and its cohomology class are both 2. A formula, involving the real α-regular conjugacy classes of G, is found to count the number of real-valued irreducible α-characters of G under the additional assumption that these characters are class functions.

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1. Introduction

Throughout this paper, G will denote a finite group. Also, all the representations considered will be taken to be over the field of complex numbers. The set of all ordinary irreducible characters of G is denoted as usual by $\mathrm{Irr}(G)$, and $\mathrm{Lin}(G)$ will denote the group of linear characters of G.

There are a number of results concerning $\mathrm{Irr}(G)$ and Frobenius–Schur indicators, three of which are reviewed here. For the first, see [3, pages 49–50].

THEOREM 1.1. *Define* $\theta_n : G \to \mathbb{Z}_{\geq 0}$ *by* $\theta_n(x) = |\{g \in G : g^n = x\}|$ *for* $n \in \mathbb{N}$. *Then*

$$\theta_n = \sum_{\chi \in \mathrm{Irr}(G)} \nu_n(\chi)\chi,$$

where

$$\nu_n(\chi) = \frac{1}{|G|} \sum_{x \in G} \chi(x^n)$$

is the nth Frobenius–Schur indicator *of* χ *and* $\nu_n(\chi) \in \mathbb{Z}$.

The second result is a consequence of this theorem (see [3, Corollary 4.6]).

COROLLARY 1.2. *Let G have exactly t involutions. Then*

$$1 + t = \sum_{\chi \in \mathrm{Irr}(G)} \nu_2(\chi)\chi(1).$$

An element of G is *real* if it is conjugate to its inverse and $\chi \in \mathrm{Irr}(G)$ is *real* if $\chi(x) \in \mathbb{R}$ for all $x \in G$. For the third result connecting these two concepts, see [3, Problem 6.13].

THEOREM 1.3. *The number of real conjugacy classes of G is equal to the number of real $\chi \in \mathrm{Irr}(G)$.*

The purpose of this paper is to find generalisations of these three results if irreducible projective characters of G are considered instead of ordinary ones. To generalise Theorem 1.1 it will be necessary to define the nth Frobenius–Schur indicator of an irreducible projective character of G. A number of remarks and examples were made and given in [2, pages 27–28] to show that this and Theorem 1.3 do not have a straightforward generalisation to the projective character situation, but our approach overcomes those difficulties.

In Section 2, basic facts about projective representations of G with 2-cocycle α will be stated. The nth Frobenius–Schur indicator of an irreducible projective character of G is then defined and interpreted for $n = 2$. Using this, the generalisations sought of the three results will be found in Section 3, although for the last two restricted to the case when both α and its cohomology class have order 2.

2. Frobenius–Schur indicators for projective characters

All of the standard facts and concepts relating to projective representations below may be found in [4, 5], or (albeit to a lesser extent) [3, Ch. 11] or [1].

DEFINITION 2.1. A 2-*cocycle* of G over $\mathbb{C}$ is a function $\alpha : G \times G \to \mathbb{C}^*$ such that $\alpha(1,1) = 1$ and $\alpha(x,y)\alpha(xy,z) = \alpha(x,yz)\alpha(y,z)$ for all $x, y, z \in G$.

The set of all such 2-cocycles of G form a group $Z^2(G,\mathbb{C}^*)$ under multiplication. Let $\delta : G \to \mathbb{C}^*$ be any function with $\delta(1) = 1$. Then $t(\delta)(x,y) = \delta(x)\delta(y)/\delta(xy)$ for all $x, y \in G$ is a 2-cocycle of G, which is called a *coboundary*. Two 2-cocycles α and β are *cohomologous* if there exists a coboundary $t(\delta)$ such that $\beta = t(\delta)\alpha$. This defines an equivalence relation on $Z^2(G,\mathbb{C}^*)$, and the *cohomology classes* $[\alpha]$ form a finite abelian group, called the *Schur multiplier* $M(G)$.

DEFINITION 2.2. Let α be a 2-cocycle of G. Then $x \in G$ is α-*regular* if $\alpha(x,y) = \alpha(y,x)$ for all $y \in C_G(x)$.

