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CHARACTERISATIONS OF EFFICIENT SETS IN VECTOR OPTIMISATION

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In this paper, we present characterisations of the sets of infima and efficient solutions and we give also a multiplier rule for these kinds of points. The results are established for a vector optimisation problem with C-convexlike criterion, C being a polyhedral cone.

1. INTRODUCTION

Scalarisation and multiplier rule are two topics very much studied in the theory of convex vector optimisation. Several results with different conditions on the data, can be found in the literature, see for instance Jahn [2]; Dinh The Luc [3]; Sawaragi-Nakayama-Tanino [6]; Wang-Li [7]; Zlobec [8]. These results are most of the time established for weakly efficient points or properly efficient points.

One of the basic questions is how to have a characterisation of efficient points. An answer to this question is given by Zlobec [8] in the finite dimensional case for convex objectives, without contraints. Here we present a similar result in the infinite dimensional case for infima. Precisely, we consider X a real topological vector space, Y a locally convex real topological vector space ordered by a closed pointed convex cone C with a non empty interior. We suppose that Y is the topological dual of Y^*.

Considering $F : X \longrightarrow Y$ and E a subset of X, the problem we are interested in, is stated under the following form

$$(P) \quad \begin{cases} \inf & F(x) \\ x \in E \end{cases}$$

where F is C-convexlike on E. The meaning of (P) is clarified by Definition 6 below.

The paper consists of four parts. In Section 2, some basic definitions are introduced. Sections 3 and 4 discuss scalararisation and multiplier rule for infima and efficient points respectively.

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2. Definitions and properties

First we introduce some notations. The positive dual cone C^* and the strictly positive dual cone $C^{\#}$ of C are defined by

$$C^* = \{\lambda \in Y^* : \quad (\forall\, y \in C) \quad \lambda y \geqslant 0\}$$
$$C^{\#} = \{\lambda \in Y^* : \quad (\forall\, y \in C \setminus \{0\}) \quad \lambda y > 0\} \subset C^* \setminus \{0\}.$$

In all the sequel, C is a polyhedral cone, that is, there exists a finite set P and a collection $\{\lambda_i, i \in P\}$ of elements of $Y^* \setminus \{0\}$, with the property

$$C = \{y \in Y : \quad (\forall i \in P) \quad \lambda_i y \geqslant 0\}.$$

For each $i \in P$, let us denote

$$P^i = P \setminus \{i\} = \{k \in P : k \neq i\}.$$

Definition 1: If E is a convex set, F is said to be C-convex on E if for any $x, y \in E$ and any $t \in [0,1]$,

$$tF(x) + (1-t)F(y) \in F(tx + (1-t)y) + C.$$

Definition 2: [7] F is said to be C-convexlike on E if for any $x, y \in E$ and any $t \in [0,1]$, there exists $z \in E$ such that

$$tF(x) + (1-t)F(y) \in F(z) + C.$$

A subset B of Y is said to be C-convex if $B + C$ is convex. So that F is C-convexlike on E if and only if $F(E)$ is C-convex. In particular, all C-convex functions on E are C-convexlike on E.

Proposition 1. [7] *If F is C-convexlike on E, then for any $x_i \in E$, $i = 1 \ldots n$ and any $\alpha_i \geqslant 0$ with $\sum_{i=1}^{n} \alpha_i = 1$ there exists $\overline{x} \in E$ such that*

$$\sum_{i=1}^{n} \alpha_i F(x_i) \in F(\overline{x}) + C.$$

The proof is recursive. In all the sequel, F is C-convexlike on E.

We now give definitions for efficiency.

Definition 3: The set of minima of the set $B \subset Y$ is the set

$$\operatorname{Min}(B) = \{y^* \in B : (\nexists y \in B) \quad y^* - y \in C \setminus \{0\}\}.$$

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DEFINITION 4: $x^* \in E$ is an efficient solution to the problem (P) if $F(x^*)$ belongs to the set $\mathrm{Min}\,(F(E))$, and the value $F(x^*)$ is called a minimum of (P).

We denote by $\mathrm{Eff}\,(P)$ the set of all efficient solutions to the problem (P) and by $\mathrm{Min}\,(P)$ the set of all minima to the problem (P).

We consider now a notion given by Dolecki-Malivert [1], Postolica [4]: the notion of infimum which is based on the upper closure of a set.

