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A characterization of virtually free groups among hyperbolic groups

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Abstract. We prove that virtually free groups are precisely the hyperbolic groups admitting a language of geodesic words containing a unique representative for each group element with bounded triangles. Equivalently, these are exactly the hyperbolic groups for which the model for the Gromov boundary defined by Silva is well defined.

1 Introduction

A group is said to be virtually free if it has a free subgroup of finite index. There are many different characterizations of virtually free groups and of different natures (see, for example [1, 2, 4, 6–9]).

In particular, in [7], Gilman, Hermiller, Holt, and Rees show that a group is virtually free if and only if it admits a generating set and a finite length-reducing rewriting system reducing every word to a geodesic, or, equivalently, a generating set and a constant $k > 0$ such that all k-locally geodesic words with respect to that generating set are geodesic. In [10], a new model for the Gromov boundary of a virtually free group was proposed. The idea is to consider the generating set from [7], and to construct a language of geodesics endowed with the prefix metric, so that the boundary obtained when taking the completion of this metric space becomes homeomorphic to the Gromov boundary. We refer the reader to Section 2 for all the relevant definitions.

Let G be a group and A a generating set. Throughout the paper, we will denote $A \cup A^{-1}$ by $\tilde{A}$, and $\tilde{A}^*$ will represent the free monoid over $\tilde{A}$. The natural surjective homomorphism will be called $\pi : \tilde{A}^* \to G$ and the set of all geodesic words over $\tilde{A}$ will be denoted by $\mathrm{Geo}_A(G)$.

Concretely, let G be a virtually free group, and L be the shortlex minimal geodesic words, for a fixed total order on $\tilde{A}$. Notice now that $\pi|_L$, which we will denote by π_L, is a bijection. Silva defined the boundary of L as

$$\partial L = \{\alpha \in \tilde{A}^\omega \mid \alpha^{[n]} \in L \text{ for every } n \in \mathbb{N}\}.$$

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Put $\widehat{L} = L \cup \partial L$ and let $d_2 : \widehat{L} \times \widehat{L} \to \widehat{L}$ be the prefix metric. Given an element $g \in G$, denote its L-representative by $\bar{g}$. Let d_1 be a visual metric on G. Silva proved the following.

Proposition 1.1 ([10, Proposition 6.2])

(1) *The mutually inverse mappings* $\pi_L : (L, d_2) \to (G, d_1) : u \mapsto u\pi$ *and* $\pi_L^{-1} : (G, d_1) \to (L, d_2) : g \mapsto \bar{g}$ *are uniformly continuous;*
(2) $(\hat{L}, d_2)$ *is the completion of* (L, d_2)*;*
(3) $(\partial L, d_2)$ *is homeomorphic to the Gromov boundary of* G*.*

It is well known by a general topology result [5, Section XIV.6] that every uniformly continuous mapping admits a continuous extension to the completion. Therefore, condition (i) ensures that both π_L and π_L^{-1} admit an extension to the completion and these extensions must be mutually inverse, that is the extension $\widehat{\pi}_L$ is bijective. In general, when referring to the extension of a mapping to the completion of the corresponding spaces, we will decorate the mapping with a hat.

In this paper, we prove that, in some sense, the only hyperbolic groups for which this model works are virtually free groups. We also show that this is equivalent to the existence of a language L of unique geodesic representatives of the elements of G with bounded triangles, i.e., such that for words $u, v \in L$ with $d_A(u\pi, v\pi) \leq 1$ and $|u \wedge v| = 0$, we have that $|u| + |v|$ is bounded by some constant $N > 0$.

Corollary 1.2 *Let* G *be a hyperbolic group. The following are equivalent:*

(1) *there is a finite set of generators* A *and a factorial language* $L \subseteq \mathrm{Geo}_A(G)$ *such that* $\widehat{\pi}_L$ *is bijective;*
(2) *there is a finite set of generators* A *and a prefix-closed language* $L \subseteq \mathrm{Geo}_A(G)$ *such that* $\widehat{\pi}_L$ *is bijective;*
(3) *there is a finite set of generators* A *and a factorial language of unique representatives* $L \subseteq \mathrm{Geo}_A(G)$ *such that* G *has bounded triangles for* L
(4) *there is a finite set of generators* A *and a prefix-closed language of unique representatives* $L \subseteq \mathrm{Geo}_A(G)$ *such that* G *has bounded triangles for* L*;*
(5) *there is a finite set of generators* A *and a factorial language of unique representatives* $L \subseteq \tilde{A}^*$ *such that* G *has bounded triangles for* L*;*
(6) *there is a finite set of generators* A *and a prefix-closed language of unique representatives* $L \subseteq \tilde{A}^*$ *such that* G *has bounded triangles for* L*;*
(7) G *is virtually free.*