Let $\beta \in [\alpha]$. Then $x \in G$ is α-regular if and only if it is β-regular. If x is α-regular then so too are x^{-1} and any conjugate of x, so from the latter one may refer to the α-regular conjugacy classes of G.

Definition 2.3. Let α be a 2-cocycle of G. Then an *α-representation* of G of *dimension n* is a function $P : G \to \mathrm{GL}(n, \mathbb{C})$ such that $P(x)P(y) = \alpha(x, y)P(xy)$ for all $x, y \in G$.

Observe that if P is an α-representation of G, then $P(g)P(x)P(g)^{-1} = f_\alpha(g, x)P(gxg^{-1})$ and $P(x)^m = p_\alpha(x, m)P(x^m)$ for all $g, x \in G$ and $m \in \mathbb{N}$, where

$$f_\alpha(g, x) = \frac{\alpha(g, xg^{-1})\alpha(x, g^{-1})}{\alpha(g, g^{-1})} \quad \text{and} \quad p_\alpha(x, m) = \prod_{i=1}^{m-1} \alpha(x, x^i) \quad \text{for } m > 1.$$

An α-representation is also called a *projective* representation of G with 2-cocycle α and its trace function is its *α-character*. Let $\mathrm{Proj}(G, \alpha)$ denote the set of all irreducible α-characters of G. The relationship between $\mathrm{Proj}(G, \alpha)$ and α-representations is much the same as that between $\mathrm{Irr}(G)$ and ordinary representations of G (see [4, page 184] for details). Next $x \in G$ is α-regular if and only if $\xi(x) \neq 0$ for some $\xi \in \mathrm{Proj}(G, \alpha)$ and $|\mathrm{Proj}(G, \alpha)|$ is the number of α-regular conjugacy classes of G.

For $[\beta] \in M(G)$ there exists $\alpha \in [\beta]$ such that $o(\alpha) = o([\beta])$ and α is a *class-function* 2-cocycle, that is, the elements of $\mathrm{Proj}(G, \alpha)$ are class functions. If α is a class-function 2-cocycle of G, then $x \in G$ is α-regular if and only if $f_\alpha(g, x) = 1$ for all $g \in G$.

The nth Frobenius–Schur indicator of $\xi \in \mathrm{Proj}(G, \alpha)$ can now be defined and agrees with the normal definition if α is trivial.

Definition 2.4. Let α be a 2-cocycle of G of finite order. Then the nth *Frobenius–Schur indicator* $\nu_n^\alpha(\xi)$ for $\xi \in \mathrm{Proj}(G, \alpha)$ and $n \in \mathbb{N}$ is given by

$$\nu_n^\alpha(\xi) = \begin{cases} \dfrac{1}{|G|} \displaystyle\sum_{x \in G} p_\alpha(x, n)\xi(x^n) & \text{if } n \equiv 0 \pmod{o(\alpha)} \\ 0 & \text{otherwise.} \end{cases}$$

If α is a 2-cocycle of finite order of G, then this allows the construction of the *α-covering group* H of G (see [4, Ch. 4, Section 1] or [1, page 191]). Let ω be a primitive $o(\alpha)$th root of unity and let $A = \langle \omega \rangle$. The set of elements of H may be taken to be $\{ar(x) : a \in A, x \in G\}$, and H is a group under the binary operation $ar(x)br(y) = ab\alpha(x, y)r(xy)$ for all $a, b \in A$ and all $x, y \in G$. This is a central extension of G:

$$1 \to A \to H \xrightarrow{\pi} G \to 1,$$

with $\pi(r(x)) = x$ for all $x \in G$. It also has the following important property. Let P be an α^i-representation of G for $i \in \mathbb{Z}$. Then $R(ar(x)) = \lambda^i(a)P(x)$ for all $a \in A$ and all $x \in G$ is an ordinary representation of H, where $\lambda \in \mathrm{Lin}(A)$ with $\lambda(\omega) = \omega$; moreover, P is irreducible if and only if R is. Here R is said to *linearise* P (or to be the *lift* of P). Let $\mathrm{Irr}(H|\lambda^i) = \{\chi \in \mathrm{Irr}(H) : \chi_A = \chi(1)\lambda^i\}$ for $i \in \mathbb{Z}$. Then the linearisation process outlined means that for each such i there exists a bijection from $\mathrm{Irr}(H|\lambda^i)$ to $\mathrm{Proj}(G, \alpha^i)$ defined by $\chi \mapsto \xi$, where $\chi(r(x)) = \xi(x)$ for all $x \in G$ and it is convenient to say that χ *linearises* ξ.

