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ABSOLUTELY CONTINUOUS MEASURES ON LOCALLY COMPACT SEMIGROUPS[1]

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ABSTRACT. Let S be a locally compact Borel subsemigroup of a locally compact semigroup G. It is shown that the algebra of all "absolutely continuous' measures on S is isometrically order isomorphic to the algebra of all measures in $M(G)$ which are "concentrated" and "absolutely continuous" on S.

§1. **Introduction.** Let G be a locally compact group $M(G)$ its measure algebra and $M_a(G)$ the absolutely continuous measures (with respect to the left Haar measure λ). It is well known that $M_a(G)$ can be identified with the group algebra $L_1(G)$. Moreover, a measure μ in $M(G)$ is absolutely continuous iff the map $a \to \mu * \varepsilon_a$ (or equivalently $a \to \varepsilon_a * \mu$) of G into $M(G)$ is norm continuous where ε_a is the Dirac measure at a.

For locally compact semigroups, $L_1(G)$ is not available due to the absence of a Haar measure. However, the absolutely continuous measures make sense. The main purpose of this paper is to show that if S is a locally compact Borel subsemigroup of a locally compact semigroup G, then the absolutely continuous measures on S are precisely the set of all measures in $M(G)$ which are "concentrated" and "absolutely continuous" on S. As a consequence, if G is a group with left Haar measure λ and $0<\lambda(S)<\infty$, then S admits absolutely continuous probability measures.

§2. **Terminologies.** Let S be a locally compact semigroup with jointly continuous multiplication, $M(S)$ the Banach algebra of all bounded regular Borel measures on S with variation norm and convolution as multiplication (see for example [8] or [9]) and $M_0(S)=\{\mu \in M(S): \mu \geq 0, \|\mu\|=1\}$ be the probability measures is $M(S)$. A measure $\mu \in M(S)$ is left (right) absolutely continuous iff the map $a \to \varepsilon_a * \mu$ ($a \to \mu * \varepsilon_a$) of S into $M(S)$ is norm continuous where ε_a is the Dirac measure at a. μ is absolutely continuous if it is both left and right absolutely continuous. Let $M_a^l(S)$, $M_a^r(S)$ and $M_a(S)=M_a^l(S) \cap M_a^r(S)$ denote the left, right and two-sided absolutely continuous measures respectively. Clearly these are norm closed subalgebras of $M(S)$. In addition, $M_a^l(S)$ ($M_a^r(S)$) is a right (left) ideal of $M(S)$. (for more detail see [4] and references cited there) For groups, these three concepts

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coincide and agree with the definition given in Hewitt and Ross [5, §14.20 and §19.27].

§3. **Main results.** From now on, unless stated otherwise, G will be a locally compact semigroup and S a locally compact Borel subsemigroup of G (for example, any subsemigroup which is either open or closed). We first present a few technical lemmas. For notations in integration on locally compact spaces, we follow Hewitt and Ross [5].

LEMMA 3.1. *Let F be a Borel measurable function on S, then there is a unique Borel measurable function $\bar{f}$ on G such that $\bar{f}=f$ on S and $\bar{f}(G-S)=0$. Moreover*

(1) *If f is bounded on S, $\bar{f}$ is bounded on G and $\|\bar{f}\|_u=\|f\|_u$*

(2) *$\overline{l_a f}=l_a\bar{f}$ outside the set $a^{-1}S-S$ for any $a \in S$.*

Here $a^{-1}S=\{x \in G: ax \in S\}$, $l_a f$ is defined by $l_a f(x)=f(ax)$, $x \in S$ and similarly for $l_a\bar{f}$. Also $G-S=\{x \in G: x \notin S\}$.

Proof. Straight forward verification, we omit the detail.

Note that each Borel subset of S is a Borel subset of G.

LEMMA 3.2. *If μ is a measure in $M(G)$, then the restriction ν of μ to the Borel subsets of S is a measure in $M(S)$. Moreover,*

(1) *$\int f\,d\nu=\int \bar{f}\,d\mu$ for any bounded Borel measurable function f on S.*

(2) *If $a \in S$ and $\mu(a^{-1}S-S)=0$, then $\int f d(\varepsilon_a * \nu)=\int \bar{f}\, d(\varepsilon_a * \mu)$ for any bounded Borel measurable function f on S.*

Consequently if μ is a measure in $M(G)$ such that $\mu(a^{-1}S-S)=0\ \forall\ a \in S$ and the map $a \to \varepsilon_a * \mu$ of S into $M(G)$ is norm continuous, then $\nu \in M_a^l(S)$.

