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THE RANGE OF THE HELGASON–FOURIER TRANSFORMATION ON HOMOGENEOUS TREES

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Let $\mathfrak{X}$ be a homogeneous tree, o be a fixed reference point in $\mathfrak{X}$, and $\mathfrak{B}_N$ be the closed ball of radius N in $\mathfrak{X}$ centred at o. In this paper we characterise the image under the Helgason–Fourier transformation $\mathcal{H}$ of $C_N(\mathfrak{X})$, the space of functions supported in $\mathfrak{B}_N$, and of $S(\mathfrak{X})$, the space of rapidly decreasing functions on $\mathfrak{X}$. In both cases our results are counterparts of known results for the Helgason–Fourier transformation on noncompact symmetric spaces.

Let $\mathfrak{X}$ be a homogeneous tree of degree $q+1$, that is, a connected graph with no loops in which every vertex is adjacent to $q+1$ other vertices. We denote by o a fixed reference point in $\mathfrak{X}$, by $|x|$ the distance of x from o, that is, the number of edges between o and x, by G the automorphism group of $\mathfrak{X}$, and by K the stabiliser of o in G. The boundary Ω of $\mathfrak{X}$ may be identified with the set of infinite geodesic rays issuing from o. We write $\mathfrak{B}_N$ and $\mathfrak{S}_N$ for the closed ball $\{x \in \mathfrak{X} : |x| \le N\}$ and the sphere $\{x \in \mathfrak{X} : |x| = N\}$. By $\mathfrak{B}_{-1}$ we mean the empty subset of $\mathfrak{X}$.

If x and y are in $\mathfrak{X}$ and ω is in Ω, we define $c(x,\omega)$ to be the confluence point of x and ω, that is, the last point lying on ω in the geodesic path $\{o, x_1, x_2, \dots, x\}$ joining o to x, and define similarly the confluence point $c(x,y)$. The height $h_\omega(x)$ of x in $\mathfrak{X}$ with respect to ω is defined by the formula

$$h_\omega(x) = 2\big|c(x,\omega)\big| - |x|.$$

Clearly, $h_\omega(x) \le |x|$. On the boundary Ω there is a natural K-invariant, G-quasi-invariant probability measure ν, and the Poisson kernel $p(go,\omega)$ is defined to be the Radon–Nikodym derivative $d\nu(g^{-1}\omega)/d\nu(\omega)$. Then

$$p(x,\omega) = q^{h_\omega(x)} \quad \forall x \in \mathfrak{X} \quad \forall \omega \in \Omega;$$

see, for example, [**4**, Chapter 2], or [**3**, Section 2]. We define $E_i(x)$ to be the set of $\{\omega' \in \Omega : |c(x,\omega')| = i\}$; then $\nu(E_i(x)) \le q^{-i}$, and

$$p(x,\omega) = \sum_{j=0}^{|x|} q^{2j-|x|} \chi_{E_j(x)}(\omega) \quad \forall x \in \mathfrak{X} \quad \forall \omega \in \Omega;$$

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see [**7**, (2.3)] or [**5**, Proposition 2.5]. We write $E(x)$ for $E_{|x|}(x)$, and define the averaging operators $\mathcal{E}_n$ on $C(\Omega)$ by the formulae $\mathcal{E}_{-1} = 0$ and, when $n \geq 0$,

$$\mathcal{E}_n \eta(\omega) = \nu\big(E(x)\big)^{-1} \int_{E(x)} \eta(\omega)\, d\nu(\omega) \quad \forall x \in \mathfrak{S}_n \quad \forall \omega \in E(x).$$

We define, for z in $\mathbb{C}$, representations π_z of G on $C(\Omega)$ by the formula

$$\big[\pi_z(g)\eta\big](\omega) = p^{1/2+iz}(go, \omega)\, \eta\big(g^{-1}o\big) \quad \forall g \in G \quad \forall \omega \in \Omega.$$