DEFINITION 5: [1] The upper closure related to C of a set $B \subset Y$ is the set

$$cl_+B = \{\omega_0 \in Y : \quad (\forall \omega \in Y, \quad \omega - \omega_0 \in C \setminus \{0\}) \quad (\exists y \in B) \quad \omega - y \in C \setminus \{0\}\}.$$

DEFINITION 6: The set of infima for the problem (P) is defined by

$$\mathrm{Inf}\,(P) = \mathrm{Min}\,(cl_+F(E)).$$

PROPOSITION 2. $cl_+(F(E) + C) = cl_+F(E)$

PROOF: $F(E) \subset F(E) + C$, so the inclusion $cl_+F(E) \subset cl_+(F(E) + C)$ is trivial.

Let $\omega_0 \in cl_+(F(E) + C)$. By definition, for any $c \in C \setminus \{0\}$, there exists $z \in F(E) + C$ such that

$$\omega_0 + c - z \in C \setminus \{0\}.$$

We can write z in the form $F(x) + y, \quad x \in E, \; y \in C$. We obtain that for any $c \in C \setminus \{0\}$, there exists $x \in E$ and $y \in C$ such that $\omega_0 + c - F(x) \in y + C \setminus \{0\} \subset C \setminus \{0\}$, and in conclusion $\omega_0 \in cl_+F(E)$. □

PROPOSITION 3. *If F is C-convexlike on E, then $cl_+F(E)$ is a convex set.*

PROOF: Let ω_1, ω_2 be elements of $cl_+F(E)$. For any $c \in C \setminus \{0\}$, there exists $x_1, x_2 \in E$ such that

$$\omega_1 + c - F(x_1) \in C \setminus \{0\} \text{ and } \omega_2 + c - F(x_2) \in C \setminus \{0\}.$$

By the C-convexlikeness of F, for all $\alpha \in [0,1]$, there exists $x_0 \in E$ such that

$$(1-\alpha)\omega_1 + \alpha\omega_2 + c - F(x_0) \in (1-\alpha)(\omega_1 + c - F(x_1)) + \alpha(\omega_2 + c - F(x_2)) + C \subset C \setminus \{0\}.$$

Thus $(1-\alpha)\omega_1 + \alpha\omega_2 \in cl_+F(E)$. □

PROPOSITION 4. [1]

$$\mathrm{Min}\,(P) = \mathrm{Inf}\,(P) \cap F(E).$$

PROOF: If $\omega_0 \in \mathrm{Inf}\,(P) \cap F(E)$, then there exists $x_0 \in E$ such that $w_0 = F(x_0)$ and there is no $F(x) \in F(E) \subset cl_+F(E)$ such that $F(x_0) - F(x) \in C \setminus \{0\}$, so $\omega_0 \in \mathrm{Min}\,(P)$.

Suppose that $\omega_0 \notin \mathrm{Inf}(P)$ but $\omega_0 \in F(E)$. Then there exists $\omega_1 \in cl_+F(E)$ such that $\omega_0 - \omega_1 \in C \setminus \{0\}$. Since $\omega_1 \in cl_+F(E)$, for $c = \omega_0 - \omega_1$, there exists $x \in E$ such that

$$\omega_1 + (\omega_0 - \omega_1) - F(x) = \omega_0 - F(x) \in C \setminus \{0\}$$

and so $\omega_0 \notin \mathrm{Min}(P)$. ∎

3. Characterisation of infima

Theorem 1. *w_0 is an infimum if and only if $\omega_0 \in cl_+F(E)$ and*

$$(\omega_0 - C \setminus \{0\}) \cap F(E) = \emptyset. \tag{1}$$

Proof: If $\omega_0 \in \mathrm{Inf}(P)$, then by definition $\omega_0 \in \mathrm{Min}(cl_+F(E))$, and in particular $(\omega_0 - C \setminus \{0\}) \cap cl_+F(E) = \emptyset$. (1) is true because $F(E) \subset cl_+F(E)$.

Suppose that $\omega_0 \notin \mathrm{Inf}(P)$ but $\omega_0 \in cl_+F(E)$. There exists $\omega_1 \in cl_+F(E)$ such that $\omega_0 - \omega_1 \in C \setminus \{0\}$. As $\omega_1 \in cl_+F(E)$, for $c = \omega_0 - \omega_1 \in C \setminus \{0\}$, there exists $x \in E$ such that

$$\omega_1 + (w_0 - \omega_1) - F(x) \in C \setminus \{0\}.$$

Therefore there exists $x \in E$ such that $w_0 - F(x) \in C \setminus \{0\}$, which contradicts (1). ∎

Definition 7: For any $\omega_0 \in cl_+F(E)$ and $i \in P$, we set

$$\Psi_i(\omega_0) = \{\lambda_j,\ j \in P^i :\ x \in E, \quad \lambda_k(\omega_0 - F(x)) \geqslant 0,\ k \in P^i \Longrightarrow \lambda_j(\omega_0 - F(x)) = 0\}.$$

Notice that if there exists $x \in E$ such that $\omega_0 - F(x) \in \mathrm{int}\, C$, then $\Psi_i(\omega_0) = \emptyset$, for all $i \in P$.

