ABSOLUTE SUMMABILITY OF SOME SERIES RELATED TO A FOURIER SERIES

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1. Definitions and notations

Let $\{p_n\}$ be a given sequence of constants, real or complex, such that $P_n = p_0+p_1+\cdots+p_n \neq 0, P_{-1} = p_{-1} = 0$, then

$$t_n = \sum_{k=0}^{n} P_k a_{n-k}/P_n, \tag{1.1}$$

defines the sequence $\{t_n\}$ of (N, p_n) means of $\sum_n a_n$. The series $\sum_n a_n$ is said to be summable $|N, p_n|$, if $\{t_n\} \in BV$, i.e., $\sum_n |t_n - t_{n-1}| < \infty$.

In the special case in which

$$p_n = \binom{n+\delta-1}{\delta-1} = \frac{\Gamma(n+\delta)}{\Gamma(n+1)\Gamma(\delta)}\,;\ \delta > -1, \tag{1.2}$$

the (N, p_n) mean reduces to the familiar (C, δ) mean. Thus $|N, p_n|$ summability is the same as $|C, \delta|$ summability, if $\{p_n\}$ is defined by (1.2).

Let $f(t)$ be a periodic function with period 2π, integrable (L) over $(-\pi, \pi)$ and

$$f(t) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t).$$

Then the allied series is

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n(t).$$

We use the following notations:

$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)\},\ \phi^*(t) = \phi(t)-\phi(+0);$$

$$\psi(t) = \tfrac{1}{2}\{f(x+t)-f(x-t)\};\ P_n^* = \sum_{k=0}^{n} |p_k|;$$

$$S_n = \frac{1}{P_n}\sum_{k=0}^{n} \frac{P_k}{k+1}\,;\ S_n^* = \frac{1}{|P_n|}\sum_{k=0}^{n} \frac{|P_k|}{k+1}\,.$$

If $P_n^* = O(|P_n|)$, $\{R_n\} \equiv \{(n+1)p_n/P_n\} \in BV$ and for some real δ,

$$|P_k| \sum_{n=k}^{\infty} \frac{1}{n^{1-\delta}|P_n|} \leqq Kk^{\delta},\ k = 1, 2, \cdots;$$

then we write $\{p_n\} \in \mathscr{C}^{\delta}$.

We put $\lambda_n^{\delta}(t) = n^{\delta} \sin nt$; $\tilde{\lambda}_n^{\delta}(t) = n^{\delta} \cos nt$; $\tau = [\pi/t]$, the greatest integer not greater than π/t.

By '$F(t) \in BV(a, b)$', we mean that $F(t)$ is a function of bounded variation in (a, b) and '$\{\mu_n\} \in B$' means that $\{\mu_n\}$ is a bounded sequence. $\Delta\mu_n = \mu_n - \mu_{n+1}$.

K denotes a positive constant, not necessarily the same at each occurrence.

2. Introduction

Concerning the $|C|$-summability of $\sum_n n^{\alpha} A_n(t)$ and $\sum_n n^{\alpha} B_n(t)$, we have the following results due to Mohanty [5].[1]

THEOREM A. *If*

$$(2.1) \qquad 0 < \alpha < 1 \text{ and } \int_0^{\pi} t^{-\alpha} |d\phi(t)| \leqq K,$$

then $\sum_n n^{\alpha} A_n(x)$ *is summable* $|C, \beta|$ *for every* $\beta > \alpha$.

THEOREM B. *If*

$$(2.2) \qquad 0 < \alpha < 1,\ \psi(+0) = 0 \text{ and } \int_0^{\pi} t^{-\alpha} |d\psi(t)| \leqq K,$$

then $\sum_n n^{a} B_n(x)$ *is summable* $|C, \beta|$ *for every* $\beta > \alpha$.

The case $\alpha = 0$ of Theorem A corresponds to an earlier result of Bosanquet [1], which follows as a special case of the following.

THEOREM C. *If* $\phi(t) \in BV(0, \pi)$ *and* $\{p_n\} \in \mathscr{C}^0$, *then* $\sum_n A_n(x)$ *is summable* $|N, p_n|$.

As pointed out in section 7 of the present paper, Theorem C is obtained by a slight modification in the proof of one of our main results given in this paper. Incidently, this provides a much shorter proof of a result due to Si-Lei ([9], Theorem 1), which is a generalisation of some of the earlier results due to Pati [6], [7], Varshney [10] and Dikshit [2], when we demonstrate in section 7 that the hypotheses used by Si-Lei imply that $\{p_n\} \in \mathscr{C}^0$.

