A HARNACK INEQUALITY FOR DEGENERATE ELLIPTIC EQUATIONS ON MINIMAL SURFACES

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Harnack inequalities are known to be of great importance in the theory of quasilinear elliptic partial differential equations. In the case of such equations defined over a domain Ω in $\mathbf{R}^n$, inequalities of this type have been proved for solutions of second-order equations in divergence form which are of either elliptic or degenerate elliptic structure. More recently Bombieri and Giusti (**2**) have proved a Harnack inequality for solutions of linear elliptic equations on a minimal surface in $\mathbf{R}^{n+1}$. The equations are of the form

$$\delta_i(a_{ij}\delta_j u) = 0, \tag{1.1}$$

where summation over $i, j = 1, \ldots, n+1$ is understood, and $\delta = (\delta_1, \ldots, \delta_{n+1})$ is the tangential derivative on S. In (**2**), the inequality is used to give much simplified proofs of some classical results on minimal surfaces, and to generalise some more recent ones.

In this paper we shall establish inequalities of Harnack type for weak solutions and supersolutions of equation (1.1), where the coefficients a_{ij} satisfy

$$\frac{\mu(x)}{\mu_0}\,|\,\xi\,|^2 \leqq a_{ij}(x)\xi_i\xi_j \leqq \mu(x)|\,\xi\,|^2 \quad \text{for all } \xi \quad \text{in } \mathbf{R}^{n+1}.$$

Here $\mu_0 > 1$ is a constant and μ, μ^{-1} are non-negative and satisfy certain integrability conditions. This means that μ can have zeros on S, and so the matrix (a_{ij}) is non-negative definite, rather than positive definite as in the elliptic case. The results in this paper combine the methods of (**2**) with the work of Murthy and Stampacchia on degenerate elliptic equations (**7**).

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Let A be an open set in $\mathbf{R}^{n+1}$, $h \in C^\infty(A)$, $|\,Dh\,|$ never zero on A. Denote the operator $\partial/\partial x_i$ by D_i, for $i = 1, \ldots, n+1$. If S is the surface $h = 0$, the normal $\nu = (\nu_1, \ldots, \nu_{n+1})$ to S (disregarding direction) is given by

$$\nu_i = \frac{D_i h}{|\,Dh\,|}.$$

We will define

$$\delta_i = D_i - \nu_i \sum_{j=1}^{n} \nu_j D_j, \quad i = 1, \ldots, n+1,$$

$$\delta f = (\delta_1 f, \ldots, \delta_{n+1} f).$$

If H^n denotes the n-dimensional Hausdorff measure on S, then for any ϕ in $C^2(\mathbf{R}^{n+1})$, such that $\phi\,|_S$ has compact support,

$$\int_S \delta_i \phi dH^n = n \int_S \phi H v_i dH^n, \quad i = 1, \ldots, n+1,$$

where H is the mean curvature of S (see (1)). If S is a minimal surface, $H \equiv 0$, and so

$$\int_S \delta_i \phi dH^n = 0, \quad i = 1, \ldots, n+1.$$

This enables us to integrate by parts on the surface, and so we can define weak derivatives on S, and thus weighted Sobolev spaces. If $\mu \in L^s(H^n, S)$ for some $s \geqq 1$, we shall write $L^p(\mu, S)$, $p \geqq 1$, for the set of functions u: $S \to R$ with

$$\| u \|_{p,\mu,S} \equiv \left(\int_S | u |^p \mu dH^n \right)^{1/p} < \infty$$

and denote by $H^{1,2}(\mu, S)$ the set of functions u in $L^p(\mu, S)$ for which there exist functions u_α, ($\alpha = 1, \ldots, n+1$), in $L^p(\mu, S)$ such that for each ϕ in $C_0^1(S)$

$$\int_S \phi(x) u_\alpha(x) dH^n = -\int_S (\delta_\alpha \phi(x)) u(x) dH^n.$$

When there is no ambiguity, we shall write $H^{1,2}(\mu)$, and omit the subscript S in writing the norm. $H^{1,2}(\mu)$ is the Banach space of equivalence classes of functions with norm

$$\| u \|_{1,2,\mu} = \| u \|_{2,\mu} + \sum_\alpha \| u_\alpha \|_{2,\mu}.$$

$H_0^{1,2}(\mu)$ will stand for the completion of $C_0^1(S)$ in $H^{1,2}(\mu)$. Local versions of these spaces are defined in the obvious way. Analogous definitions are made for the unweighted Sobolev spaces $H^{1,2}(S)$, $H_0^{1,2}(S)$.