2 Preliminaries

Let G be a hyperbolic group. Given $g, h, p \in G$, we define the Gromov product of g and h taking p as basepoint by

$$(g|h)_p^A = \frac{1}{2}(d_A(p, g) + d_A(p, h) - d_A(g, h)).$$

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We will often write $(g|h)$ to denote $(g|h)_1^A$, when the generating set is clear from context. Notice that, in the free group case, we have that $(g|h) = |g \wedge h|$, where $u \wedge v$ denotes the longest common prefix between u and v.

The Gromov boundary of a hyperbolic group can be defined in several ways. For example, it can be obtained by taking equivalence classes of Gromov sequences or by considering the space of geodesic rays up to equivalence, where two rays are considered equivalent if the Hausdorff distance between them is finite. The Gromov boundary can also be seen as the completion of the group when endowed with a visual metric, that we will now define (see [3, Section III.H.3] for details).

Put

$$\rho_{p,\gamma}^A(g,h) = \begin{cases} e^{-\gamma(g|h)_p^A} & \text{if } g \neq h \\ 0 & \text{otherwise} \end{cases}$$

for all $p, g, h \in G$.

Given $p \in G$, $\gamma > 0$ and $T \geq 1$, a visual metric on G is one satisfying the inequalities:

$$\frac{1}{T}\rho_{p,\gamma}^A(g,h) \leq d(g,h) \leq T\rho_{p,\gamma}^A(g,h). \tag{2.1}$$

It is well known that all visual metrics yield equivalent completions and that the topology induced by these metrics is the Gromov topology. In free groups, an example of a visual metric is the prefix metric: the distance between two reduced words $u, v \in F_n$ is defined as

$$d(u,v) = \begin{cases} 2^{-|u\wedge v|} \text{ if } u \neq v \\ 0 \text{ otherwise} \end{cases},$$

where $|u \wedge v|$ represents the length of the longest common prefix between u and v.

As mentioned above, in [10], a new model for the boundary of a virtually free group is defined by taking a special language of geodesics L endowed with the prefix metric and taking its completion. The fact that L is factorial, that is, a factor of a word in L is still a word in L, plays an important role in [10]. The need of at least requiring closure under taking prefixes is evident by the definition of the boundary. The following proposition shows this condition is sufficient to ensure that ∂L is indeed the completion of L and it forms a compact space.

Proposition 2.1 *Let L be a prefix-closed language endowed with the prefix metric d_2. Then, $(\widehat{L}, d_2)$ is the completion of (L, d_2) and it is compact.*

Proof We start by showing that $(\widehat{L}, d_2)$ is complete. Let $(\alpha_n)_n$ be a Cauchy sequence in $(\widehat{L}, d_2)$. Then

$$\forall\, M \in \mathbb{N}\;\; \exists\, N \in \mathbb{N}\;\; \forall\, m,n > N\;\; (|\alpha_n \wedge \alpha_m| > M).$$

Thus, for all $k \in \mathbb{N}$, $\alpha_n^{[k]}$ stabilizes when $n \to \infty$ and $\lim_{n\to\infty} \alpha_n^{[k]}$ is a prefix of $\lim_{n\to\infty} \alpha_n^{[k+1]}$. Thus, there is a unique word β in $\tilde{A}^\infty$ such that $\beta^{[k]} = \lim_{n\to\infty} \alpha_n^{[k]}$ and so $\beta = \lim_{n\to\infty} \alpha_n$. Since L is prefix-closed, then, for all $k \in \mathbb{N}$, $\beta^{[k]} = \lim_{n\to\infty} \alpha_n^{[k]} \in L$, and so $\beta \in \widehat{L}$. Hence $(\widehat{L}, d_2)$ is complete. Notice that L is clearly dense in $\widehat{L}$, since for all $N \in \mathbb{N}$ and $\alpha \in \widehat{L}$, $\alpha^{[N]} \in L$ and $|\alpha \wedge \alpha^{[N]}| = N$, and so $(\widehat{L}, d_2)$ is the completion of (L, d_2).