It is clear that $\pi_z = \pi_{z+\tau}$, where $\tau = 2\pi/\log q$. We write $\mathbb{T}$ for the torus $\mathbb{R}/\tau\mathbb{Z}$, which we usually identify with the interval $[-\tau/2, \tau/2)$. The Poisson transformation $\mathcal{P}^z : C(\Omega) \to C(\mathfrak{X})$ is given by the formula

$$\mathcal{P}^z \eta(x) = \big\langle \pi_z(x)\mathbf{1}, \eta \big\rangle = \int_\Omega p^{1/2+iz}(x, \omega)\, \eta(\omega)\, d\nu(\omega).$$

The spherical function ϕ_z on $\mathfrak{X}$ is defined to be $\mathcal{P}^z \mathbf{1}$. It is known that

$$\phi_z(x) = \begin{cases} \Big(\dfrac{q-1}{q+1}\,|x| + 1\Big) q^{-|x|/2} & \forall z \in \tau\mathbb{Z} \\ \Big(\dfrac{q-1}{q+1}\,|x| + 1\Big) q^{-|x|/2}(-1)^{|x|} & \forall z \in \tau/2 + \tau\mathbb{Z} \\ \mathbf{c}(z) q^{(iz-1/2)|x|} + \mathbf{c}(-z) q^{(-iz-1/2)|x|} & \forall z \in \mathbb{C} \setminus (\tau/2)\mathbb{Z}, \end{cases}$$

where $\mathbf{c}$ is the meromorphic function given by

$$\mathbf{c}(z) = \frac{q^{1/2}}{q+1} \frac{q^{1/2+iz} - q^{-1/2-iz}}{q^{iz} - q^{-iz}} \quad \forall z \in \mathbb{C} \setminus (\tau/2)\mathbb{Z}.$$

Now

$$\phi_0(x) = \int_\Omega p^{1/2}(x, \omega)\, d\nu(\omega) = \sum_{x \in \mathfrak{S}_n} \int_{E(x)} q^{h_\omega(x)/2}\, d\nu(\omega),$$

whence

$$\sum_{x \in \mathfrak{S}_n} q^{h_\omega(x)/2} \leq 2(n+1)\, q^{-n/2} \quad \forall n \in \mathbb{N}. \tag{1}$$

It should perhaps be remarked that we use a different parametrisation of the representations and spherical functions from Figà-Talamanca and his collaborators (for example, [**5**] and [**4**]): our ϕ_z corresponds to their $\phi_{1/2+iz}$, and π_z and $\mathbf{c}(z)$ are similarly reparametrised. Similar comments apply to the intertwining operators considered below. Our parametrisation makes the analogy with the semisimple Lie group case more transparent.

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The Helgason–Fourier transform $\widetilde{f}$ of a finitely supported function f on $\mathfrak{X}$ is the function on $\mathbb{T} \times \Omega$ defined by the formula

$$\widetilde{f}(s,\omega) = \left[\pi_s(f)\mathbf{1}\right](\omega) = \sum_{x \in \mathfrak{X}} f(x)\, p^{1/2+is}(x,\omega).$$

The Helgason–Fourier transformation $\mathcal{H}$ is the linear operator that maps f to $\widetilde{f}$. The following inversion and Plancherel formulae hold (see [**5**, Chapter 3 Section IV and Chapter 5 Section IV], or [**4**, Chapter II Section 6]). If f is finitely supported on $\mathfrak{X}$, then

$$f(x) = \int_{\mathbb{T}} \int_{\Omega} p^{1/2-is}(x,\omega)\, \widetilde{f}(s,\omega)\, d\nu(\omega)\, d\mu(s) \quad \forall x \in \mathfrak{X}.$$

If f_1 and f_2 are finitely supported, then

$$\sum_{x \in \mathfrak{X}} f_1(x)\, \overline{f_2}(x) = \int_{\mathbb{T}} \int_{\Omega} \widetilde{f}_1(s,\omega)\, \overline{\widetilde{f}_2(s,\omega)}\, d\nu(\omega)\, d\mu(s).$$