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Now x is α-regular if and only if $\omega^i r(x)$ and $\omega^j r(x)$ are not conjugate for all i and j with $0 \le i < j \le o(\alpha) - 1$. So for counting purposes there are exactly $o(\alpha)$ conjugacy classes of H that map under π to the conjugacy class of an α-regular element of G and fewer than this for an element that is not α-regular. If $o(\alpha) = o([\alpha])$, then $A \le H'$ and the mapping $\alpha^i \mapsto [\alpha^i] = [\alpha]^i$ for $i = 0, \ldots, o(\alpha) - 1$ is a bijection.

LEMMA 2.5. *Let α be a 2-cocycle of G of finite order and let H be the α-covering group of G. If $r(x) \in H$ is real, then so too is x. Conversely if $x \in G$ is real, then $r(x)$ is real if and only if there exists $g \in G$ such that $gxg^{-1} = x^{-1}$ and $f_\alpha(g, x) = \alpha(x, x^{-1})^{-1}$.*

PROOF. If $r(x)$ is real with $r(g)r(x)r(g)^{-1} = r(x)^{-1}$, it follows that $f_\alpha(g, x)r(gxg^{-1}) = \alpha(x, x^{-1})^{-1}r(x^{-1})$, so that in particular $gxg^{-1} = x^{-1}$ and x is real. The converse is now obvious. □

LEMMA 2.6. *Let α be a 2-cocycle of G of finite order and let H be the α-covering group of G. Let $\xi \in \mathrm{Proj}(G, \alpha^i)$ for $i \in \mathbb{Z}$ and let $\chi \in \mathrm{Irr}(H|\lambda^i)$ linearise ξ. Then $\nu_n^{\alpha^i}(\xi) = \nu_n(\chi)$.*

PROOF. Using the notation introduced, $r(x)^n = p_\alpha(x, n)r(x^n)$ for $n \in \mathbb{N}$. So from Theorem 1.1,

$$\begin{aligned} \nu_n(\chi) &= \frac{1}{|H|} \sum_{a \in A, x \in G} \chi(a^n p_\alpha(x, n) r(x^n)) \\ &= \frac{1}{|H|} \sum_{a \in A, x \in G} \lambda^i(a^n) p_{\alpha^i}(x, n) \xi(x^n) = \nu_n(\lambda^i)\nu_n^{\alpha^i}(\xi) = \nu_n^{\alpha^i}(\xi), \end{aligned}$$

since $\nu_n(\lambda^i) = \nu_1(\lambda^{ni})$ from Theorem 1.1, so that $\nu_n(\lambda^i) = 1$ if $o(\lambda^{ni}) = 1$ and is 0 otherwise. □

Let α be a 2-cocycle of G of finite order and let H be the α-covering group of G. Consider another transversal of A in $H, \{s(x) : x \in G\}$ with $s(1) = 1$, where $s(x) = \delta(x)r(x)$ for $\delta(x) \in A$. This gives rise to a new 2-cocycle $\beta \in [\alpha]$ with $\beta = t(\delta)\alpha$ and for which $o(\beta)$ divides $o(\alpha)$. Let $\chi \in \mathrm{Irr}(H|\lambda^i)$. Then χ linearises $\xi \in \mathrm{Proj}(G, \alpha^i)$ and $\xi' \in \mathrm{Proj}(G, \beta^i)$, where $\xi'(x) = \lambda^i(\delta(x))\xi(x)$ for all $x \in G$. Now $s(x)^n = r(x)^n$ for $n \equiv 0 \pmod{o(\alpha)}$ and so, from the proof of Lemma 2.6, $\nu_n^{\alpha^i}(\xi) = \nu_n^{\beta^i}(\xi')$ for $n \equiv 0 \pmod{o(\alpha)}$. If $o(\alpha) = o([\alpha])$, then $o(\beta) = o(\alpha)$ and H is also the β-covering group of G.