In view of Theorem C and the corresponding result for $|C|$-summability due to Bosanquet [1], it is natural to expect from Theorem A and Theorem B that the hypotheses (2.1) and (2.2) may lead to $|N, p_n|$ summability of $\sum_n n^{\alpha} A_n(t)$ and

[1] We write $\int_0^{\pi}$ for $\lim_{\varepsilon \to +0} \int_{\varepsilon}^{\pi}$.

$\sum_n n^\alpha B_n(t)$, respectively, if $\{p_n\} \in \mathscr{C}^\alpha$ and that such results may include as a special case Theorem A or Theorem B. The object of the present paper is to show that this is indeed true. That Theorem A and Theorem B are special cases of our Theorem 1 and Theorem 2, respectively follows when we observe that

$$\{p_n\} \equiv \left\{\binom{n+\beta-1}{\beta-1}\right\} \in \mathscr{C}^\alpha, \ \beta > \alpha > 0,$$

and $|N, p_n|$ for such a $\{p_n\}$ is the same as $|C, \beta|$.

3. The main results

We prove the following.

THEOREM 1. *If* (2.1) *holds and* $\{p_n\} \in \mathscr{C}^\alpha$ *then* $\sum_n n^\alpha A_n(x)$ *is summable* $|N, p_n|$.

THEOREM 2. *If* (2.2) *holds and* $\{p_n\} \in \mathscr{C}^\alpha$ *then* $\sum_n n^\alpha B_n(x)$ *is summable* $|N, p_n|$.

4. Lemmas

LEMMA 1. *If* $0 < m \leqq n$, *and* $0 < \alpha < 1$, *then uniformly in* $0 < t \leqq \pi$,

$$\left|\sum_{k=m}^{n} k^{\alpha-1} \exp(ikt)\right| \leqq Kt^{-\alpha}.$$

PROOF. The lemma follows, when we observe that

$$\begin{aligned} \left|\sum_{k=m}^{n} k^{\alpha-1} \exp(ikt)\right| &\leqq \sum_{k=m}^{\tau} k^{\alpha-1} + K\tau^{\alpha-1} \max_{\tau<\nu\leqq n} \left|\sum_{k=\tau+1}^{\nu} \exp(ikt)\right| \\ &\leqq K\tau^{\alpha}. \end{aligned}$$

LEMMA 2. *If* $P_n^* = O(|P_n|)$, *then uniformly in* $0 < t \leqq \pi$,

$$\left|\sum_{k=0}^{\nu} P_k(n-k)^{\alpha-1} \exp i(n-k)t\right| \leqq Kt^{-\alpha}|P_\nu|,$$

where $0 \leqq \nu < n$ *and* $0 < \alpha < 1$.

PROOF. We have by Abel's transformation and Lemma 1,

$$\begin{aligned} &\left|\sum_{k=0}^{\nu} P_k(n-k)^{\alpha-1} \exp i(n-k)t\right| \\ &\quad \leqq \sum_{k=0}^{\nu-1} |p_{k+1}| \left|\sum_{\mu=0}^{k} (n-\mu)^{\alpha-1} \exp i(n-\mu)t\right| + |P_\nu| \left|\sum_{\mu=0}^{\nu} (n-\mu)^{\alpha-1} \exp i(n-\mu)t\right| \\ &\quad \leqq Kt^{-\alpha} P_\nu^*. \end{aligned}$$

The lemma now follows when we appeal to the hypothesis: $P_n^* = O(|P_n|)$.

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LEMMA 3. *If* $\{p_n\}$ *is any sequence such that* $P_n^* = O(|P_n|)$, *then uniformly in* $0 < t \leqq \pi$

$$\left|\sum_{k=0}^{v} P_k \exp i(n-k)t\right| \leqq Kt^{-1}|P_v|,$$

where $0 \leqq v$ *and* n *is any integer.*

The proof of Lemma 3 is similar to that of Lemma 2.

LEMMA 4. *For any sequence* $\{p_n\}$ *such that* $P_n^* = O(|P_n|)$, *the statement* $\{S_n\} \in BV$ *implies* $\{S_n^*\} \in B$.

Lemma 4 is the same as Lemma 2 in [3].

LEMMA 5. *If* $\alpha > 0$, $\eta > 0$, *then necessary and sufficient conditions that*

$$\int_0^\eta t^{-\alpha}|d\psi(t)| \leqq K \text{ and } \psi(+0) = 0,$$

are that (i) $t^{-\alpha}\psi(t) \in BV(0, \eta)$,[2] *and* (ii) $t^{-\alpha-1}|\psi(t)|$ *should be integrable in* $(0, \eta)$.