Definition 8: For any $\triangle \subset \{\lambda_i,\ i \in P\}$ and $\omega_0 \in cl_+F(E)$, we define

$$\mathcal{E}(\omega_0, \triangle) = \{x \in E : \quad (\forall \lambda_i \in \triangle) \quad \lambda_i(\omega_0 - F(x)) = 0\}.$$

The following result gives a sufficient condition for $\omega_0 \in cl_+F(E)$ to be in $\mathrm{Inf}(P)$.

Theorem 2. *Let $\omega_0 \in cl_+F(E)$. If there exists $\lambda \in co\Big(\bigcup_{i \in P} \lambda_i \setminus \bigcup_{i \in P} \Psi_i(\omega_0)\Big)$, $\lambda \neq 0$ such that*

$$\Big(\forall x \in \bigcap_{i \in P} \mathcal{E}(\omega_0, \Psi_i(\omega_0))\Big), \quad \lambda(F(x) - \omega_0) \geqslant 0 \tag{2}$$

then $\omega_0 \in \operatorname{Inf}(P)$.

PROOF: Suppose that $\omega_0 \notin \operatorname{Inf}(P)$. Then as $\omega_0 \in cl_+F(E)$, $(\omega_0 - C \setminus \{0\}) \cap F(E) \neq \emptyset$. Also there exists $x_0 \in E$ such that $\omega_0 - F(x_0) \in C \setminus \{0\}$. C is pointed, so $F(x_0) - \omega_0 \notin C$ and therefore

$$(3) \qquad (\exists i_0 \in P) \quad \lambda_{i_0}(F(x_0) - \omega_0) < 0.$$

By assumption, we can write λ as a convex combination,

$$(4) \qquad \lambda = \sum_{\lambda_{j_k} \in \triangle} \mu_{j_k} \lambda_{j_k}$$

where $\triangle \subset \{\lambda_i, i \in P\}$, $(\forall \lambda_{j_k} \in \triangle) \quad \lambda_{j_k} \notin \bigcup_{i \in P} \Psi_i(\omega_0)$, $\mu_{j_k} > 0$, $\sum_{\lambda_{j_k} \in \triangle} \mu_{j_k} = 1$.

Case $\lambda_{i_0} \in \triangle$. Since $\omega_0 - F(x_0) \in C$, for any $i \in P$, $\lambda_i(F(x_0) - \omega_0) \leqslant 0$. We deduce from (3) and (4) that $\lambda(F(x_0) - \omega_0) < 0$. Furthermore, for any $\lambda_j \in \Psi_i(\omega_0)$, using $\omega_0 - F(x_0) \in C$, we have $\lambda_j(\omega_0 - F(x_0)) = 0$. Therefore $x_0 \in \mathcal{E}(\omega_0, \Psi_i(\omega_0))$ and $\lambda(F(x_0) - \omega_0) < 0$ contradicting (2).

Case $\lambda_{i_0} \notin \triangle$. Take for example $\lambda_{j_1} \notin \bigcup_{i \in P} \Psi_i(\omega_0)$. We have in particular $\lambda_{j_1} \notin \Psi_{i_0}(\omega_0)$ and by definition of $\Psi_{i_0}(\omega_0)$, there exists $x_1 \in E$ such that

$$(5) \qquad (\forall i \in P^{i_0}) \quad \lambda_i(\omega_0 - F(x_1)) \geqslant 0, \qquad \lambda_{j_1}(\omega_0 - F(x_1)) > 0$$

and therefore $\lambda(\omega_0 - F(x_1)) > 0$. Since F is C-convexlike on E, for all $\alpha \in [0,1]$, there exists $x_\alpha \in E$ such that

$$\begin{aligned} \omega_0 - F(x_\alpha) &\in \alpha\omega_0 + (1-\alpha)\omega_0 - \alpha F(x_1) - (1-\alpha)F(x_0) + C \\ &\in (1-\alpha)(\omega_0 - F(x_0)) + \alpha(\omega_0 - F(x_1)) + C. \end{aligned}$$