Using this notation, $\{s(x) : x \in G\}$ can be chosen to be *conjugacy-preserving*, that is, $s(x)$ and $s(y)$ are conjugate in H whenever x and y are conjugate in G (see [5, Lemma 4.1.1] or [1, Proposition 1.1]) and this choice makes β a class-function 2-cocycle.

The next result is an immediate corollary of Lemma 2.6 from [3, page 58].

COROLLARY 2.7. *Let α be a 2-cocycle of G with $o(\alpha) = o([\alpha]) = 2$. Let $\xi \in \mathrm{Proj}(G, \alpha)$. Then $\nu_2^\alpha(\xi) = 0$ or ± 1. Moreover, $\nu_2^\alpha(\xi) = 0$ if and only if ξ is nonreal, $\nu_2^\alpha(\xi) = 1$ if and only if ξ is afforded by a real α-representation, and $\nu_2^\alpha(\xi) = -1$ if and only if ξ is real but is not afforded by any real α-representation of G.*

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Lemma 2.6 also explains why the second Frobenius–Schur indicator is defined to be 0 when $o(\alpha) > 2$, but another rationale follows. If $\alpha(x, y) \notin \mathbb{R}$ and P is an α-representation of G, then at least one of the three matrices $P(x), P(y)$ and $P(xy)$ must contain a nonreal entry.

EXAMPLE 2.8. Consider the elementary abelian group $G = C_p \times C_p$ for p a prime number, which has $M(G) \cong C_p$ (see [4, Proposition 10.7.1]). Let α be any 2-cocycle of G with $o([\alpha]) = p$. Then the only α-regular element of G is the identity element and consequently the only element $\xi \in \mathrm{Proj}(G, \alpha)$ has $\xi(1) = p$ and $\xi(x) = 0$ for $x \neq 1$ (see [5, Theorem 8.2.21]). So ξ is integer-valued, but is not afforded by any real α-representation for $p \geq 3$ from the remark preceding this example. If $o(\alpha) \geq 3$ and is finite, let H be the α-covering group of G and let $\chi \in \mathrm{Irr}(H|\lambda)$ linearise ξ. Then χ is nonreal since λ is nonreal.

It can be concluded from Example 2.8 that the results of Corollary 2.7 do not hold in general for any group G with a 2-cocycle of finite order greater than 2 and in this case $v_2^\alpha(\xi) = 0$ for all $\xi \in \mathrm{Proj}(G, \alpha)$ can only be interpreted as meaning that each ξ is not afforded by any real α-representation of G.

It should be noted that in general the value of $v_n^\alpha(\xi)$ for $n \equiv 0 \pmod{o(\alpha)}$ depends upon the choice of α, even if $o(\alpha) = o([\alpha]) = 2$, as the next example illustrates.

EXAMPLE 2.9. Let $G = C_2 \times C_2$. It is well known that G has two *Schur representation groups* (also known as *covering groups*) up to isomorphism, namely D and Q, the dihedral and quaternion groups of order 8, respectively. The character tables of these two groups are identical, and the irreducible characters χ and χ' of degree 2 of each linearise $\xi \in \mathrm{Proj}(G, \alpha)$ and $\xi' \in \mathrm{Proj}(G, \alpha')$ respectively, where α and α' are the 2-cocycles of G constructed from D and Q of order 2 with $o([\alpha]) = o([\alpha']) = 2$. Now ξ and ξ' are identical and integer-valued from Example 2.8; however, $v_2^\alpha(\xi) = v_2(\chi) = 1$, whereas $v_2^{\alpha'}(\xi') = v_2(\chi') = -1$.

Using Lemma 2.6 other results concerning v_n carry over to v_n^α, as in the next lemma.