The Helgason–Fourier transformation extends to an isometric mapping from $L^2(\mathfrak{X})$ into $L^2(\mathbb{T} \times \Omega, \mu \times \nu)$, so $\mathcal{H}$ is injective on $L^2(\mathfrak{X})$. Its range is then the subspace of $L^2(\mathbb{T} \times \Omega, \mu \times \nu)$ of the functions F which satisfy the symmetry condition

$$\int_{\Omega} p^{1/2-is}(x,\omega)\, F(s,\omega)\, d\nu(\omega) = \int_{\Omega} p^{1/2+is}(x,\omega)\, F(-s,\omega)\, d\nu(\omega) \tag{2}$$

for every x in $\mathfrak{X}$ and almost every s in $\mathbb{T}$. Here, μ denotes the Plancherel measure, whose density with respect to Lebesgue measure is given by $c_G \left|\mathbf{c}(s)\right|^{-2}$ (see, for example, [**5**] or [**4**]). We note that $\mathbf{c}^{-1}$ is smooth on $\mathbb{T}$.

The space of functions supported in $\mathfrak{B}_N$ is written $C_N(\mathfrak{X})$. A function f on $\mathfrak{X}$ is said to be rapidly decreasing if, for every k in $\mathbb{N}$, there exists a constant C_k such that

$$\left|f(x)\right| \le C_k \left(|x| + 1\right)^{-k} q^{-|x|/2} \quad \forall x \in \mathfrak{X}$$

(see, for example, [**1**]). The space of rapidly decreasing functions is denoted by $S(\mathfrak{X})$.

The aim of this paper is to characterise the image under $\mathcal{H}$ of the spaces $C_N(\mathfrak{X})$ and $S(\mathfrak{X})$. After a preliminary version of this paper was completed, we learned that a similar characterisation of the range of $C_N(\mathfrak{X})$, involving the horocyclical Radon transformation $\mathcal{R}$ on $\mathfrak{X}$, was obtained independently by Tarabusi, Cohen, and Colonna [**2**]; these authors also describe the the image under $\mathcal{R}$ of certain spaces of "slowly vanishing functions" on $\mathfrak{X}$. We refer to [**3**, Section 2] for a discussion of the relationship between $\mathcal{R}$ and $\mathcal{H}$.

1. Functions with finite support

It is easy to see that, if f is in $C_N(\mathfrak{X})$, then the following conditions hold:

(i) $\widetilde{f}$ is continuous on $\mathbb{T} \times \Omega$ (indeed, $\widetilde{f}$ is in $C^\infty(\mathbb{T} \times \Omega)$ in the sense of Theorem 2 below);

(ii) $\widetilde{f}$ extends to a τ-periodic entire function of exponential type N uniformly in ω, that is, there exists C such that

$$\left|\widetilde{f}(z,\omega)\right| \le C\, q^{|\mathrm{Im}\, z| N} \quad \forall \omega \in \Omega \quad \forall z \in \mathbb{C};$$

(iii) $\widetilde{f}$ satisfies the symmetry condition (2);

(iv) $\widetilde{f}$ is N-cylindrical in ω, that is, for s fixed, $\widetilde{f}(s,\omega)$ is constant on the sets $E(x)$ for every x in $\mathfrak{S}_N$.

Conditions (i)–(iii) are the analogues of the conditions that describe the Paley–Wiener space for the Helgason–Fourier transformation (see [**6**]). The content of the following theorem is that (i)–(iii) characterise the image of $C_N(\mathfrak{X})$ under $\mathcal{H}$.

Theorem 1. *A function $F : \mathbb{T} \times \Omega \to \mathbb{C}$ is the Helgason–Fourier transform of a function f in $\mathbb{C}_N(\mathfrak{X})$ if and only if F satisfies conditions* (i)–(iii).

Proof: Clearly only the "if" implication requires proof. It should be noted that, contrary to the symmetric space case and to the case of radial functions on $\mathfrak{X}$, the proof is not obtained by contour integration arguments alone, but also involves a counting argument.