Now suppose the isoperimetric inequality holds, that is, for any subset E of S,

$$(H^n(E))^{(n-1)/n} \leqq \gamma H^{n-1}(\partial E), \tag{2.1}$$

where $H^n(E)$ denotes the n-dimensional measure of E, ∂E the boundary of E, and γ is a constant depending only on n. Then we have a Sobolev inequality for functions u in the space $H_0^{1,p}(S)$,

Theorem 2.1. (See (1).) *Let $n \geqq 2$ and $1 \leqq p < n$. Then for all u in $H_0^{1,p}(S)$, $u \in L^{pn/(n-p)}(S)$ and*

$$\left(\int_S | u |^{np/(n-p)} dH^n \right)^{1/p-1/n} \leqq C(n, p) \left(\int_S | \delta u |^p dH^n \right)^{1/p}. \tag{2.2}$$

By using the techniques of (7), we can derive a similar inequality in the case of weighted Sobolev spaces, under suitable hypotheses on μ. Let

$$\mu \in L^\infty(H^n, S), \ \mu^{-1} \in L^t(H^n, S),$$

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where $1/t \leqq 2/n < 1+1/t$. For convenience we write

$$(p^*)^{-1} = 1/p - 1/n,\ (p^\#)^{-1} = (1+1/t)/p - 1/n,$$

and $1/\tau = 1+1/t$.

First we shall prove that $H_0^{1,2}(\mu, S) \subset H_0^{1,2\tau}(S)$, with a continuous embedding. Now $1 \leqq 2\tau < n$, and $\tau + \tau/t = 1$. Let $u \in H_0^{1,2}(\mu, S)$; then

$$\| u \|_{2\tau} = \left(\int_S | u |^{2\tau} \mu^\tau \mu^{-\tau} dH^n \right)^{1/(2\tau)} \leqq \| u \|_{2,\mu} \| \mu^{-1} \|_t^{\frac{1}{2}}$$

and

$$\| \, | \delta u | \, \|_{2\tau} \leqq \| \, | \delta u | \, \|_{2,\mu} \| \mu^{-1} \|_t^{\frac{1}{2}}.$$

From (2.2), we see that

$$\| u \|_{(2\tau)^*} \leqq C(n) \| \mu^{-1} \|_t^{\frac{1}{2}} \| \, | \delta u | \, \|_{2,\mu}.$$

But $(2\tau)^* = 2^\#$ and so

$$\| u \|_{2^\#} \leqq C \| \, | \delta u | \, \|_{2,\mu}.$$

If $\mu \in L^\infty(H^n, S)$ then

$$\| u \|_{2^\#,\mu} \leqq \| u \|_{2^\#} \| \mu \|_S^{\frac{1}{2}^\#}$$

and thus

$$\| u \|_{2^\#,\mu} \leqq C \| \, | \delta u | \, \|_{2,\mu}. \tag{2.3}$$

Now suppose that Ω is a subset of $\mathbf{R}^n$, $f \in C^2(\Omega)$, and let S be the surface given by

$$S = \{(x, f(x)) \colon x \in \Omega\}. \tag{2.4}$$

If f satisfies the minimal surface equation, then S is a minimal surface. If E is any measurable subset of S, and $\tilde{E}$ denotes the projection of E onto Ω then, cf. (**4**), there is a constant $\kappa > 0$ such that

$$L^n(\tilde{E}) = \kappa H^n(E)$$

where L^n denotes n-dimensional Lebesgue measure. Suppose $\mu \in L^\infty(S, H^n)$, then

$$H_\mu^n(E) \equiv \left| \int_E \mu dH^n \right| \leqq \| \mu \|_{\infty,E} \left| \int_E dH^n \right|$$

and so $H_\mu^n(E) \leqq KL^n(\tilde{E})$, where K depends on n, μ. Similarly

$$H^n(E) = \left| \int_E \mu\mu^{-1} dH^n \right| \leqq \left(\int_E \mu^{t'} dH^n \right)^{1/t'} \left(\int_E \mu^{-t} dH^n \right)^{1/t}$$