With (5) and because $\omega_0 - F(x_0) \in C$, we have

$$(6) \qquad (\forall i \in P^{i_0}) \quad \lambda_i(\omega_0 - F(x_\alpha)) \geqslant 0.$$

On the other hand , for α small enough, we have from (3)

$$(7) \qquad \lambda_{i_0}(\omega_0 - F(x_\alpha)) > 0,$$

and from (5)

$$(8) \qquad \lambda_{j_1}(\omega_0 - F(x_1)) > 0.$$

From (6) and (7), we deduce that $\omega_0 - F(x_\alpha) \in C$ and $x_\alpha \in \mathcal{E}(\omega_0, \Psi_i(\omega_0))$ for $i \in P$. Also from (7) and (8), $\lambda(\omega_0 - F(x_\alpha)) > 0$. This gives a contradiction with (2). ∎

The following result gives a necessary condition for $\omega_0 \in cl_+F(E)$ to be in $\operatorname{Inf}(P)$.

Theorem 3. *If* ω_0 *is an infimum and* $\bigcup_{i\in P}\lambda_i \neq \bigcup_{i\in P}\Psi_i(\omega_0)$ *then there exists* $\lambda \in co\Big(\bigcup_{i\in P}\lambda_i \setminus \bigcup_{i\in P}\Psi_i(\omega_0)\Big)$, $\lambda \neq 0$ *such that*

$$(9) \qquad \Big(\forall x \in \bigcap_{i\in P}\mathcal{E}(\omega_0,\Psi_i(\omega_0))\Big) \quad \lambda(F(x)-\omega_0) \geqslant 0.$$

Proof: We suppose that $\bigcap_{i\in P}\mathcal{E}(\omega_0,\Psi_i(\omega_0)) \neq \emptyset$. First we show that

$$(10) \qquad co\Big[F\Big(\bigcap_{i\in P}\mathcal{E}(\omega_0,\Psi_i(\omega_0))\Big)-\omega_0\Big] \cap \Big(-\operatorname{int}\Big(\bigcup_{i\in P}\lambda_i \setminus \bigcup_{i\in P}\Psi_i(\omega_0)\Big)^*\Big) = \emptyset.$$

Notice that

$$(11) \qquad \operatorname{int}\Big(\bigcup_{i\in P}\lambda_i \setminus \bigcup_{i\in P}\Psi_i(\omega_0)\Big)^* \supset \operatorname{int} C \neq \emptyset.$$

Suppose that (10) is false, then there exists $v \in co\Big[F\Big(\bigcap_{i\in P}\mathcal{E}(\omega_0,\Psi_i(\omega_0))\Big)-\omega_0\Big]$ such that for any $\lambda_k \notin \bigcup_{i\in P}\Psi_i(\omega_0)$, $\lambda_k v < 0$. We can find such a λ_k because of the assumption $\bigcup_{i\in P}\lambda_i \neq \bigcup_{i\in P}\Psi_i(\omega_0)$. We can write v as $v = \sum_{j\in J}\mu_j(F(x_j)-\omega_0) = \sum_{j\in J}\mu_j F(x_j)-\omega_0$ where J is finite, $\mu_j > 0$, $\sum_{j\in J}\mu_j = 1$, and $x_j \in \bigcap_{i\in P}\mathcal{E}(\omega_0,\Psi_i(\omega_0))$. By the C-convexlikeness of F, there exists $x^* \in E$ such that $v \in F(x^*)-\omega_0+C$ and then

$$(12) \qquad \Big(\forall \lambda_k \notin \bigcup_{i\in P}\Psi_i(\omega_0)\Big) \quad \lambda_k(F(x^*)-\omega_0) < 0.$$

Now, we consider $\lambda_k \in \bigcup_{i\in P}\Psi_i(\omega_0)$. There exists $r \in P$, such that $\lambda_k \in \Psi_r(\omega_0)$, $(k \neq r)$ and since $x_j \in \bigcap_{i\in P}\mathcal{E}(\omega_0,\Psi_i(\omega_0))$, we have in particular

$$\text{for any} \quad j \in J, \quad x_j \in \mathcal{E}(\omega_0,\Psi_r(\omega_0)).$$

This implies that $\lambda_k(F(x_j)-\omega_0) = 0$ and by using the C-convexlikeness of F,

$$(13) \qquad \Big(\forall \lambda_k \in \bigcup_{i\in P}\Psi_i(\omega_0)\Big) \quad \lambda_k(F(x^*)-\omega_0) \leqslant 0.$$

From (12) and (13), $\omega_0 - F(x^*) \in C \setminus \{0\}$, which contradicts $\omega_0 \in \operatorname{Inf}(P)$, and therefore (10) is true.