Lemma 5 is given in [5].

5. Proof of Theorem 1

We have

$$\begin{aligned} t_n - t_{n-1} &= \sum_{k=0}^{n-1} \left(\frac{P_k}{P_n} - \frac{P_{k-1}}{P_{n-1}}\right) a_{n-k} \\ &= \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} (P_n p_k - p_n P_k) a_{n-k}. \end{aligned}$$

Integrating by parts, we have

$$\begin{aligned} n^\alpha A_n(x) &= \frac{2}{\pi}\int_0^\pi \phi(t) n^\alpha \cos nt\, dt \\ &= -\frac{2}{\pi}\int_0^\pi n^{\alpha-1} \sin nt\, d\phi(t). \end{aligned}$$

Thus for the series $\sum_n n^\alpha A_n(x)$,

$$\begin{aligned} \sum_n |t_n - t_{n-1}| &= \frac{2}{\pi}\sum_n \left|\int_0^\pi \left\{\frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} (P_n p_k - p_n P_k)\lambda_{n-k}^{\alpha-1}(t)\right\} d\phi(t)\right| \\ &\leqq \int_0^\pi \sum_n \left|\frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} (P_n p_k - p_n P_k)\lambda_{n-k}^{\alpha-1}(t)\right| |d\phi(t)|. \end{aligned}$$

Since $\int_0^\pi t^{-\alpha}|d\phi(t)| \leqq K$, in order to prove Theorem 1 it is sufficient to show that uniformly in $0 < t \leqq \pi$

[2] That is, in the interval $0 < t \leqq \eta$.

(5.1) $$\sum \equiv t^{\alpha}\sum_{n}\left|\frac{1}{P_nP_{n-1}}\sum_{k=0}^{n-1}(P_np_k-p_nP_k)\lambda_{n-k}^{\alpha-1}(t)\right| \leqq K.$$

Now (cf. [2], p. 168)

(5.2) $$\begin{aligned}\Sigma &\leqq t^{\alpha}\sum_{n=1}^{\infty}\frac{1}{(n+1)|P_{n-1}|}\left|\sum_{k=0}^{n-1}P_k(R_k-R_n)\lambda_{n-k}^{\alpha-1}(t)\right| \\ &\quad+t^{\alpha}\sum_{n=1}^{\tau}\frac{1}{(n+1)|P_{n-1}|}\left|\sum_{k=0}^{n-1}p_k\lambda_{n-k}^{\alpha}(t)\right| \\ &\quad+t^{\alpha}\sum_{n=\tau+1}^{\infty}\frac{1}{(n+1)|P_{n-1}|}\left|\sum_{k=0}^{\tau-1}p_k\lambda_{n-k}^{\alpha}(t)\right| \\ &\quad+t^{\alpha}\sum_{n=\tau+1}^{\infty}\frac{1}{(n+1)|P_{n-1}|}\left|\sum_{k=\tau}^{n}p_k\lambda_{n-k}^{\alpha}(t)\right| \\ &= \Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4,\end{aligned}$$

say. By a change of order of summation, we have

(5.3) $$\begin{aligned}\Sigma_1 &= t^{\alpha}\sum_{n=1}^{\infty}\frac{1}{(n+1)|P_{n-1}|}\left|\sum_{\nu=0}^{n-1}\Delta R_{\nu}\sum_{k=0}^{\nu}P_k\lambda_{n-k}^{\alpha-1}(t)\right| \\ &\leqq K\sum_{n=1}^{\infty}\frac{1}{(n+1)|P_{n-1}|}\sum_{\nu=0}^{n-1}|\Delta R_{\nu}||P_{\nu}| \\ &= K\sum_{\nu=0}^{\infty}|\Delta R_{\nu}||P_{\nu}|\sum_{n=\nu+1}^{\infty}\frac{1}{(n+1)|P_{n-1}|} \\ &\leqq K\sum_{\nu=0}^{\infty}|\Delta R_{\nu}| \leqq K,\end{aligned}$$

by virtue of Lemma 2 and the hypothesis that $\{p_n\}\in\mathscr{C}^{\alpha}$.

Since $|\sin(n-k)t| \leqq n\,t$ for relevant k, we have

(5.4) $$\Sigma_2 \leqq t^{\alpha+1}\sum_{n=1}^{\tau}\frac{n^{\alpha+1}}{n|P_{n-1}|}P_{n-1}^{*} \leqq K,$$

by virtue of the hypothesis that $\{p_n\}\in\mathscr{C}^{\alpha}$.