$$\leqq \left(\int_E \mu^{t'-1} \mu dH^n \right)^{1/t'} \left(\int_E \mu^{-t} dH^n \right)^{1/t}$$

$$\leqq \| \mu \|_{\infty,S}^{1/t} \| \mu^{-1} \|_{t,S} (H_\mu^n(E))^{1/t'}.$$

Thus

$$(K' L^n(\tilde{E}))^{t'} \leqq H_\mu^n(E) \leqq KL^n(\tilde{E}). \tag{2.5}$$

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In the work that follows we shall have occasion to use the following two theorems proved in (2). The second is an analogue of the John-Nirenberg lemma used in the proof of Harnack inequalities in a domain Ω in $\mathbf{R}^n$.

Theorem 2.2. *Let S be an oriented boundary of least area in B_R the $(n+1)$-ball of radius R. Then for every f in $C^1(B_R)$,*

$$\min_{k\in\mathbf{R}}\left\{\int_{B_{\beta R}} |f-k|^{n/(n-1)}\,dH^n\right\}^{(n-1)/n} \leqq 2\gamma\int_{B_R} |\delta f|\,dH^n,$$

where β is a constant depending only on the dimension $n+1$.

Before stating the next theorem we need some notation. Let X be a topological space, m a regular positive Borel measure on X, and K_r, $0\leqq r\leqq 1$, a family of non-empty open subsets of X such that

(i) $K_s\subset K_r$ if $s\leqq r$,

(ii) $0<m(K_r)<\infty$ if $0\leqq r\leqq 1$.

Let u: $X\to R$ be m-measurable, and if $p\neq 0$ let

$$|u|_{p,r}\equiv\left(\frac{1}{m(K_r)}\int_{K_r} u^p dm\right)^{1/p},$$

$$|u|_{\infty,r}=\operatorname*{ess\,sup}_{K_r} u,$$

$$|u|_{-\infty,r}=\operatorname*{ess\,inf}_{K_r} u.$$

Theorem 2.3. *Let $0<\theta_0,\ \theta_1\leqq\infty$, and let*

$$|u|_{\theta_0,1}<+\infty,\ |u|_{-\theta_1,1}>0.$$

Suppose there exist constants $\alpha>1$, p_0, $0<p_0\leqq\frac{1}{2}\min(\theta_0,\ \theta_1)$, and $Q>0$ such that for $0\leqq\rho\leqq r\leqq 1$

$$|u|_{\theta_0,\rho}\leqq\{Q(r-\rho)^\alpha\}^{1/\theta_0-1/p}|u|_{p,r},$$

$$|u|_{-\theta_1,\rho}\leqq\{Q(r-\rho)^\alpha\}^{1/p-1/\theta_1}|u|_{-p,r},$$

and further suppose that

$$A=\sup_r\inf_k\left\{\frac{1}{m(K_r)}\int_{K_r}|\log u/k|\,dm\right\}<\infty.$$

Then

$$|u|_{\theta_0,0}\leqq\{m(K_1)/(m(K_0)\}^{1/\theta_0+1/\theta_1}\exp\{c_2Q^{-2}(1/p_0+A)\}|u|_{-\theta_1,0},$$

where c_2 depends only on α.

3

In this section we shall obtain inequalities for weak solution and super-solutions of the equation

$$\delta_i(a_{ij}\delta_j u)=0 \tag{3.1}$$

over a minimal surface S. As we are interested in local estimates we may assume that S is contained in the open $(n+1)$-ball of radius R, B_R. We suppose that $a_{ij}(x) = a_{ji}(x)$ are measurable functions satisfying

$$(\mu(x)/\mu_0)|\xi|^2 \leqq a_{ij}(x)\xi_i\xi_j \leqq \mu(x)|\xi|^2 \quad \text{for all } \xi \text{ in } \mathbf{R}^{n+1}, \tag{3.2}$$

where $\mu \in L^\infty(H^n, S)$, $\mu^{-1} \in L^t(H^n, S)$ and t satisfies $1/t \leqq 2/n < 1 + 1/t$.

A function u is called a weak solution (supersolution) of (3.1) if it lies in $H_0^{1,2}(\mu, S)$ and

$$\int_S a_{ij}\delta_i\phi\delta_j u dH^n = 0 \quad (\geqq 0)$$

for all bounded non-negative functions ϕ in $H_0^{1,2}(\mu, S)$. We let $S_r = B_r \cap S$.