Proof. If $\mu \in M(G)$, it is clear that its restriction ν is a bounded measure on the Borel sets in S. Regularity of ν follows from that of μ (by [5, Theorem 11.32 and §11.34] while taking note that an open set in S need not be open in G).

Next, if f is the characteristic function of a Borel set B in S, then $\bar{f}$ is the characteristic function of B in G. Hence (1) holds for such f and the same is true for all bounded Borel measurable functions f on S.

Finally, let $a \in S$ and f bounded Borel measurable on S. By Lemma 3.1, $\overline{l_a f}= l_a\bar{f}$ outside $a^{-1}S-S$ with $\mu(a^{-1}S-S)=0$. Therefore $\int f\,d\varepsilon_a * \nu=\int l_a f\,d\nu=\int \overline{l_a f}\,d\mu= \int l_a\bar{f}\,d\mu=\int \bar{f}\,d\varepsilon_a * \mu$ which established (2). Now if $\mu \in M(G)$ satisfies $\mu(a^{-1}S-S)= 0\ \forall\ a \in S$, then $\forall\ a, b \in S$

$$\begin{aligned}\|\varepsilon_a * \nu-\varepsilon_b * \nu\| &= \sup\left\{\left|\int fd(\varepsilon_a * \nu-\varepsilon_b * \nu)\right|: f \in C_o(S),\ \|f\|_u \leq 1\right\}\\ &= \sup\left\{\left|\int \bar{f}d(\varepsilon_a * \mu-\varepsilon_b * \mu)\right|: f \in C_o(S),\ \|f\|_u \leq 1\right\}\\ &\leq \sup\{\|\bar{f}\|_u \cdot \|\varepsilon_a * \mu-\varepsilon_b * \mu\|: f \in C_o(S),\ \|f\|_u \leq 1\}\\ &\leq \|\varepsilon_a * \mu-\varepsilon_b * \mu\|.\end{aligned}$$

Here $C_o(S)$ is the space of all continuous functions on S which vanish at infinity. Therefore $\nu \in M_a^l(S)$ if the map $a \to \varepsilon_a * \mu$ of S into $M(G)$ is norm continuous.

REMARKS. (1) The measure ν is uniquely determined by the condition that $\int f\,d\nu = \int \bar{f}\,d\mu$ for any $f \in C_o(S)$ (Riesz Representation Theorem)

(2) We do not have to assume that $l_a f \in C_o(S)$ if $f \in C_o(S)$ and $a \in S$. In any case $l_a f$ is bounded Borel measurable on S.

The converse of Lemma 3.2 is true. In fact, we have a stronger result.

LEMMA 3.3. *If ν is a measure in $M(S)$, then there is a unique measure $\mu \in M(G)$ such that $|\mu|(S')=0$ and $\int \phi\,d\mu = \int (\phi/S)\,d\nu$ for any $\phi \in C_o(G)$. In fact $\mu(B)=\nu(B \cap S)$ for any Borel set B in G. Moreover, for any $a \in S$, $\phi \in C_o(G)$, $\int \phi\,d\varepsilon_a * \mu = \int \phi/S\,d\varepsilon_a * \mu$.* Consequently, if $\nu \in M_a^l(S)$, *then the map $a \to \varepsilon_a * \mu$ of S into $M(G)$ is norm continuous.*