We will now prove that $(\widehat{L}, d_2)$ is a totally bounded metric space and that suffices, as it is complete. Since $\tilde{A}$ is finite, for a given $n \in \mathbb{N}$, there are finitely many words in $\tilde{A}^*$ of length n and so, finitely many of such words in L. Let $\varepsilon > 0$ and take $k \in \mathbb{N}$ such that $2^{-k} < \varepsilon$. Let $V = \{x \in L \,\big|\, |x| = k\}$, which is finite. It is clear that

$$\widehat{L} = \bigcup_{x \in V} B(x, \varepsilon),$$

since for every $\alpha \in \widehat{L}$, we have that $\alpha \in B(\alpha^{[k]}; \varepsilon)$. ∎

3 A characterization of virtually free groups

We want to prove that the hyperbolic groups admitting a factorial language of unique representatives $L \subseteq \mathrm{Geo}_A(G)$ such that $\pi_L : L \to G$ and $\pi_L^{-1} : G \to L$ are mutually inverse uniformly continuous mappings that extend to homeomorphisms $\widehat{\pi_L}$ and $\widehat{\pi_L^{-1}}$ is precisely the class of virtually free groups.

Let L be such a language. We observe that $\pi_L : (L, d_2) \to (G, d_1) : u \mapsto u\pi$ is always uniformly continuous. Indeed, let $g, h \in G$ and put $u = \bar{g} \wedge \bar{h}$. Since $d_A(g, h) \leq d_A(u\pi, g) + d_A(u\pi, h)$, we have that

$$\begin{aligned}
(g|h) &= \frac{1}{2}(d_A(1, g) + d_A(1, h) - d_A(g, h)) \\
&= \frac{1}{2}(|u| + d_A(u\pi, g) + |u| + d_A(u\pi, h) - d_A(g, h)) \\
&\geq |u| = |\bar{g} \wedge \bar{h}|.
\end{aligned}$$

Thus,

$$\forall\, M \in \mathbb{N}\;\; \exists N \in \mathbb{N} \big(|\bar{g} \wedge \bar{h}| > N \Rightarrow (g|h) > M\big)$$

and we are done. This means that π_L admits a unique continuous extension $\widehat{\pi_L} : (\hat{L}, d_2) \to (\hat{G}, d_1)$ to the completion. However, it might happen that $\widehat{\pi_L}$ is not bijective, even for virtually free groups. For example, let $G = \mathbb{Z} \times \mathbb{Z}_2 = \langle a, b \mid a^2, aba^{-1}b^{-1} \rangle$. However, Proposition 1.1 ensures that there is a generating set for which this happens. Then, the words ab^∞ and b^∞ represent the same element in ∂G since they are Hausdorff close geodesic rays.

We say that G has bounded triangles for the language $L \subseteq A^*$ if there is a homomorphism $\pi : A^* \to G$ such that $\pi|_L$ is bijective and given words $u, v \in L$ with $d_A(u\pi, v\pi) \leq 1$ and $|u \wedge v| = 0$, we have that $|u| + |v|$ is bounded by some constant $N > 0$.

Lemma 3.1 *Let G be a hyperbolic group with bounded triangles for some factorial language $L \subseteq A^*$ and constant $N > 0$. Then, for any words $u, v \in L$ with $|u \wedge v| = 0$, we have that $|u| + |v| \leq N d_A(u, v)$.*

Proof We prove this by induction on $d_A(u\pi, v\pi)$. If $d_A(u\pi, v\pi) \leq 1$, we are done. So, assume that the result holds for all $u, v \in L$ with $|u \wedge v| = 0$ such that $d_A(u\pi, v\pi) \leq k$ for some integer $k > 1$ and let $w, w' \in L$ be words such that $|w \wedge w'| = 0$ and $d_A(w\pi, w'\pi) = k + 1$. Take a geodesic path $\xi = x_1 \ldots x_{k+1}$ connecting $w\pi$ and $w'\pi$ and let $\zeta = \overline{(w\xi^{[k]})\pi} \in L$. Now, if ζ does not share a prefix neither with w nor with w', then $|w| + |w'| \leq |w| + |\zeta| + |\zeta| + |w'| \leq kN + N = Nd_A(w\pi, w'\pi)$. Suppose w.l.o.g. that ζ shares a prefix with w (and so, not with w'). Then let $\alpha = \zeta \wedge w$ and put $w = \alpha w''$ and $\zeta = \alpha\zeta'$. Since L is factorial, then ζ' and w'' are words in L such that $|w'' \wedge \zeta'| = 0$ and $d_A(w''\pi, \zeta'\pi) = k$. By hypothesis, we have that $|w''| + |\zeta'| = |w| - |\alpha| + |\zeta| - |\alpha| \leq Nk$. Also, $|w'| + |\zeta| \leq N$. Since $|\alpha| \leq |\zeta|$, we have that $|w| + |w'| \leq |w'| + |\zeta| + |w| - |\alpha| + |\zeta| - |\alpha| \leq N(k + 1)$. ■