Theorem 3.1. *Let S be a minimal surface and u a positive solution in B_R of* (3.1). *Then*

$$\operatorname*{ess\,sup}_{S_r} u \leqq C \operatorname*{ess\,inf}_{S_r} u$$

for $r \leqq \beta R$, where β is the constant of Theorem 2.2, *and C depends on n, μ, μ_0, r, t.*

For supersolutions we have

Theorem 3.2. *Let S be a minimal surface, u a positive supersolution in B_R of* (3.1). *Then*

$$|u|_{\theta_0, r} \leqq C \operatorname*{ess\,inf}_{S_r} u$$

for $r < \beta R$. Here C depends on n, μ, μ^{-1}, μ_0, K, t and θ_0, where $\theta_0 < 2^\#/2$.

These theorems are known for $S = \mathbf{R}^n$, see for example (3).

First we shall prove Theorem 3.1. Let u be a solution of (3.1) and put $v = u^{p/2}$, where $p \neq 0, 1$. Then by direct calculation, v satisfies

$$\delta_i(a_{ij}\delta_j v) + (2/p - 1)v^{-1}a_{ij}\delta_i v\delta_j v = 0. \tag{3.3}$$

Let ϕ be a non-negative test function with compact support in S. Multiplying (3.3) by $\phi^2 v$ and integrating over S gives

$$\int_S \phi^2 v\delta_i(a_{ij}\delta_j v)dH^n + (2/p - 1)\int_S \phi^2 a_{ij}\delta_i v\delta_j v dH^n = 0.$$

Since S is a minimal surface, $\int_S \delta_i f dH^n = 0$, for any function f with compact support on S, and it follows that

$$-\int_S \phi v a_{ij}\delta_i\phi\delta_j v dH^n + (1/p - 1)\int_S \phi^2 a_{ij}\delta_i v\delta_j v dH^n = 0,$$

hence

$$\frac{(p-1)}{|p|}\int_S \phi^2 a_{ij}\delta_i v\delta_j v dH^n + \operatorname{sgn} p \int_S \phi v a_{ij}\delta_i\phi\delta_j v dH^n = 0.$$

If $p>1$, $(p-1)/|p|>0$, and so

$$\int_S \phi^2 a_{ij}\delta_i v\delta_j v dH^n \leqq \frac{p}{(p-1)}\left|\int_S \phi v a_{ij}\delta_i\phi\delta_j v dH^n\right|.$$

If $p<1$, $(p-1)/|p|<0$, and this, together with (3.2) gives

$$0 \geqq \frac{p-1}{|p|}\int_S \phi^2 a_{ij}\delta_i v\delta_j v dH^n \geqq -\left|\int_S \phi v a_{ij}\delta_i\phi\delta_j v dH^n\right|.$$

Using the Cauchy-Schwarz inequality:

$$|a_{ij}\delta_i\phi\delta_j v| \leqq (a_{ij}\delta_i\phi\delta_j\phi)^{\frac{1}{2}}(a_{ij}\delta_i v\delta_j v)^{\frac{1}{2}}$$

together with Hölder's inequality, we get, in either case

$$\int_S \phi^2 a_{ij}\delta_i v\delta_j v dH^n \leqq \frac{p^2}{(p-1)^2}\int_S v^2 a_{ij}\delta_i\phi\delta_j\phi dH^n. \tag{3.4}$$

We now use the structure condition (3.2) to give

$$\frac{1}{\mu_0}\int_S \phi^2 |\delta v|^2 \mu dH^n \leqq \left(\frac{p}{p-1}\right)^2\int_S v^2|\delta\phi|^2 \mu dH^n.$$

Since $|\delta(\phi v)|^2 \leqq 2(|\phi\delta v|^2+|v\delta\phi|^2)$, this yields

$$\int_S |\delta(\phi v)|^2 \mu dH^n \leqq 2\left(1+\frac{\mu_0 p^2}{(p-1)^2}\right)\int_S v^2|\delta\phi|^2 \mu dH^n,$$

which, together with the Sobolev inequality (2.3), gives

$$\left(\int_S |\phi v|^{2^\#} \mu dH^n\right)^{2/2^\#} \leqq C\left(1+\frac{\mu_0 p^2}{(p-1)^2}\right)\int_S v^2|\delta\phi|^2 \mu dH^n.$$