We use Eidelheit's separation theorem [**2**] for the convex sets

$$S = -\Big(\bigcup_{i\in P} \lambda_i \setminus \bigcup_{i\in P} \Psi_i(\omega_0)\Big)^* \quad \text{with} \quad \operatorname{int} S \neq \emptyset \quad \text{and}$$

$$T = co[F\Big(\bigcap_{i\in P} \mathcal{E}(\omega_0, \Psi_i(\omega_0))\Big) - \omega_0].$$

(10) is equivalent to the existence of $\lambda \in Y^* \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that

$$\begin{cases} (\forall s \in S,\ \forall t \in T) & \lambda s \leqslant \alpha \leqslant \lambda t \\ (\forall s \in \operatorname{int} S) & \lambda s < \alpha. \end{cases}$$

S is a non empty cone, thus $\alpha = 0$ and

$$\lambda \in (-S)^* = \Big(\bigcup_{i\in P} \lambda_i \setminus \bigcup_{i\in P} \Psi_i(\omega_0)\Big)^{**} = co\Big(\bigcup_{i\in P} \lambda_i \setminus \bigcup_{i\in P} \Psi_i(\omega_0)\Big)$$

because $\Big(\bigcup_{i\in P} \lambda_i \setminus \bigcup_{i\in P} \Psi_i(\omega_0)\Big)$ is a finite set whose convex hull is closed. We obtain (9). □

From the two previous theorems, we deduce the following characterisation of the infima.

THEOREM 4. *$\omega_0 \in cl_+F(E)$ is an infimum if and only if we have one of the two following conditions*

1.

$$\bigcup_{i\in P} \lambda_i = \bigcup_{i\in P} \Psi_i(\omega_0). \tag{14}$$

2. *There exists $\lambda \in co\Big(\bigcup_{i\in P} \lambda_i \setminus \bigcup_{i\in P} \Psi_i(\omega_0)\Big)$, $\lambda \neq 0$ such that*

$$\Big(\forall x \in \bigcap_{i\in P} \mathcal{E}(\omega_0, \Psi_i(\omega_0))\Big) \quad \lambda(F(x) - \omega_0) \geqslant 0. \tag{15}$$

PROOF: For any point ω_0, we have $\bigcup_{i\in P}\lambda_i = \bigcup_{i\in P}\Psi_i(\omega_0)$ or $\bigcup_{i\in P}\lambda_i \neq \bigcup_{i\in P}\Psi_i(\omega_0)$. In the second case, we proved the necessary part in Theorem 3.

On the other hand, by Theorem 2 and Condition 2 we obtain that ω_0 is an infimum. It remains to show that Condition 1 is also sufficient. Suppose that $\omega_0 \notin \operatorname{Inf}(P)$. There exists $\omega \in cl_+F(E)$ such that $\omega_0 - \omega \in C\setminus\{0\}$ and in particular there exists $i_0 \in P$ such that $\lambda_{i_0}(\omega_0 - \omega) > 0$, because C is pointed. Also for any $i \neq i_0$, $\lambda_{i_0} \notin \Psi_i(\omega_0)$ and by definition $\lambda_{i_0} \notin \bigcup_{i\in P}\Psi_i(\omega_0)$. ∎

We give now a multiplier rule for infima in the case where the set E of constraints has the following form

$$E = \{x \in S : G(x) \in -K\}$$

with $S \subset X$ and $G : X \longrightarrow Z$, Z being a real topological vector space with order induced by a convex cone K with a non empty interior. We suppose that C is polyhedral and (F, G) is $C \times K$-convexlike on S.

THEOREM 5. *Let $\omega_0 \in cl_+F(E)$. If there exists $\lambda \in co\Big(\bigcup_{i\in P}\lambda_i \setminus \bigcup_{i\in P}\Psi_i(\omega_0)\Big)$, $\lambda \neq 0$ and $\mu \in K^*$ such that*

$$\Big(\forall x \in \bigcap_{i\in P}\mathcal{E}(\omega_0, \Psi_i(\omega_0))\Big)\quad \lambda(F(x) - \omega_0) + \mu G(x) \geqslant 0. \tag{16}$$

then ω_0 is an infimum.