Proof. Let $\nu \in M(S)$ be non-negative. The map $\phi \to \int (\phi/S)\,d\nu$ is clearly a non-negative bounded linear functional on $C_o(G)$. Hence there is a non-negative measure $\mu \in M(G)$ such that $\int \phi\,d\mu = \int (\phi/S)\,d\nu$ for any $\phi \in C_o(G)$. We shall prove that $\mu(B)=\nu(B \cap S)$ for any Borel set B in G. Observe that if B is open in G, then $B \cap S$ is open hence Borel in S. Now $\{B \subset G : B \cap S \text{ is a Borel set in } S\}$ is a σ-ring containing all open sets in G. It follows that $B \to \nu(B \cap S)$ is a Borel measure on G. Let U be open in G, then the characteristic function χ_U of U in G is lower semi-continuous (see [5, §11.8] for definition). Therefore $\mu(U)=\int \chi_U\,d\mu = \sup\{\int \phi\,d\mu : \phi \in C_o(G), 0 \le \phi \le \chi_U\} = \sup\{\int (\phi/S)\,d\nu : \phi \in C_o(G), 0 \le \phi \le \chi_U\} \le \nu(U \cap S)$. On the other hand, by regularity of ν, given $\varepsilon > 0$, there is some compact set $F \subset U \cap S$ such that $\nu(U \cap S) < \nu(F) + \varepsilon = \nu(F \cap S) + \varepsilon$ ([5, §11.32]). Since $F \subset U$, there is some $\phi \in C_o(G)$, $0 \le \phi \le 1$ such that $\phi(F)=1$ and $\phi(U')=0$ (Kelley [6, Theorem 18, p. 146]). Hence $\chi_F \le \phi \le X_U$ and $\nu(U \cap S) < \nu(F \cap S) + \varepsilon \le \int (\phi/S)\,d\nu + \varepsilon \le \mu(U) + \varepsilon$. Hence $\mu(U)=\nu(U \cap S)$. If $F \subset G$ is closed, write $F = G - U$, U open in G, then $\mu(F)=\mu(G)-\mu(U)=\nu(S)-\nu(S \cap U)=\nu(S-U)=\nu(F \cap S)$. In general, let B be any Borel set in G, then $\mu(B)=\sup\{\mu(F) : F \subset B, F \text{ compact}\} = \sup\{\nu(F \cap S) : F \subset B, F \text{ compact}\} \le \nu(B \cap S)$. By regularity of ν, given $\varepsilon > 0$, there is some $F \subset B \cap S$, F compact such that $\nu(B \cap S) < \nu(F) + \varepsilon = \nu(F \cap S) + \varepsilon = \mu(F) + \varepsilon$. Therefore $\mu(B)=\nu(B \cap S)$. In particular $\mu(S')=0$. In general, write $\nu = \nu_1 - \nu_2$ where ν_1, ν_2 are non-negative measures in $M(S)$ and let μ_1, μ_2 be the correrponding measures in $M(G)$. Then $\mu = \mu_1 - \mu_2 \in M(G)$ has the required properties. Finally, if $\phi \in C_o(G)$, $a \in S$, then $\int \phi\,d\varepsilon_a * \mu = \int l_a \phi\,d\mu = \int ((l_a \phi)/S)\,d\nu = \int l_a(\phi/S)\,d\nu = \int (\phi/S)\,d\varepsilon_a * \nu$ which also implies that $\|\varepsilon_a * \mu - \varepsilon_b * \mu\| \le \|\varepsilon_a * \nu - \varepsilon_b * \nu\|$ for any a, $b \in S$. This completes the proof.

REMARKS. (1) It can also be proved directly that the Borel measure $B \to \nu(B \cap S)$ on G is regular so that $\mu(B)=\nu(B \cap S)$ for any Borel set B in G once the same equality is established for open sets in G.

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(2) An analogue of the above construction is given in Hewitt and Ross [5, §11.45] with the *additional assumption* that S be closed in G (but without any semigroup structure for G or S). This assumption is required to make sure that $\phi/S \in C_o(G)$ if $\phi \in C_o(G)$ so $\int (\phi/S)\, d\nu$ is finite even if ν is not bounded. In any case ϕ/S is bounded Borel measurable on S and $\int (\phi/S)\, d\nu$ is finite for $\nu \in M(S)$. Thus the assumption that S be closed in G is not necessary in our case.

THEOREM 3.4. *$M_a^l(S)$ is isometrically order isomorphic to the subalgebra of all measures μ in $M(G)$ such that $|\mu|\,(S')=0$ and the map $a\to\varepsilon_a * \mu$ of S into $M(G)$ is norm continuous.*

Proof. Let M_l be the set of all measures in $M(G)$ such that $|\mu|\,(S')=0$ and the map $a\to\varepsilon_a * \mu$ of S into $M(G)$ is norm continuous. Clearly M_l is a linear subspace of $M(G)$. Let $\mu_1,\ \mu_2 \in M_l$ observe that if $y \in S$, then $S'y^{-1} \cap S=\phi$. Hence $|\mu * \nu|\,(S')\le|\mu| * |\nu|\,(S')=\int\int \chi_{S'}(xy)\, d\,|\mu|\,(x)\, d\,|\nu|\,(y)=\int_S |\mu|\,(S'y^{-1} \cap S)\, d\,|\nu|\,(y)=0$, while $\varepsilon_a * (\mu * \nu)=(\varepsilon_a * \mu) * \nu$. This shows that M_l is a subalgebra of $M(G)$.