Now let ϕ have compact support in S_r, $\phi \equiv 1$ on S_ρ, $\rho<r$, $|\delta\phi|<2/(r-\rho)$. Then substituting this ϕ into the equation we obtain

$$\left(\int_{S_\rho} |u|^{p2^\#/2} \mu dH^n\right)^{2/2^\#} \leqq \frac{C}{(r-\rho)^2}\left(1+\frac{\mu_0 p^2}{(p-1)^2}\right)\int_{S_r} u^p \mu dH^n. \tag{3.5}$$

We now use the well-known iteration scheme due to Moser (6). First suppose $p>0$. Let $\rho_\nu = (1-\tau)r+\tau r2^{-\nu}$, so that $\rho_0 = r$, $\rho_\infty = (1-\tau)r$, and let $\lambda = 2^\#/2$. Then (3.5) gives

$$\left(\int_{S_{\rho_{\nu+1}}} u^{p\lambda^{\nu+1}}\mu dH^n\right)^{1/\lambda^{\nu+1}}$$
$$\leqq \left\{\frac{C}{(\rho_\nu-\rho_{\nu+1})^2}\left(1+\frac{p^2\lambda^{2\nu}\mu_0}{(p\lambda^\nu-1)^2}\right)\right\}^{\lambda^{-\nu}}\left(\int_{S_{\rho_\nu}} u^{p\lambda^\nu}\mu dH^n\right)^{1/\lambda^\nu}. \tag{3.6}$$

Iterating the right-hand side ν times gives

$$\left(\int_{S_{\rho_{\nu+1}}} u^{p\lambda^{\nu+1}}\mu dH^n\right)^{1/\lambda^{\nu+1}} \leqq \frac{C^{\sum_0^\nu \lambda^{-j}}}{\prod_{j=0}^{\nu}(\rho_j-\rho_{j+1})^{2\lambda^{-j}}} \prod_{j=0}^{\nu}\left(1+\frac{p^2\lambda^{2j}\mu_0}{(p\lambda^j-1)^2}\right)^{\lambda^{-j}} \int_{S_r} u^p \mu dH^n.$$

As $\nu\to\infty$, $\rho_\nu\to(1-\tau)r$ and so

$$\sup_{S_{(1-\tau)r}} u^p \leqq \frac{C^{\sum_0^\infty \lambda^{-j}}}{\prod_0^\infty(\rho_j-\rho_{j+1})^{2\lambda^{-j}}} \prod_0^\infty\left(1+\frac{p^2\lambda^{2j}\mu_0}{(p\lambda^j-1)^2}\right)^{\lambda^{-j}} \int_{S_r} u^p\mu dH^n.$$

Since $\lambda>1$, $\Sigma\lambda^{-j}$ is convergent to $2^\#/(2^\#-2)$. Also $\rho_j-\rho_{j+1} = \tau r 2^{-(j+1)}$, so

$$\prod_0^\infty(\rho_j-\rho_{j+1})^{2\lambda^{-j}} = (\tau r)^{\Sigma 2\lambda^{-j}} \prod_0^\infty 2^{-(j+1)2\lambda^{-j}} = C(\tau r)^{2^\# 2/(2^\#-2)}.$$

To estimate the product $\prod_0^\infty\left(1+\frac{p^2\lambda^{2j}\mu_0}{(p\lambda^j-1)^2}\right)^{\lambda^{-j}}$, we suppose

(i) $0<p<1/\sqrt{\mu_0}$;

(ii) $|p\lambda^j-1| \geqq 1/n$ for each j. (3.7)

Then the product converges and is bounded by a constant independent of p. Thus, letting $\sigma = 22^\#/(2^\#-2)$, we have, for $0<\tau<1$,

$$\sup_{S_{(1-\tau)r}} u^p \leqq \frac{C}{\tau^\sigma r^\sigma}\int_{S_r} u^p\mu dH^n.$$

Now suppose $p<0$, $|p|<1/\sqrt{\mu_0}$. As we have already shown (3.5) holds. We can assume, with no loss in generality, that $u \geqq \varepsilon>0$ (see Moser (6)). As before, putting $\rho_\nu = (1-\tau)r+\tau r 2^{-\nu}$, (3.6) holds. Iterating as before, and taking account of the fact that $p<0$, we get

$$(\min_{S_{(1-\tau)r}} u^{|p|})^{-1} \leqq \frac{C}{(\tau r)^\sigma}\int_{S_r} u^{-|p|}\mu dH^n$$

that is,

$$\min_{S_{(1-\tau)r}} u^{|p|} \geqq \left\{\frac{C}{(\tau r)^\sigma}\int_{S_r} u^{-|p|}\mu dH^n\right\}^{-1}$$