PROOF: Let $\omega_0 \in cl_+F(E)$ such that $\omega_0 \notin \operatorname{Inf}(P)$. Using Theorem 2, we have for any $\lambda \in co\Big(\bigcup_{i\in P}\lambda_i \setminus \bigcup_{i\in P}\Psi_i(\omega_0)\Big)$, $\lambda \neq 0$ $\Big(\exists x_0 \in \bigcap_{i\in P}\mathcal{E}(\omega_0, \Psi_i(\omega_0))\Big)$ $\lambda(F(x_0) - \omega_0) < 0$ and for any $\mu \in K^*$, $\mu G(x_0) \leqslant 0$ so $\lambda(F(x_0) - \omega_0) + \mu G(x_0) < 0$, which contradicts (16). ∎

Let us define the set $\Gamma(\omega_0)$ for a point $\omega_0 \in cl_+F(E)$:

$$\Gamma(\omega_0) = \{x \in E : \quad \omega_0 - F(x) \in C\}.$$

We remark that $\Gamma(\omega_0)$ is a subset of $\bigcap_{i\in P}\mathcal{E}(\omega_0, \Psi_i(\omega_0))$.

THEOREM 6. *If $\omega_0 \in \operatorname{Inf}(P)$ and $\bigcup_{i\in P}\lambda_i \neq \bigcup_{i\in P}\Psi_i(\omega_0)$ then there exists $\lambda \in co\Big(\bigcup_{i\in P}\lambda_i \setminus \bigcup_{i\in P}\Psi_i(\omega_0)\Big)$, $\mu \in K^*$ $(\lambda, \mu) \neq (0, 0)$ such that*

$$\Big(\forall x \in \bigcap_{i\in P}\mathcal{E}(\omega_0, \Psi_i(\omega_0))\Big)\quad \lambda(F(x) - \omega_0) + \mu G(x) \geqslant 0. \tag{17}$$

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If there exists $x_0 \in S$ *such that* $G(x_0) \in -\operatorname{int} K$ *then we obtain (17) with* $\lambda \neq 0$. *If in addition,* $\Gamma(\omega_0) \neq \emptyset$, *then for any* $x_0 \in \Gamma(\omega_0)$, $\mu G(x_0) = 0$ *and* $x_0 \in \operatorname{Eff}(P)$.

PROOF: Let $\omega_0 \in \operatorname{Inf}(P)$. We define the set M of $Y \times Z$ by

$$M = \{(F(x) - \omega_0 + y, G(x) + z) : x \in \bigcap_{i \in P} \mathcal{E}(\omega_0, \Psi_i(\omega_0)),$$
$$y \in \operatorname{int}\left(\bigcup_{i \in P} \lambda_i \setminus \bigcup_{i \in P} \Psi_i(\omega_0)\right)^*, \ z \in K\}$$

We show that $(0_Y, 0_Z) \notin coM$. Otherwise there exist $x_1, x_2 \in \bigcap_{i \in P} \mathcal{E}(\omega_0, \Psi_i(\omega_0))$, $y_1, y_2 \in \operatorname{int}\left(\bigcup_{i \in P} \lambda_i \setminus \bigcup_{i \in P} \Psi_i(\omega_0)\right)^*$, $z_1, z_2 \in K$ and $\alpha \in [0,1]$ such that

$$(0,0) = (1-\alpha)(F(x_1) - \omega_0 + y_1, G(x_1) + z_1) + \alpha(F(x_2) - \omega_0 + y_2, G(x_2) + z_2).$$

By using the $C \times K$-convexlikeness of (F, G), there exist $x_0 \in S$, $c_0 \in C$ and $k_0 \in K$ such that

$$(0,0) = (F(x_0) + c_0 - \omega_0 + (1-\alpha)y_1 + \alpha y_2, G(x_0) + k_0 + (1-\alpha)z_1 + \alpha z_2).$$

Setting $y = (1-\alpha)y_1 + \alpha y_2 + c_0 \in \operatorname{int}\left(\bigcup_{i \in P} \lambda_i \setminus \bigcup_{i \in P} \Psi_i(x^*)\right)^*$ and $z = (1-\alpha)z_1 + \alpha z_2 + k_0 \in K$, we have

$$\begin{cases} y = \omega_0 - F(x_0) \\ z = -G(x_0). \end{cases}$$

(18) For any $\lambda_k \notin \bigcup_{i \in P} \Psi_i(\omega_0)$, we have $\lambda_k(\omega_0 - F(x_0)) > 0$.