LEMMA 2.10. *Let α be a 2-cocycle of G of finite order. Let $\xi \in \mathrm{Proj}(G, \alpha)$ and let $\mu \in \mathrm{Lin}(G)$ with μ^n trivial for $n \in \mathbb{N}$. Then $v_n^\alpha(\xi) \in \mathbb{Z}$ and $v_n^\alpha(\mu\xi) = v_n^\alpha(\xi)$.*

PROOF. Let H be the α-covering group of G and $\chi \in \mathrm{Irr}(H|\lambda)$ linearise ξ. Then $v_n^\alpha(\xi) \in \mathbb{Z}$ from Lemma 2.6 and Theorem 1.1. Now let $\nu \in \mathrm{Lin}(H)$ linearise μ. Then $\nu\chi$ linearises $\mu\xi$ and ν^n is trivial, so $v_n^\alpha(\mu\xi) = v_n(\nu\chi) = v_n(\chi) = v_n^\alpha(\xi)$ using [3, Lemma 4.8] and Lemma 2.6. □

3. Frobenius–Schur indicator applications

Let α be a 2-cocycle of G of finite order and define

$$\theta_n^\alpha = \sum_{\xi \in \mathrm{Proj}(G,\alpha)} v_n^\alpha(\xi)\xi.$$

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From Lemma 2.10, θ_n^α is an integral linear combination of α-characters of G and so $\theta_n^\alpha(x) = 0$ if x is not α-regular. If, in addition, α is a class-function 2-cocycle, then θ_n^α is a class function. If $o(\alpha) = 1$, then $\theta_n^\alpha = \theta_n$ as in Theorem 1.1.

By analogy with the definition in Theorem 1.1, define $\theta_n^+ : G \to \mathbb{Z}_{\geq 0}$ by

$$\theta_n^+(x) = |\{g \in G : p_\alpha(g, n) = 1 \text{ and } g^n = x\}|$$

for $n \in \mathbb{N}$. This function is used in the generalisation of Theorem 1.1.

THEOREM 3.1. *Let α be a 2-cocycle of G with $o(\alpha) = o([\alpha])$ of finite order m and let $n \in \mathbb{N}$ with $n \equiv 0 \pmod m$. Then*

$$\sum_{i=1}^{m-1} \theta_n^{\alpha^i} = m\theta_n^+ - \theta_n.$$

PROOF. Let H be the α-covering group of G. Then, using Theorem 1.1 and Lemma 2.6,

$$\begin{aligned}\theta_n(r(x)) &= m|\{g \in G : p_\alpha(g, n) = 1 \text{ and } g^n = x\}| \\ &= \sum_{\chi \in \mathrm{Irr}(H)} v_n(\chi)\chi(r(x)) = \sum_{\psi \in \mathrm{Irr}(G)} v_n(\psi)\psi(x) + \sum_{i=1}^{m-1} \sum_{\xi \in \mathrm{Proj}(G,\alpha^i)} v_n^{\alpha^i}(\xi)\xi(x) \\ &= \theta_n(x) + \sum_{i=1}^{m-1} \theta_n^{\alpha^i}(x)\end{aligned}$$

for all $x \in G$. □

Continuing with the notation and hypotheses in Theorem 3.1, suppose $g \in G$ with $g^n = x$ and let $y \in C_G(x)$. Then

$$f_\alpha(y, x)p_\alpha(g, n)r(x) = (r(y)r(g)r(y)^{-1})^n = p_\alpha(ygy^{-1}, n)r(x).$$

Now if m is a prime number and x is not α-regular, then $r(x)$ is conjugate to $ar(x)$ for all $a \in A$. So if $r(y)r(x)r(y)^{-1} = ar(x)$, then the mapping $g \mapsto ygy^{-1}$ defines a bijection from $\{g \in G : p_\alpha(g, n) = 1$ and $g^n = x\}$ to $\{g \in G : p_\alpha(g, n) = a$ and $g^n = x\}$, which explains why $m\theta_n^+(x) = \theta_n(x)$ in this scenario.

The next result is a special case of Theorem 3.1 that generalises Corollary 1.2.

COROLLARY 3.2. *Let α be a 2-cocycle of G with $o(\alpha) = o([\alpha]) = 2$. Let H be the α-covering group of G and let H and G have exactly t and s involutions, respectively. Then*

$$t - s = \sum_{\xi \in \mathrm{Proj}(G,\alpha)} v_2^\alpha(\xi)\xi(1).$$

PROOF. Using Corollary 1.2 and the proof of Theorem 3.1,

$$\sum_{\xi \in \mathrm{Proj}(G,\alpha)} v_2^\alpha(\xi)\xi(1) = \theta_2(r(1)) - \theta_2(1) = t - s.$$ □

The final aim is to generalise Theorem 1.3, which involves an analysis of the real conjugacy classes of G.