Since $\mathcal{H}$ is injective, $\mathcal{H}\big(C_N(\mathfrak{X})\big)$ has dimension equal to the cardinality $|\mathfrak{B}_N|$ of $\mathfrak{B}_N$, and it suffices to show that the space of functions on $\mathbb{T} \times \Omega$ which satisfy conditions (i)–(iii) has dimension at most (and therefore exactly) $|\mathfrak{B}_N|$.

To do this, we recast the symmetry condition (2) in a more suitable form. Using the representations π_z of G defined above, we may rewrite (2) in the form

$$\big\langle \pi_{-s}(x)\mathbf{1}, F(s,\cdot)\big\rangle = \big\langle \pi_s(x)\mathbf{1}, F(-s,\cdot)\big\rangle \quad \forall x \in \mathfrak{X} \quad \forall s \in \mathbb{T}.$$

Let I_z denote the normalised intertwining operators between the representations π_z and π_{-z}; see [4] or [**7**]. Then $I_s \pi_s I_{-s} = \pi_{-s}$, so

$$\begin{aligned}\big\langle \pi_s(x)\mathbf{1}, F(-s,\cdot)\big\rangle &= \big\langle I_s\pi_s(x)I_{-s}\mathbf{1}, F(s,\cdot)\big\rangle \\ &= \big\langle \pi_s(x)\mathbf{1}, I_s^* F(s,\cdot)\big\rangle.\end{aligned}$$

The set of functions $\big\{\pi_s(x)\mathbf{1} : x \in \mathfrak{X}\big\}$ span a dense subspace of $L^2(\Omega)$, because π_s is irreducible, and $I_s^* = I_s^{-1} = I_{-s}$, so we conclude that

$$F(-s,\omega) = I_s^* F(s,\omega) = I_{-s} F(s,\omega). \tag{3}$$

Next we use the fact that $F(\cdot,\omega)$ is entire of exponential type N, and the Paley–Wiener theorem on $\mathbb{Z}$ (which involves contour integration), to write

$$F(s,\omega) = \sum_{k\in\mathbb{Z}} F(k,\omega)\, q^{isk},$$

where $F(k,\omega) = 0$ unless $-N \le k \le N$, so that (3) becomes

$$\sum_{k\in\mathbb{Z}} F(k,\omega)\, q^{-iks} = \sum_{k\in\mathbb{Z}} (I_{-s}F)(k,\omega)\, q^{iks}.$$

Now we apply the difference operator $\mathcal{D}_n$, defined to be $\mathcal{E}_n - \mathcal{E}_{n-1}$ (see [7]), to both sides of this equation: setting $F_n(k,\omega) = \mathcal{D}_n F(k,\omega)$, so that $F_n(k,\omega) = 0$ unless $-N \le k \le N$, we see that

$$\sum_{k\in\mathbb{Z}} F_n(k,\omega)\, q^{-iks} = \sum_{k\in\mathbb{Z}} \left(I_{-s}F\right)_n(k,\omega)\, q^{iks}. \tag{4}$$

If $\mathcal{D}_n F = F$ then $I_z F = c(n,z)F$, where

$$c(n,-s) = \begin{cases} 1 & \text{if } n = 0 \\ \dfrac{1-q^{-1-2is}}{1-q^{-1+2is}}\, q^{2isn} & \text{if } n \ge 1 \end{cases}$$

(see [**7**, p. 383]). A straightforward computation shows that

$$\begin{aligned} c(n,-s) &= \left(1-q^{-2is-1}\right) q^{2isn} \sum_{l=0}^{\infty} q^{(2is-1)l} \\ &= -q^{2is(n-1)-1} + \left(1-q^{-2}\right) \sum_{l=0}^{\infty} q^{2is(l+n)-l} \end{aligned}$$

when $n \ge 1$. Inserting these expressions for $c(n,z)$ in (4) we obtain, when $n = 0$, that $F_0(k,\omega) = F_0(-k,\omega)$, and when $n \ge 1$,

$$\sum_{k\in\mathbb{Z}} F_n(k,\omega)\, q^{-iks} = \sum_{k\in\mathbb{Z}} q^{iks} \left[q^{2is(n-1)-1} + \left(1-q^{-2}\right) \sum_{l=0}^{\infty} q^{2is(l+n)-l} \right] F_n(k,\omega).$$