For any $\lambda_k \in \bigcup_{i \in P} \Psi_i(\omega_0)$, we have $\lambda_k(\omega_0 - F(x_1)) = 0$ and $\lambda_k(\omega_0 - F(x_2)) = 0$,

$$\text{also } \lambda_k(\omega_0 - F(x_0)) = \lambda_k y \geqslant 0.$$

(18) and (19) contradict $\omega_0 \in \operatorname{Inf}(P)$.

We use now Eidelheit's separation theorem [2]. There exists $\lambda \in Y^*, \mu \in Z^*$, $(\lambda, \mu) \neq (0,0)$ such that $\forall x \in \bigcap_{i \in P} \mathcal{E}(\omega_0, \Psi_i(\omega_0))$, $\forall y \in \left(\bigcup_{i \in P} \lambda_i \setminus \bigcup_{i \in P} \Psi_i(\omega_0)\right)^*$, $\forall z \in K$

(20) $$\lambda(F(x) - \omega_0 + y) + \mu(G(x) + z) \geqslant 0.$$

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Let $x_0 \in \bigcap_{i \in P} \mathcal{E}(\omega_0, \Psi_i(\omega_0))$ be fixed and $z_0 = -G(x_0) \in K$. Then (20) becomes:

$$\forall y \in \left(\bigcup_{i \in P} \lambda_i \setminus \bigcup_{i \in P} \Psi_i(\omega_0)\right)^*, \ \lambda(F(x_0) - \omega_0) \geqslant -\lambda y$$

so $\lambda \in \left(\bigcup_{i \in P} \lambda_i \setminus \bigcup_{i \in P} \Psi_i(\omega_0)\right)^{**} = co\left(\bigcup_{i \in P} \lambda_i \setminus \bigcup_{i \in P} \Psi_i(\omega_0)\right)$ because $\left(\bigcup_{i \in P} \lambda_i \setminus \bigcup_{i \in P} \Psi_i(\omega_0)\right)$ is a finite set whose convex hull is closed. Taking x_0, y_0 fixed, (20) becomes:

$$(\forall z \in K) \ \lambda(F(x_0) - \omega_0 + y_0) + \mu G(x_0) \geqslant -\mu z,$$

from which $\mu \in K^*$. Finally, taking $y = 0$, $z = 0$, we obtain (17).

If there exists $x_0 \in S$ such that $G(x_0) \in -\operatorname{int} K$, then if $\mu \neq 0 \quad \mu G(x_0) < 0$. If we want (17), we need $\lambda \neq 0$.

If $\Gamma(\omega_0) \neq \emptyset$, then there exists $x_0 \in S$ such that $\omega_0 - F(x_0) \in C$ and $G(x_0) \in -K$. $x_0 \in \bigcap_{i \in P} \mathcal{E}(\omega_0, \Psi_i(\omega_0))$, because if we take $\lambda_k \in \Psi_i(\omega_0)$, as $\omega_0 - F(x_0) \in C$,

$$\text{we have} \quad \lambda_k(\omega_0 - F(x_0)) = 0.$$

$y = \omega_0 - F(x_0) \in \left(\bigcup_{i \in P} \lambda_i \setminus \bigcup_{i \in P} \Psi_i(\omega_0)\right)^*$ because for any $k \in P$, $\lambda_k y \geqslant 0$ and this is so for the particular $\lambda_k \in \bigcup_{i \in P} \lambda_i \setminus \bigcup_{i \in P} \Psi_i(\omega_0)$. Taking x_0, y in (20), we obtain:

$$(\forall z \in K) \quad \mu G(x_0) \geqslant -\mu z \ \text{ and so } \ \mu G(x_0) \geqslant 0,$$

but $G(x_0) \in -K$ so $\mu G(x_0) = 0$. By definition of $\Gamma(\omega_0)$, $\omega_0 - F(x_0) \in C$ and if $\omega_0 - F(x_0) \in C \setminus \{0\}$, then $\omega_0 \notin \operatorname{Inf}(P)$; so $\omega_0 = F(x_0) \in \operatorname{Min}(P)$ and using Proposition 4, we obtain $x_0 \in \operatorname{Eff}(P)$. ∎

In [**3**, Theorem 2.10, p.91], Luc considers the problems

$$(VP) \begin{cases} \min & F(x) \\ x \in E \end{cases} \quad \text{and} \quad (SP)_\lambda \begin{cases} \min & \lambda F(x) \\ x \in E \end{cases}$$

where E is a convex set, F is C-convex on E and $\lambda \in C^* \setminus \{0\}$. Denoting by $\operatorname{WEff}(VP)$ the set of weakly efficient points

$$\operatorname{WEff}(VP) = \{x^* \in E : \quad (\nexists x \in E) \quad F(x^*) - F(x) \in \operatorname{int} C\}$$

he presents a theorem in the following form:

Theorem 7. $x^* \in \operatorname{WEff}(VP)$ *if and only if there exists* $\lambda \in C^* \setminus \{0\}$ *such that* x^* *is an optimal solution of the problem* $(SP)_\lambda$.