Theorem 3.2 *Let G be a hyperbolic group, A be a generating set and L be a factorial language of unique representatives $L \subseteq \mathrm{Geo}_A(G)$. Then the following are equivalent:*

(1) *$\widehat{\pi}_L$ is bijective;*
(2) *G has bounded triangles for L.*

Proof Assume 1. Then, $\widehat{\pi_L}$ is a continuous bijection between compact spaces and so it must be a homeomorphism. Hence, it has a continuous inverse $\widehat{\pi_L}^{-1}$. This is a continuous map between compact spaces, and so it is uniformly continuous and its restriction to G, π_L^{-1}, is also uniformly continuous. Thus, there is some integer $N_L > 0$ such that $|u \wedge v| = 0 \Rightarrow (u\pi|v\pi) \leq N_L$ for all $u, v \in L$. Take words $u, v \in L$, with $d_A(u\pi, v\pi) \leq 1$ and $|u \wedge v| = 0$. Then

$$(u\pi|v\pi) = \frac{1}{2}(|u| + |v| - d_A(u\pi, v\pi)) \leq N_L$$

and so $|u| + |v| \leq 2N_L + d_A(u\pi, v\pi) \leq 2N_L + 1$.

Now, we will prove the converse. Assume that G admits a set of generators A and a factorial language of unique representatives $L \subseteq \mathrm{Geo}_A(G)$ such that given words $u, v \in L$ with $d_A(u\pi, v\pi) \leq 1$ and $|u \wedge v| = 0$, we have that $|u| + |v|$ is bounded by some constant $C > 0$. Let $g, h \in G$ and put $u = |\bar{g} \wedge \bar{h}|$. Consider the L-geodesics α, β, γ connecting $u\pi$ to g, $u\pi$ to h and g to h, respectively. Since G is hyperbolic, then $[[\alpha, \beta, \gamma]]$ is δ-thin. Put $q_0 = g, q_n = h$ and $q_i = g(\gamma^{[i]}\pi)$, for $i \in \{1, \ldots, n-1\}$. Since $q_0 \in \alpha$ and $q_n \in \beta$, there are some $j \in \{0, \ldots, n-1\}$, $p_1 \in \alpha$ and $p_2 \in \beta$ such that $d_A(q_j, p_1) \leq \delta$ and $d_A(q_{j+1}, p_2) \leq \delta$. Now,

$$\begin{aligned}
(g|h) =& \frac{1}{2}(d_A(1, g) + d_A(1, h) - d_A(g, h)) \\
=& \frac{1}{2}(|u| + d_A(u\pi, p_1) + d_A(p_1, g) + |u| + d_A(u\pi, p_2) \\
&+ d_A(p_2, h) - d_A(g, q_j) - 1 - d_A(q_{j+1}, h)) \\
=& |u| + \frac{1}{2}(d_A(u\pi, p_1) + d_A(u\pi, p_2)) + \frac{1}{2}(d_A(p_1, g) - d_A(g, q_j)) \\
&+ \frac{1}{2}(d_A(p_2, h) - d_A(h, q_{j+1})) - \frac{1}{2}.
\end{aligned}$$

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We have that $\frac{1}{2}(d_A(p_1, g) - d_A(g, q_j)) \le \frac{\delta}{2}$, because

$$d_A(p_1, g) \le d_A(p_1, q_j) + d_A(q_j, g) \le \delta + d_A(q_j, g).$$

Similarly, $\frac{1}{2}(d_A(p_2, h) - d_A(h, q_{j+1})) \le \frac{\delta}{2}$. So,

$$(g|h) \le |\bar{g} \wedge \bar{h}| + \delta - \frac{1}{2} + \frac{1}{2}(d_A(u\pi, p_1) + d_A(u\pi, p_2)).$$

But now, consider the subpath $\alpha' \subseteq \alpha$ from $u\pi$ to p_1 and the subpath $\beta' \subseteq \beta$ from $u\pi$ to p_2. They are labeled by words in L, since L is factorial, sharing no prefix and ending at distance at most $2\delta + 1$ since

$$d_A(p_1, p_2) \le d_A(p_1, q_j) + d_A(q_j, q_{j+1}) + d_A(q_{j+1}, p_2) \le 2\delta + 1.$$

By Lemma 3.1, we have that $|\alpha'| + |\beta'| \le (2\delta + 1)C$, and so $\frac{1}{2}(d_A(u\pi, p_1) + d_A(u\pi, p_2)) \le \frac{(2\delta+1)C}{2}$ and

$$(g|h) \le |\bar{g} \wedge \bar{h}| + \delta - \frac{1}{2} + \frac{(2\delta + 1)C}{2}.$$