Using the inequality (2.5), we have $H_\mu^n(S_r) \leqq Kr^n$, and hence if (3.7) is satisfied, we have

$$\sup_{S_{(1-\tau)r}} u^p \leqq \frac{C}{\tau^\sigma r^{\sigma-n}}\frac{1}{H_\mu^n(S_r)}\int_{S_r} u^p\mu dH^n = \frac{Cr^{n-\sigma}}{\tau^\sigma}|u|_{p,r}^p \tag{3.8}$$

and a similar inequality for $\min_{S_{(1-\tau)r}} u^p$.

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In order to eliminate the second condition in (3.7) we notice that this holds for some p in any interval $(a, \lambda a)$. For each such p, and for q satisfying $p<q<\lambda p$, using $|u|_{p,r} \leqq |u|_{q,r}$, (3.8) gives

$$\sup_{S_{(1-\tau)r}} u^q \leqq \left(\frac{Cr^{n-\sigma}}{\tau^\sigma}\right)^{q/p} |u|^q_{q,r} \leqq \frac{Cr^{(n-\sigma)\lambda}}{\tau^{\sigma\lambda}} |u|^q_{q,r}.$$

Thus we have shown that if $p \neq 0$, $0<p<1/\sqrt{\mu_0}$, and u is a positive solution of (3.1) then there is a constant C, depending on n, μ_0, μ, K, r, t such that

$$\sup_{S_{(1-\tau)r}} u^p \leqq \frac{C}{\tau^{\sigma\lambda}} |u|^p_{p,r},$$

$$\min_{S_{(1-\tau)r}} u^p \geqq \left(\frac{C}{\tau^{\sigma\lambda}}\right)^{-1} |u|^p_{-p,r}. \tag{3.9}$$

Now write $(1-\tau)r = \rho$; then $r = (1+s)\rho$ for some $s \in [0, 1]$. In the notation of Theorem 2.3, put

$$\theta_0 = \theta_1 = \infty,$$

$$K_\tau = S_{(1+\tau)\rho}, \text{ where } 0<\rho<\beta R/4$$

(here β is the constant of Theorem 2.2),

$$m = H^n_\mu,$$

$$\alpha = \lambda\sigma.$$

Then we see that (3.9) puts us in a position to apply Theorem 2.3, provided we can show

$$A = \sup_{\rho \leqq r \leqq 2\rho} \inf_{k \in R} \left\{\frac{1}{H^n_\mu(S_r)} \int_{S_r} |\log u/k| \,\mu dH^n\right\} < \infty.$$

Now since μ is a bounded function

$$\int_{S_r} |\log u/k| \,\mu dH^n \leqq \|\mu\|_\infty \int_{S_r} |\log u/k| \,dH^n$$

$$\leqq \|\mu\|_\infty \left(\int_{S_{2\rho}} |\log u/k|^{n/(n-1)} dH^n\right)^{(n-1)/n} \left(\int_{S_{2\rho}} dH^n\right)^{1/n}$$

Putting $w = \log u$, and using Theorem 2.2, we find

$$\min_{k \in \mathbf{R}} \int_{S_{2\rho}} |\log u/k| \,\mu dH^n \leqq C\rho \,\|\mu\|_\infty \int_{S_{2\rho/\beta}} |\delta w| \,dH^n.$$

But w satisfies the differential equation

$$\delta_i(a_{ij}\delta_j w) + a_{ij}\delta_i w \delta_j w = 0.$$

If we multiply this by a test function ϕ^2, and integrate by parts then

$$\int_S \phi^2 a_{ij}\delta_i w \delta_j w dH^n - 2\int_S \phi w a_{ij}\delta_i \phi \delta_j w dH^n = 0.$$

Using the Cauchy-Schwarz inequality we get

$$\int_S \phi^2 a_{ij}\delta_i w\delta_j w dH^n \leqq 4\int_S a_{ij}\delta_i\phi\delta_j\phi dH^n,$$

which together with the structure conditions gives

$$\int_S \phi^2 \mid \delta w\mid^2 \mu dH^n \leqq 4\mu_0 \int_S \mid \delta\phi\mid^2 \mu dH^n.$$