In the next section, we present a result of this type for efficient points.

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4. Characterisation of efficient points

For a fixed point $x^* \in E$ and $i \in P$, let us denote

$$\varphi_i(x^*) = \{\lambda_j, j \in P^i : x \in E, \quad \lambda_k(F(x^*) - F(x)) \geqslant 0, k \in P^i \Longrightarrow \lambda_j(F(x^*) - F(x)) = 0\},$$

and for any $\triangle \subset \{\lambda_i,\ i \in P\}$ and $x^* \in E$, we define

$$\mathcal{D}(x^*, \triangle) = \{x \in E : \quad (\forall \lambda_i \in \triangle) \quad \lambda_i(F(x^*) - F(x)) = 0\}.$$

Notice that if there exists $x \in E$ such that $F(x^*) - F(x) \in \operatorname{int} C$, then $\varphi_i(x^*) = \emptyset$, for any $i \in P$.

If we set $F(x^*) = \omega_0$, then the sets $\varphi_i(x^*)$ and $\mathcal{D}(x^*, \triangle)$ are identical to $\Psi_i(\omega_0)$ and $\mathcal{E}(\omega_0, \triangle)$ so that using Proposition 4 and the results of the previous section, we obtain a characterisation of the efficient points.

Corollary 1. *$x^* \in E$ is efficient if and only if we have one of the two following conditions*

1.

$$\bigcup_{i \in P} \lambda_i = \bigcup_{i \in P} \varphi_i(x^*). \tag{21}$$

2. *There exists* $\lambda \in co\Big(\bigcup_{i \in P} \lambda_i \setminus \bigcup_{i \in P} \varphi_i(x^*)\Big)$, $\lambda \neq 0$ *such that*

$$\Big(\forall x \in \bigcap_{i \in P} \mathcal{D}(x^*, \varphi_i(x^*))\Big) \quad \lambda(F(x) - F(x^*)) \geqslant 0. \tag{22}$$

Proof: We use Theorem 4 and the previous remarks. ∎

We give now a multiplier rule for efficient points in the case where the set E of contraints has the form

$$E = \{x \in S : G(x) \in -K\}$$

with $S \subset X$ and $G : X \longrightarrow Z$, Z being a real topological vector space with order induced by a convex cone K with a non-empty interior. We suppose that C is polyhedral and (F, G) is $C \times K$-convexlike on S.

Corollary 2. *Let $x^* \in E$. If there exists* $\lambda \in co\Big(\bigcup_{i \in P} \lambda_i \setminus \bigcup_{i \in P} \varphi_i(x^*)\Big)$, $\mu \in K^*$, $(\lambda, \mu) \neq (0, 0)$ *such that*

$$\Big(\forall x \in \bigcap_{i \in P} \mathcal{D}(x^*, \varphi_i(x^*))\Big) \quad \lambda(F(x) - F(x^*)) + \mu(G(x) - G(x^*)) \geqslant 0 \tag{23}$$

and $\mu G(x^*) = 0$ *then* $x^* \in \mathrm{Eff}\,(P)$.

PROOF: We use Theorem 6 and the previous remarks. $\square$

COROLLARY 3. *If* x^* *is efficient and* $\bigcup_{i\in P} \lambda_i \neq \bigcup_{i\in P} \varphi_i(x^*)$ *then there exists* $\lambda \in co\Big(\bigcup_{i\in P} \lambda_i \setminus \bigcup_{i\in P} \varphi_i(x^*)\Big)$, $\mu \in K^*$, $(\lambda,\mu) \neq (0,0)$ *such that*

$$\Big(\forall x \in \bigcap_{i\in P} \mathcal{D}(x^*, \varphi_i(x^*))\Big) \quad \lambda(F(x) - F(x^*)) + \mu(G(x) - G(x^*)) \geqslant 0 \tag{24}$$

and $\mu G(x^*) = 0$.

PROOF: We use Theorem 6, the fact that $x^* \in \Gamma(F(x^*))$ and the previous remarks. $\square$

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