This implies that π_L^{-1} is uniformly continuous since

$$\forall M \in \mathbb{N} \ \exists N \in \mathbb{N} \big((g|h) > N \Rightarrow |\bar{g} \wedge \bar{h}| > M\big).$$

Therefore, it extends to some continuous mapping $\widehat{\pi_L^{-1}}$. Hence, $\widehat{\pi}_L \widehat{\pi_L^{-1}}$ is a continuous extension of the identity to the completion. Since that extension is unique, it must be the identity on $\hat{L}$. Similarly, $\widehat{\pi_L^{-1}} \widehat{\pi}_L$ must be the identity on $\hat{G}$ and so $\widehat{\pi}_L \widehat{\pi_L^{-1}}$ are mutually inverse continuous mappings, thus homeomorphisms. ∎

Remark 3.3 Notice that in the proof of 1 $\Longrightarrow$ 2 factoriality is not needed and that this implication holds for any language L.

Proposition 3.4 *Let G be a group. Then G is virtually free if and only if there is a generating set A and a prefix-closed language $L \subseteq \tilde{A}^*$ such that $\pi|_L$ is bijective and G has bounded triangles for L.*

Proof Let $L \subseteq \tilde{A}^*$ be a prefix-closed language such that $\pi|_L$ is bijective and G has bounded triangles for L with constant N. Since L is prefix closed, the undirected graph T (obtained by identifying inverse edges) with vertex set $V = L$ having an edge (labeled by $a \in A$) between x and y if and only if $y = xa$ is a tree, i.e., between any pair of vertices, there is one and only one reduced path in T. We have that T is a subgraph of the Cayley graph $\Gamma = \Gamma_X(G)$. Hence, for all $u, v \in L$, we have that $d_\Gamma(u, v) \le d_T(u, v)$. Now, let $u, v \in L$ and put $w = u \wedge v$, $u = wu'$ and $v = wv'$. We have that $d_T(u, v) = |u'| + |v'| \le N d_\Gamma(u\pi, v\pi)$. Therefore, Γ is quasi-isometric to a tree, which by [1, Theorem 4.7(B2)], implies that G is virtually free.

If G is virtually free, then, by Proposition 1.1, there is a finite set of generators A and a factorial language $L \subseteq \tilde{A}^*$ such that $\widehat{\pi}_L$ is bijective; which, by Theorem 3.2, implies

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that there is a finite set of generators A and a factorial (hence, prefix-closed) language of unique representatives $L \subseteq \tilde{A}^*$ such that G has bounded triangles for L. ■

Combining the results above, we obtain Corollary 1.2.

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References

[1] Y. Antolín, *On Cayley graphs of virtually free groups*. Groups Complex. Cryptol. **3**(2011), 301–327.

[2] V. Araújo and P. V. Silva, *Geometric characterizations of virtually free groups*. J. Algebra Appl. **16**(2017), no. 9, 13.

[3] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 319, Springer-Verlag, Berlin, 1999.

[4] V. Diekert and A. Weiß, *Context-free groups and Bass-Serre theory*. In: J. González-Meneses, M. Lustig and E. Ventura (eds.), Algorithmic and geometric topics around free groups and automorphisms, Advanced Courses in Mathematics-CRM Barcelona, Birkhäuser/Springer, Cham, 2017, pp. 43–110.

[5] J. Dugundji, *Topology*., Allyn and Bacon Series in Advanced Mathematics, Allyn and Bacon, Inc., Boston, MA, Reprinting of the 1966 original, 1978.

[6] E. Ghys and P. de la Harpe, *Sur les Groupes Hyperboliques d'après Mikhail Gromov*, Birkhäuser, Boston, MA, 1990.

[7] R. H. Gilman, S. Hermiller, D. F. Holt, and S. Rees, *A characterization of virtually free groups*. Arch. Math. (Basel) **89**(2007), no. 4, 289–295.

[8] A. Karrass, A. Pietrowski, and D. Solitar, *Finite and infinite cyclic extensions of free groups*. J. Australian Math. Soc. **16**(1973), 458–466.

[9] D. E. Muller and P. E. Schupp, *Groups, the theory of ends, and context-free languages*. J. Comput. System Sci. **26**(1983), no. 3, 295–310.

[10] P. V. Silva, *Fixed points of endomorphisms of virtually free groups*. Pacific J. Math. **263**(2013), no. 1, 207–240.

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