Taking $\phi \equiv 1$ in $S_{2\rho/\beta}$, $\phi = 0$ outside $S_{4\rho/\beta}$, $\mid \delta\phi \mid \leqq \beta/\rho$,

$$\int_{S_{2\rho/\beta}} \mid \delta w\mid^2 \mu dH^n \leqq 4C\mu_0 \parallel \mu \parallel_\infty \rho^{n-1}. \tag{3.10}$$

Now if E is any subset of $S_{2\rho/\beta}$

$$\begin{aligned}\int_E \mid \delta w\mid dH^n &= \int_E \mid \delta w\mid \mu^{\frac12}\mu^{-\frac12} dH^n \\ &\leqq \left(\int_E \mid \delta w\mid^2 \mu dH^n\right)^{\frac12}\left(\int_E \mu^{-1} dH^n\right)^{\frac12} \\ &\leqq C \parallel \mu^{-1}\parallel_t^{\frac12}\left(\int_E \mid \delta w\mid^2 \mu dH^n\right)^{\frac12} \\ &\leqq C\parallel \mu^{-1}\parallel_t^{\frac12} \rho^{(n-1)/2}\end{aligned}$$

by (3.10).

Thus

$$\min_{k\in\mathbf{R}} \int_{S_{2\rho}} \mid \log u/k\mid \mu dH^n \leqq C\parallel \mu^{-1}\parallel_t^{\frac12} \rho^{(n+1)/2} < \infty$$

and so A is bounded, and Theorem 3.1 is proved.

If u is a positive supersolution, then u satisfies

$$\delta_i(a_{ij}\delta_j u) \leqq 0.$$

Putting $v = u^{p/2}$, we obtain

$$\delta_i(a_{ij}\delta_j v) + (2/p-1)v^{-1}a_{ij}\delta_i v\delta_j v \begin{cases}\leqq 0 \text{ if } p>0\\ \geqq 0 \text{ if } p<0.\end{cases}$$

If $p<0$, we can proceed as in the proof of Theorem 3.1 to get

$$\min_{S_{(1-\tau)r}} u^{\mid p\mid} \geqq \left\{\frac{C}{(\tau r)^\sigma}\int_{S_r} u^{-\mid p\mid}\mu dH^n\right\}^{-1},$$

provided $\mid p\mid < 1/\sqrt{\mu_0}$. If $p>0$, we have

$$\delta_i(a_{ij}\delta_j v) + (2/p-1)v^{-1}\delta_i v\delta_j v \leqq 0.$$

Multiplying by $\phi^2 v$ and integrating by parts this gives

$$\frac{p-1}{p}\int_S \phi^2 a_{ij}\delta_i v\delta_j v dH^n + \int_S \phi v a_{ij}\delta_i\phi\delta_j v dH^n \geqq 0. \tag{3.11}$$

If $0<p<1$, $(p-1)/p<0$, hence

$$\frac{1-p}{p}\int_S \phi^2 a_{ij}\delta_i v\delta_j v dH^n \leqq \int_S \phi v \mid a_{ij}\delta_i\phi\delta_j\phi \mid dH^n,$$

giving

$$\int_S \phi^2 a_{ij}\delta_i v\delta_j v dH^n \leqq \frac{p^2}{(p-1)^2}\int_S v^2 a_{ij}\delta_i\phi\delta_j\phi dH^n.$$

If $p>1$, $(p-1)/p>0$, so (3.11) yields only trivial information. In the case $0<p<1$, using the structure relations we have

$$\int_S \phi^2 \mid \delta v \mid^2 \mu dH^n \leqq \frac{p^2\mu_0}{(p-1)^2}\int_S v^2 \mid \delta\phi \mid^2 \mu dH^n.$$