Taking the Fourier coefficients of both sides, we obtain

$$F_0(k,\omega) = -F_0(-k,\omega) \tag{5}$$

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and, when $n \geq 1$,

$$F_n(k,\omega) = -q^{-1}F_n(-2n-k+2,\omega) + \left(1-q^{-2}\right)\sum_{l=0}^{\infty} q^{-l}F_n(-2n-k-2l,\omega), \tag{6}$$

for every k in $\mathbb{Z}$ and ω in Ω.

For fixed ω, we consider the identities (5) and (6) as a system of equations in the unknowns $F_n(k,\omega)$. It is easily verified that

(a) if $n > N$, $F_n(k,\omega) = 0$ for every k (so that the function F is in fact N-cylindrical, and (iv) is a consequence of (i)–(iii));
(b) if $0 \leq n \leq N$, $F_n(k,\omega) = 0$ when $k > N+2-2n$;
(c) for given n and N, the functions $F_n(k,\omega)$, where $1-n \leq k \leq N+2-2n$, are determined in terms of the functions $F_n(j,\omega)$, where $-N \leq j \leq -n$.

Set $b_n = N+1-n$ when $0 \leq n \leq N$. Then, for fixed n and ω, there are at most b_n independent $F_n(k,\omega)$'s, and the remaining $F_n(k,\omega)$'s are determined by these.

Now for any given k and n, $\mathcal{D}_n F_n(k,\omega) = F_n(k,\omega)$, so the independent $F_n(k,\omega)$'s can be chosen in at most d_n independent ways, where d_n is the dimension of the space $\{\eta \in C(\Omega) : \mathcal{D}_n\eta = \eta\}$. We therefore conclude that the dimension of the space of functions F satisfying (i)–(iii) is at most

$$\sum_{n=0}^{N} (N+1-n)\, d_n.$$

But, when $n \geq 1$, $\mathcal{D}_n\eta = \eta$ if and only if η is constant on the sets $E(x)$ for every x in $\mathfrak{S}_n$ and η has zero average on the sets $E(y)$ for every y in $\mathfrak{S}_{n-1}$, while, when $n = 0$, $\mathcal{D}_0\eta = \eta$ if and only if η is constant on Ω. Thus $d_n = e_n - e_{n-1}$, where $e_n = |\mathfrak{S}_n|$ when $n \geq 0$ and $e_{-1} = 0$, and therefore

$$\sum_{n=0}^{N}(N+1-n)d_n = \sum_{k=0}^{N}\sum_{n=0}^{k} d_n = \sum_{k=0}^{N} e_k = \dim \mathcal{H}(C_N(\mathfrak{X})),$$

as required. ∎

2. Rapidly decreasing functions

We now describe the image of the space $S(\mathfrak{X})$ under $\mathcal{H}$. We say that a function $F : \mathbb{T}\times\Omega \to \mathbb{C}$ is in the space $C^\infty(\mathbb{T}\times\Omega)$ if the function $\partial_s^l F(s,\omega)$ is in $C(\mathbb{T}\times\Omega)$ for every l in $\mathbb{N}$, and for every l and k in $\mathbb{N}$ there exists a constant $C_{k,l}$ such that

$$\left\|\partial_s^k(F - \mathcal{E}_n F)\right\|_\infty \leq C_{k,l}(n+1)^{-l} \quad \forall n \in \mathbb{N}\cup\{-1\}.$$

The symbol $C^\infty(\mathbb{T}\times\Omega)^\flat$ denotes the subspace of $C^\infty(\mathbb{T}\times\Omega)$ of functions which satisfy the symmetry condition (2).