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LEMMA 3.3. *Let α be a class-function 2-cocycle of G with $o(\alpha) = o([\alpha]) = 2$. Let H be the α-covering group of G with its associated central subgroup $A = \langle -1 \rangle$ and transversal $\{r(x) : x \in G\}$. Let $x \in G$ be real. Then $r(x)$ is nonreal if and only if x is α-regular and $\alpha(x, x^{-1}) = -1$.*

PROOF. If x is α-regular, then $r(x)$ is real if and only if $\alpha(x, x^{-1}) = 1$ from Lemma 2.5. On the other hand, if x is not α-regular, then there exists $y \in C_G(x)$ such that $r(y)r(x^{-1})r(y)^{-1} = -r(x^{-1})$. Now if $gxg^{-1} = x^{-1}$, then either $f_\alpha(g, x)$ or $f_\alpha(yg, x)$ equals $\alpha(x, x^{-1})^{-1}$ and so $r(x)$ is real from Lemma 2.5. □

Let P be an α-representation of G of dimension n. Then for all $g, x \in G$, $P(g)P(x)P(x^{-1})P(g)^{-1}$ equals $f_\alpha(g, x)f_\alpha(g, x^{-1})\alpha(gxg^{-1}, gx^{-1}g^{-1})I_n$, but it also equals $\alpha(x, x^{-1})I_n$. Thus if α is a class-function 2-cocycle of G and x is α-regular, then $\alpha(x, x^{-1}) = \alpha(gxg^{-1}, gx^{-1}g^{-1})$ for all $g \in G$. In the context of Lemma 3.3 and using this result, let k_0, k^+ and k^- denote the number of conjugacy classes C of G that are respectively (a) real and not α-regular, (b) real and α-regular with $\alpha(x, x^{-1}) = 1$ for all $x \in C$, and (c) real and α-regular with $\alpha(x, x^{-1}) = -1$ for all $x \in C$.

THEOREM 3.4. *Let α be a class-function 2-cocycle of G with $o(\alpha) = o([\alpha]) = 2$. Then the number of real elements of* $\mathrm{Proj}(G, \alpha)$ *is $k^+ - k^-$.*

PROOF. Let H be the α-covering group of G. The number of real conjugacy classes of G and H is $k_0 + k^+ + k^-$ and $k_0 + 2k^+$, respectively, from Lemma 3.3 and previous remarks. Thus from Theorem 1.3 the number of real elements of $\mathrm{Proj}(G, \alpha)$ is the second number minus the first. □

If α' is a 2-cocycle of G with $o(\alpha') = o([\alpha']) = 2$, then we may let H be the α'-covering group of G. As explained after Lemma 2.6: (a) there exists a change of transversal so that the resultant 2-cocycle α of G is a class-function 2-cocycle with $o(\alpha) = 2$ and $\alpha \in [\alpha']$; (b) the numbers of real elements of $\mathrm{Proj}(G, \alpha)$ and $\mathrm{Proj}(G, \alpha')$ are equal, with this number given by Theorem 3.4.

EXAMPLE 3.5. Every element of the symmetric group S_4 is real, $M(S_4) \cong C_2$ and S_4 has two Schur representation groups up to isomorphism (see [6, Theorem 1]). One is the binary octahedral group, and the three elements of $\mathrm{Proj}(S_4, \alpha)$ constructed from this group, for a class-function 2-cocycle α with $o(\alpha) = o([\alpha]) = 2$, are all real (see [6, page 70]), so $k^+ = 3$ and $k^- = 0$. The other Schur representation group is $\mathrm{GL}(2, 3)$, and only one element of $\mathrm{Proj}(S_4, \alpha')$ constructed from this group, for a class-function 2-cocycle α' with $o(\alpha') = 2$ and $\alpha' \in [\alpha]$, is real (see [2, Remark (ii), pages 27–28] or [6, page 56]), so here $k^+ = 2$ and $k^- = 1$.

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