Next, we observe that

(5.5) $$\Sigma_3 \leqq Kt^{\alpha}P_{\tau}^{*}\sum_{n=\tau+1}^{\infty}\frac{1}{n^{\alpha-1}|P_{n-1}|} \leqq K,$$

by virtue of the hypothesis that $\{p_n\}\in\mathscr{C}^{\alpha}$.

Since $p_k = (k+1)^{-1}R_kP_k$, we have by Abel's transformation

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$$
\begin{aligned}
&|\sum_{k=\tau}^{n} p_k \lambda_{n-k}^{\alpha}(t)| \\
&\quad = \left| \sum_{k=\tau}^{n-1} \Delta \left(\frac{R_k}{k+1}\right) \sum_{v=\tau}^{k} P_v \lambda_{n-v}^{\alpha}(t) + \frac{R_n}{n+1} \sum_{v=\tau}^{n-1} P_v \lambda_{n-v}^{\alpha}(t) \right| \\
&\quad \leqq K n^{\alpha} t^{-1} \sum_{k=\tau}^{n-1} \left\{ \frac{|\Delta R_k|}{k+1} + \frac{|R_{k+1}|}{(k+1)(k+2)} \right\} |P_k| + K n^{\alpha-1} t^{-1} |P_{n-1}|
\end{aligned}
$$

by Abel's Lemma, Lemma 3 and the hypothesis that $\{p_n\} \in \mathscr{C}^{\alpha}$.

Thus, finally

$$
\begin{aligned}
\Sigma_4 &\leqq K t^{\alpha-1} \sum_{n=\tau+1}^{\infty} \frac{1}{n^{1-\alpha}|P_{n-1}|} \sum_{k=\tau}^{n} \frac{|P_k|}{k} \left\{ |\Delta R_k| + \frac{1}{k} \right\} \\
&\quad + K t^{\alpha-1} \sum_{n=\tau+1}^{\infty} \frac{1}{n^{2-\alpha}} \\
&\leqq K t^{\alpha-1} \sum_{k=\tau}^{\infty} \left\{ |\Delta R_k| + \frac{1}{k} \right\} \frac{|P_k|}{k} \sum_{n=k+1}^{\infty} \frac{1}{n^{1-\alpha}|P_{n-1}|} + K \qquad (5.6) \\
&\leqq K t^{\alpha-1} \sum_{k=\tau}^{\infty} \left\{ |\Delta R_k| + \frac{1}{k} \right\} k^{\alpha-1} + K \\
&\leqq K \sum_{k=\tau}^{\infty} |\Delta R_k| + K \leqq K,
\end{aligned}
$$

by virtue of the hypothesis that $\{p_n\} \in \mathscr{C}^{\alpha}$.

Combining (5.2)–(5.6), we prove (5.1), which completes the proof of Theorem 1.

6. Proof of Theorem 2

Integrating by parts and observing that $\psi(+0) = 0$, we have [5]

$$
\begin{aligned}
n^{\alpha} B_n(x) &= \frac{2}{\pi} \int_0^{\pi} \psi(t) n^{\alpha} \sin nt \, dt \\
&= -\frac{2}{\pi} \psi(\pi) n^{\alpha-1} \cos n\pi + \frac{2}{\pi} \int_0^{\pi} n^{\alpha-1} \cos nt \, d\psi(t).
\end{aligned}
$$

As in the proof of Theorem 1, for the series $\sum_n n^{\alpha} B_n(x)$, we have

$$
\begin{aligned}
\sum_n |t_n - t_{n-1}| &\leqq |\psi(\pi)| \sum_n \left| \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} (P_n p_k - p_n P_k) \tilde{\lambda}_{n-k}^{\alpha-1}(\pi) \right| \\
&\quad + \int_0^{\pi} \sum_n \left| \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} (P_n p_k - p_n P_k) \tilde{\lambda}_{n-k}^{\alpha-1}(t) \right| |d\psi(t)|.
\end{aligned}
$$

Since $\int_0^\pi t^{-\alpha}|d\psi(t)| \leqq K$, in order to prove Theorem 2, it is sufficient to show that uniformly in $0 < t \leqq \pi$

$$(6.1) \qquad \sum_n \left| \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} (P_n p_k - p_n P_k) \tilde{\lambda}_{n-k}^{\alpha-1}(t) \right| \leqq K.$$

The proof of (6.1) follows from the preceeding section, when one observes that the proof of (5.1) with a slight modification remains valid even if $\tilde{\lambda}_{n-k}^{\alpha-1}(t)$ is replaced by $\tilde{\lambda}_{n-k}^{\alpha-1}(t)$.