If $\mu \in M_l$, let $\nu \in M_a^l(S)$ be the restriction of μ to the Borel sets of S as in Lemma 3.2. Define a map $T: M_l\to M_a^l(S)$ by $T\mu=\nu$. Clearly T is bounded linear. In fact $\|T_\mu\|=\sup\{|\int f\, d\nu| : f \in C_o(S),\ \|f\|_u\le 1\}=\sup\{|\int \bar{f}\, d\mu| : f \in C_o(S),\ \|f\|_u\le 1\}\le\|\mu\|$. Next, if $\nu \in M(S)$, by Lemma 3.3, there is a measure $\mu \in M_l$ such that $\mu(B)=\nu(B \cap S)$ for any Borel set B in G. Hence if B is a Borel subset of S, $T\mu(B)=\mu(B)=\nu(B \cap S)=\nu(B)$ or $T\mu=\nu$ and T is onto. Let $\phi \in C_o(G)$ and $f=\phi/S$, then $\phi=\bar{f}$ on S. Hence if $\mu_1,\ \mu_2 \in M_l$ and $T\mu_1=T\mu_2$, we have $\int \phi\, d\mu_1=\int \bar{f}\, d\mu_1=\int \bar{f}\, d\mu_2=\int \phi\, d\mu_2$ (since $|\mu_1|$ and $|\mu_2|$ vanish on S') which implies that $\mu_1=\mu_2$ and T is one-to-one. To show that T is a homomorphism, observe that if $f \in C_o(S)$, $y \in G$, the function $x\to\bar{f}(xy)$ is bounded Borel measurable on G and $\int_G \bar{f}(xy)\, d\mu_1(x)=\int_S \bar{f}(sy)\, d\nu_1(s)$ (because $\mu_1(B)=\nu_1(B \cap S)$). Consequently

$$
\begin{aligned}
\int f\, dT\mu_1 * T\mu_2 &= \int\int f(st)\, dT\mu_1(s)\, dT\mu_2(t) \\
&= \int\int \overline{l_s f}\, d\mu_2\, dT\mu_1(s) \\
&= \int\int l_s\bar{f}\, d\mu_2\, dT\mu_1(s) \\
&= \int\int \bar{f}(sy)\, dT\mu_1(s)\, d\mu_2(y) \\
&= \int\int \bar{f}(xy)\, d\mu_1(x)\, d\mu_2(y) \\
&= \int \bar{f}\, d\mu_1 * \mu_2 = \int f\, dT(\mu_1 * \mu_2)
\end{aligned}
$$

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Therefore $T(\mu_1 * \mu_2)=T\mu_1 * T\mu_2$ and T is an isomorphism which is evidently order preserving.

Finally since $\|\mu\|=\sup\{|\int \phi \, d\mu| : \phi \in C_o(G), \|\phi\|_u \leq 1\}=\sup\{|\int (\phi/S) \, d\nu| : \phi \in C_o(G), \|\phi\|_u \leq 1\} \leq \|\nu\|=\|T_\mu\|$, T is an isometry. This completes the proof.

REMARK. There are of course right handed and two sided versions of the results in this section. We omit the details.

§4. **Consequences and comments.** Since the group algebra $L_1(G)=M_a(G)$ of a locally compact group G plays an important role in abstract harmonic analysis, the quenstion naturally arises whether there exist absolutely continuous probability measures on a locally compact semigroup. In general, the answer is negative. Take $S=[0, 1]$ with usual topology and multiplication defined by $ab=a$, $\forall\, a, b \in S$. Then S is a compact semigroup. If $\nu \in M_a^l(S)$, $\nu \geq 0$, $\|\nu\|=1$, then $\varepsilon_a * \nu=\varepsilon_a$ for any $a \in S$. Left absolute continuity of ν implies that the unit ball in $C[0, 1]$ is equicontinuous hence norm compact by Ascoli's Theorem. This is certainly impossible. Thus not every (locally) compact semigroup admits absolutely continuous probability measure. However, most locally compact Borel subsemigroup of a locally compact group do. More precisely, if G is a locally compact group with left Haar measure λ and S is a locally compact Borel subsemigroup of G, then S admits absolutely continuous probability measures provided that there is some Borel set B in G, $B \subset S$ such that $0<\lambda(B)<\infty$. For if μ is the measure in $M_a(G)$ which corresponds to the normalized characteristic function $\phi=\lambda(B)^{-1}\chi_B \in L_1(G)$, then $T\mu=\nu$ is an absolutely continuous probability measure on S.

In [4, Theorem 6.3], G. Hart shows that if G is a locally compact abelian group and S a (locally compact) Borel subsemigroup of G, then $\{\mu \in L_1(G) : |\mu|\,(G-S)=0\} \subset M_a(S)$ (identifying μ and $T\mu$). Lemma 3.2 is an extension of this result (apart from commutativity, which is really not necessary there) since if $\mu \in M_a(G)=L_1(G)$, then the maps $a \rightarrow \varepsilon_a * \mu$ and $a \rightarrow \mu * \varepsilon_a$ are norm continuous on G hence on S ([5, Theorem 20.4]) and $G-S \supset a^{-1}S-S$ $(Sa^{-1}-S)$ for any $a \in S$.

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