THEOREM 2. *The Helgason–Fourier transformation is an isomorphism from the space $S(\mathfrak{X})$ onto the space $C^\infty(\mathbb{T}\times\Omega)^\flat$.*

PROOF: We show first that if f is in $S(\mathfrak{X})$, then $\widetilde{f}$ is in $C^\infty(\mathbb{T}\times\Omega)$.

For any n in $\mathbb{N}$, define the averaging operator $\varepsilon_n : C(\mathfrak{X}) \to C(\mathfrak{X})$ by the formula

$$\varepsilon_n f(x) = \left|\mathfrak{Z}(n,x)\right|^{-1} \sum_{y\in\mathfrak{Z}(n,x)} f(x) \quad \forall x \in \mathfrak{X},$$

where

$$\mathfrak{Z}(n,x) = \begin{cases} \{x\} & \text{if } |x| \le n \\ \{y \in \mathfrak{X} : |x| = |y|,\ \left|c(x,y)\right| \ge n\} & \text{if } |x| > n. \end{cases}$$

The operators ε_n were introduced in [7], where it was shown that the Poisson transformation intertwines $\mathcal{E}_n$ and ε_n, that is, for every η in $C(\Omega)$ we have

$$\varepsilon_n \mathcal{P}^z(\eta) = \mathcal{P}^z(\mathcal{E}_n\eta) \quad \forall n \in \mathbb{N} \quad \forall z \in \mathbb{C}.$$

The identity clearly holds when η is replaced by a function F in $C(\mathbb{T}\times\Omega)$, so $\mathcal{H}^{-1}\left(\mathcal{E}_n\widetilde{f}\right) = \varepsilon_n f$ by Fourier inversion, and, equivalently,

$$\mathcal{E}_n\mathcal{H}f = \mathcal{H}\varepsilon_n f. \tag{7}$$

Assume now that f is in $S(\mathfrak{X})$ so that, for every l in $\mathbb{N}$, there exists a constant C_l such that

$$\left|f(x)\right| \le C_l\left(|x|+1\right)^{-l} q^{-|x|/2} \quad \forall x \in \mathfrak{X}. \tag{8}$$

Using (7) and the expression of the Poisson kernel in terms of the height function h_ω, for all k and l in $\mathbb{N}$ we may write

$$\begin{aligned}\partial_s^k\big(\widetilde{f} - \mathcal{E}_N\widetilde{f}\big)(s,\omega) &= \partial_s^k\left(\sum_{x\in\mathfrak{X}} q^{(1/2+is)h_\omega(x)}\left(f - \varepsilon_N f\right)(x)\right) \\ &= \sum_{x\in\mathfrak{X}} i^k h_\omega(x)^k q^{(1/2+is)h_\omega(x)}\left(f - \varepsilon_N f\right)(x) \\ &= \sum_{x\in\mathfrak{X}\setminus\mathfrak{B}_N} i^k h_\omega(x)^k q^{(1/2+is)h_\omega(x)}\left(f - \varepsilon_N f\right)(x),\end{aligned}$$

since $f(x) = \varepsilon_N f(x)$ when x is in $\mathfrak{B}_N$. Because $h_\omega(x) \le |x|$, and (8) also holds when

f is replaced by $\varepsilon_N f$, we find from (1) that

$$\begin{aligned}\left|\partial_s^k\left(\widetilde{f} - \mathcal{E}_N\widetilde{f}\right)(s,\omega)\right| &\le \sum_{x\in\mathfrak{X}\setminus\mathfrak{B}_N} |x|^k q^{h_\omega(x)/2}\, 2C_{k+l+3}\left(|x|+1\right)^{-k-l-3} q^{-|x|/2}\\ &\le 2C_{k+l+3}\sum_{n=N+1}^{\infty}(n+1)^{-l-3}q^{-n/2}\sum_{x\in\mathfrak{S}_n} q^{h_\omega(x)/2}\\ &\le 4C_{k+l+3}\sum_{n=N+1}^{\infty}(n+1)^{-l-2}\\ &\le 4C_{k+l+3}(N+1)^{-l};\end{aligned}$$

see also [1] where an analogous result for the Radon transformation is proved.