This completes the proof of Theorem 2.

7.

In view of Lemma 5, our Theorem 1 and Theorem 2 are equivalent to the following, respectively (cf. Theorem 1a and Theorem 2a due to Mohanty [5] and Theorem IV and Theorem III due to Salem and Zygmund [8]).

THEOREM 1′. *If* $\{p_n\} \in \mathscr{C}^\alpha$ *and*

$$(2.1)' \qquad 0 < \alpha < 1,\ t^{-\alpha}\phi^*(t) \in BV(0,\pi) \ \text{and} \int_0^\pi t^{-\alpha-1}|\phi^*(t)|dt \leqq K,$$

then $\sum_n n^\alpha A_n(x)$ *is summable* $|N, p_n|$.

THEOREM 2′. *If* $\{p_n\} \in \mathscr{C}^\alpha$ *and*

$$(2.2)' \qquad 0 < \alpha < 1,\ t^{-\alpha}\psi(t) \in BV(0,\pi) \ \text{and} \int_0^\pi t^{-\alpha-1}|\psi(t)|dt \leqq K,$$

then $\sum_n n^\alpha B_n(x)$ *is summable* $|N, p_n|$.

Under a condition similar to the last condition of (2.1)′ with $\alpha = 0$, recently the present author has deduced from the proof given in [2], a result concerning $|N, p_n|$ summability of a series associated with $\sum_n A_n(t)$ in [4].

It follows from the proof of Theorem 1 that in order to prove Theorem C, it is sufficient to prove (5.1) uniformly in $0 < t \leqq \pi$, when $\alpha = 0$. Using the technique of proof of Lemma 2, we observe that if $\{p_n\} \in \mathscr{C}^\alpha$ and $0 \leqq v < n$, then

$$\left|\sum_{k=0}^{v} P_k \lambda_{n-k}^{-1}(t)\right| \leqq K|P_v|,$$

since $\sum_{k=a}^{b} |\lambda_k^{-1}(t)| \leqq K$ for any $b \geqq a > 0$. Therefore $\sum_1 \leqq K$ in the case $\alpha = 0$ also. The proofs of $\sum_2 \leqq K$, $\sum_3 \leqq K$ and $\sum_4 \leqq K$, when $\alpha = 0$, run exactly parallel to those given in (5.4)–(5.6).

This completes the proof of Theorem C.

Finally, to demonstrate that the hypotheses used by Si-Lei for the proof of his Theorem 1 in [9], imply that $\{p_n\} \in \mathscr{C}^\alpha$, we have the following.

If $\{p_n\}$ *is any sequence such that* $P_n^* = O(|P_n|)$, *then* $\{S_n\} \in BV$ *implies that*

$$|P_k| \sum_{n=k+1}^{\infty} \frac{1}{n|P_{n-1}|} \leqq K, \quad k = 0, 1, 2, \cdots.$$

Since $P_n^* = O(|P_n|)$,

$$\begin{aligned}|P_k| \sum_{n=k+1}^{M} \frac{1}{n|P_{n-1}|} &\leqq K|P_k| \sum_{n=k+1}^{M} \frac{|P_{n-1}|}{n} (P_{n-1}^*)^{-2} \\ &\leqq K|P_k| \sum_{n=k+1}^{M-1} \{(P_{n-1}^*)^{-2} - (P_n^*)^{-2}\} \sum_{\nu=1}^{n} \frac{|P_{\nu-1}|}{\nu} \\ &\quad + \frac{K}{P_k^*} \sum_{\nu=1}^{k+1} \frac{|P_{\nu-1}|}{\nu} + K \frac{|P_k|}{(P_{M-1}^*)^2} \sum_{\nu=1}^{M} \frac{|P_{\nu-1}|}{\nu} \\ &\leqq K|P_k| \sum_{n=k+1}^{M} \frac{|p_n|}{P_n^* P_{n-1}^*} S_{n-1}^* + KS_k^* + KS_{M-1}^* \\ &\leqq K|P_k| \sum_{n=k+1}^{M} \left(\frac{1}{P_{n-1}^*} - \frac{1}{P_n^*} \right) + K \leqq K, \quad M \to \infty,\end{aligned}$$

since by Lemma 4, $\{S_n^*\} \in B$.

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