As before, using Sobolev's inequality, and choosing ϕ suitably we see that

$$\left(\int_{S_\rho} u^{p2^\#/2}\mu dH^n\right)^{2/2^\#} \leqq \frac{C}{(r-\rho)^2}\left(1+\frac{p^2\mu_0}{(p-1)^2}\right)\int_{S_r} u^p\mu dH^n,$$

and in general for $p(2^\#/2)^v<1$ we can obtain, again putting $\lambda = 2^\#/2$

$$\left(\int_{S_\rho} u^{p\lambda^{v+1}}\mu dH^n\right)^{\lambda^{-1}} \leqq \frac{C}{(r-\rho)^2}\left(1+\frac{\lambda^{2v}p^2\mu_0}{(p\lambda^v-1)^2}\right)\int_{S_r} u^{p\lambda^v}\mu dH^n,$$

but since $\lambda>1$ there is only a finite number, say j, of λ such that $p\lambda<1$. For these λ we find $p\lambda^{v+1}<\lambda$. Let $\theta_0 = \max\{p\lambda^{v+1}\colon p\lambda^{v+1}<\lambda\}$, that is $\theta_0 = p\lambda^{j+1}$. Then

$$\left(\int_{S_\rho} u^{\theta_0}\mu dH^n\right)^{p/\theta_0} \leqq \left\{\frac{C}{(r-\rho)^2}\left(1+\frac{\lambda^{2j}p^2\mu_0}{(p\lambda^j-1)^2}\right)\right\}^{\lambda^{-j}}\left(\int_{S_r} u^{p\lambda^j}dH^n\right)^{\lambda^{-j}}.$$

Defining ρ_v as before and iterating on the right-hand side we get

$$\left(\int_{S_{\rho_{j+1}}} u^{\theta_0}\mu dH^n\right)^{p/\theta_0} \leqq \frac{C^{\sum^j \lambda^{-v}}}{\prod_0^j (\tau r 2^{-(v+1)})^{2\lambda^{-v}}}\prod_0^j\left(1+\frac{\lambda^{2v}p^2\mu_0}{(p\lambda^v-1)^2}\right)^{\lambda^{-v}}\int_{S_r} u^p\mu dH^n.$$

Now

$$\prod_0^j (\tau r 2^{-(v+1)})^{2\lambda^{-v}} = C(\tau r)^{2\sum^j \lambda^{-v}}.$$

But

$$\sum_0^j \lambda^{-v} = \sum_0^\infty \lambda^{-v} - \sum_{j+1}^\infty \lambda^{-v} = \frac{2^\#}{2^\#-2} - \frac{1}{\lambda^{j+1}}\left(\frac{2^\#}{(2^\#-2)}\right)$$
$$= \frac{\sigma}{2}\left(1-\frac{p}{\theta_0}\right).$$

Thus, since $(1-\tau)r<\rho_{j+1}$

$$\left(\int_{S_{(1-\tau)r}} u^{\theta_0}\mu dH^n\right)^{p/\theta_0} \leqq \frac{C}{(\tau r)^{\sigma(1-p/\theta_0)}}\int_{S_r} u^p\mu dH^n.$$

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Since $C(\lambda(E))^{t'} \leqq H_\mu^n(E)$, hence

$$\frac{Cr^{nt'}}{H_\mu^n(S_{(1-\tau)r})}\left(\int_{S_{(1-\tau)r}} u^{\theta_0}\mu dH^n\right)^{p/\theta_0} \leqq \frac{Cr^{n-\sigma(1-p/\theta_0)}}{\tau^{\sigma(1-p/\theta_0)}}\left(\frac{1}{H_\mu^n(S_r)}\int_{S_r} u^p \mu dH\right).$$

Thus, putting $\theta_0 = p\lambda^{j+1} < \lambda$, $\theta_1 = \infty$, and K_τ, m as in the proof of Theorem 3.1, we can apply Theorem 2.3, provided $A < \infty$.

Writing $w = \log u$, w satisfies

$$\delta_i(a_{ij}\delta_j w) + a_{ij}\delta_i w \delta_j w \leqq 0.$$

Multiplying by ϕ^2, and integrating by parts gives

$$\int_S \phi^2 a_{ij}\delta_i w \delta_j w dH^n - 2\int_S \phi a_{ij}\delta_i\phi\delta_j w dH^n \leqq 0,$$

hence A is bounded exactly as before. Thus

$$|\, u\, |_{\theta_0, r} \leqq \left(\frac{H_\mu^n(S_{(1+\tau)r})}{H_\mu^n(S_\rho)}\right)^{1/\theta_0} \exp\left\{C\left(\frac{1}{p_0} + A\right)\right\} \min_{S_r} u,$$

where C depends on n, μ, μ^{-1}, K, t, μ_0, giving Theorem 3.2.

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