To prove the reverse inclusion, take F in $C^\infty(\mathbb{T}\times\Omega)$ which satisfies the symmetry condition (2), so that F is the Helgason–Fourier transform of a function f in $L^2(\mathfrak{X})$ by the Plancherel theorem. We shall show that f is in $S(\mathfrak{X})$. Take x in $\mathfrak{X}$, and choose N to be the integer part of $|x|/3$. Then

$$|x|/3 < N+1 \le |x|/3 + 1. \tag{9}$$

Write

$$f = \mathcal{H}^{-1}(F - \mathcal{E}_N F) + \mathcal{H}^{-1}(\mathcal{E}_N F) = \mathcal{H}^{-1}F_N + \mathcal{H}^{-1}G_N,$$

say. We consider $\mathcal{H}^{-1}F_N$ and $\mathcal{H}^{-1}G_N$ separately.

First we estimate $\mathcal{H}^{-1}F_N$. By assumption, if k is in $\mathbb{N}$, then

$$\|F_N\|_\infty \le A_{k+1}\,(N+2)^{-k-1},$$

so

$$\begin{aligned}\left|\mathcal{H}^{-1}F_N(x)\right| &\le \sum_{j=0}^{N}\int_{\mathbb{T}}\int_{E_j(x)} q^{j-|x|/2}\left|F_N(s,\omega)\right| d\nu(\omega)\, d\mu(s)\\ &\le \sum_{j=0}^{N} q^{j-|x|/2}\,\nu\left(E_j(x)\right) A_{k+1}\,(N+2)^{-k-1}\\ &\le A_{k+1}\,(N+2)^{-k} q^{-|x|/2}\\ &\le 3^k A_{k+1}\left(|x|+1\right)^{-k} q^{-|x|/2}\end{aligned}$$

from the inversion formula, the normalisation of the Plancherel measure, and (9).

Now we estimate $\mathcal{H}^{-1}G_N$. Since $\mathcal{E}_N$ commutes with differentiation with respect to s we have

$$\left|\partial_s^k G_N(s,\omega)\right| \le B_k \quad \forall s\in\mathbb{T} \quad \forall\omega\in\Omega.$$

Recalling that the function $G_N(s,\cdot)$ is constant on the sets $E(y)$ for all y in $\mathfrak{S}_N$, we denote by x_N the point in $\mathfrak{S}_N$ on the geodesic path $[o,x]$, by $G_N(s,x_N)$ the value that $G_N(s,\cdot)$ takes on the set $E(x_N)$, and by $H_N(s,\omega)$ the difference $G_N(s,\omega)-G_N(s,x_N)$. Note that $E(x_N)=\bigcup_{j\geq N} E_j(x)$, and that $H_N(s,\omega)=0$ when ω is in $E(x_N)$. Therefore, using the explicit formula for the Poisson kernel, and the integral representation of the spherical functions, we deduce from the inversion formula that

$$\begin{aligned}
\mathcal{H}^{-1}G_N(x) &= \int_{\mathbb{T}}\int_{\Omega} p^{1/2-is}(x,\omega)\left[H_N(s,\omega)+G_N(s,x_N)\right] d\nu(\omega)\, d\mu(s) \\
&= \sum_{j=0}^{|x|}\int_{\mathbb{T}}\int_{E_j(x)} q^{(1/2-is)\left(2j-|x|\right)} H_N(s,\omega)\, d\nu(\omega)\, d\mu(s) \\
&\quad + \int_{\mathbb{T}}\int_{\Omega} p^{1/2-is}(x,\omega)\, G_N(s,x_N)\, d\nu(\omega)\, d\mu(s) \\
&= \sum_{j=0}^{N-1}\int_{\mathbb{T}}\int_{E_j(x)} q^{(1/2-is)\left(2j-|x|\right)} H_N(s,\omega)\, d\nu(\omega)\, d\mu(s) \\
&\quad + \int_{\mathbb{T}} \phi_{-s}(x)\, G_N(s,x_N)\, d\mu(s) \\
&= \sum_{j=0}^{N} I_{j,N}(x) \quad \forall x\in\mathfrak{X}\setminus\mathfrak{B}_{N-1},
\end{aligned}$$

where

$$I_{j,N}(x)=\int_{\mathbb{T}}\int_{E_j(x)} q^{(1/2-is)\left(2j-|x|\right)} H_N(s,\omega)\, d\nu(\omega)\, d\mu(s)$$

if $0\leq j\leq N-1$ and

$$I_{N,N}(x)=\int_{\mathbb{T}} \phi_{-s}(x)\, G_N(s,x_N)\, d\mu(s).$$

We claim that for every l in $\mathbb{N}$ there exists a constant C_l, which depends on l, q and B_k (where $0\leq k\leq l+1$), but not on f, x, or N, such that

$$\left|I_{j,N}(x)\right| \leq C_l\left(|x|+1\right)^{-l-1} q^{-|x|/2} \quad \forall j\in\{0,1,\dots,N\}.$$

Assuming our claim, the estimate required to finish the proof of the theorem follows immediately: indeed, from (8) we conclude that

$$\left|\mathcal{H}^{-1}G_N(x)\right| \leq (N+1)\, C_l\left(|x|+1\right)^{-l-1} q^{-|x|/2} \leq C_l\left(|x|+1\right)^{-l} q^{-|x|/2}.$$

To finish, we must prove our claim. We estimate $I_{j,N}$ where $0\leq j\leq N-1$. To deal with $I_{N,N}$ one argues similarly, using the explicit expression of the spherical

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functions ϕ_s. Recalling that $d\mu(s) = c_G|\mathbf{c}(s)|^{-2}ds$, and noting that all the functions involved are smooth in s, we integrate by parts and find that $I_{j,N}$ is equal to

$$c_G \frac{q^{j-|x|/2}\, i^{l+1}}{\left(2j-|x|\right)^{l+1}\log^{l+1} q} \int_{\mathbb{T}} q^{-is\left(2j-|x|\right)}\, \partial_s^{l+1}\Big(|\mathbf{c}(s)|^{-2}\int_{E_j(x)} H_N(s,\omega)\, d\nu(\omega)\Big)\, ds.$$

By Leibniz's rule, this is a linear combination with coefficients $c_G\binom{l+1}{k}$ of $l+2$ terms of the form

$$\frac{q^{j-|x|/2}\, i^{l+1}}{\left(2j-|x|\right)^{l+1}\log^{l+1} q} \int_{\mathbb{T}} q^{-is\left(2j-|x|\right)}\, \partial_s^{l+1-k}\left(|\mathbf{c}(s)|^{-2}\right)\int_{E_j(x)} \partial_s^k H_N(s,\omega)\, d\nu(\omega)\, ds.$$

Using the estimate $\nu\big(E_j(x)\big) \le q^{-j}$, it is easily shown that the absolute value of each term is bounded above by

$$\begin{aligned}
&\frac{q^{j-|x|/2}}{\left(|x|-2j\right)^{l+1}\log^{l+1} q} \int_{\mathbb{T}} \left|\partial_s^{l+1-k}\left(|\mathbf{c}(s)|^{-2}\right)\right|\, 2B_k\, \nu\big(E_j(x)\big)\, ds \\
&\qquad \le \frac{2B_k\, 3^{l+1} q^{-|x|/2}}{|x|^{l+1}\log^{l+1} q} \int_{\mathbb{T}} \left|\partial_s^{l+1-k}\left(|\mathbf{c}(s)|^{-2}\right)\right|\, ds,
\end{aligned}$$

and the required estimate for $I_{j,N}$ follows. ∎

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