GROUP ALGEBRAS WITH CENTRAL RADICALS

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1. Introduction. It is well known that when the characteristic p ($\neq 0$) of a field divides the order of a finite group, the group algebra possesses a non-trivial radical and that, if p does not divide the order of the group, the group algebra is semi-simple. A group algebra has a centre, a basis for which consists of the class-sums. The radical may be contained in this centre; we obtain necessary and sufficient conditions for this to happen.

Notations. If G is a group and if H is a subgroup of G, then we denote the order of H and the index of H in G by $|H|$ and $|G:H|$ respectively. G' and G'' are the first and second derived groups of G and e is the identity of G. We consider all group algebras over a fixed algebraically closed field K of characteristic p, $A(G)$ being the group algebra of G over K and $N(G)$ being the corresponding radical. $Z[A(G)]$ is the centre of $A(G)$. If I is a linear subspace of $A(G)$, its dimension is written as dim I.

Before stating our main result, let us recall the definition and some properties of Frobenius groups (Cf. [**2**, p. 587]). A group G containing a subgroup Q which is its own normalizer and which has trivial intersection with any distinct conjugate is said to be a Frobenius group. By a celebrated theorem of Frobenius the elements of G not in any conjugate of Q, together with e, form a normal subgroup M called the regular subgroup of G for the subgroup Q. The inner automorphism of G induced by an element of G not in M induces a regular automorphism of M. Thus we have $G = QM$, $Q \cap M = \{e\}$ and if, for $x \in G$, $Q \cap x^{-1}Qx \neq \{e\}$, then $x \in Q$. If $a \in Q$ ($a \neq e$), then every element of M may be written in the form $x^{-1}a^{-1}xa$ ($x \in M$). To show this it is sufficient to prove that the cardinal number of $\{x^{-1}a^{-1}xa \mid x \in M\}$ is equal to $|M|$, and this follows from the remark that, if

$$x^{-1}a^{-1}xa = y^{-1}a^{-1}ya \quad (x, y \in M),$$

then

$$a^{-1}(xy^{-1})a = xy^{-1},$$

and this implies that

$$xy^{-1} = e$$

from which we have that

$$x = y.$$

We remark here for future reference that, if Q is abelian, then $M = G'$.

We prove the following theorem.

THEOREM. *Let G be a group. Then $N(G) \subseteq Z[A(G)]$ if and only if G is of one of the following three types:*

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(i) *G has order prime to p.*
(ii) *G is abelian.*
(iii) *If P is a p-Sylow subgroup of G, then G′P is a Frobenius group with G′ as the regular subgroup of G′P under the inner automorphisms induced by elements of P.*

It is clearly sufficient to prove the following lemma.

LEMMA 1. *Let G be a non-abelian group whose order is divisible by p. Then* $N(G) \subseteq Z[A(G)]$ *if and only if condition (iii) of the theorem is satisfied.*

2. Lemmas on group algebras. Since the lemmas in this section are either known or easy to prove, only outlines of proofs are given.

LEMMA 2. *Let G be a group and H a normal subgroup of index n. Let I be an ideal of A(H) such that* $x^{-1}Ix = I \ (x \in G)$. *Let I generate an ideal J of A(G). Then, if*

$$G = Ha_1 \cup Ha_2 \cup \ldots \cup Ha_n \quad (a_1 = e)$$

is a coset decomposition, we have:
(i) $J = Ia_1 + Ia_2 + \ldots + Ia_n$.
(ii) $\dim J = n \dim I$.
(iii) $I = J \cap A(H)$.
(iv) *J is nilpotent if and only if I is nilpotent.*

Proof. To prove (i) it is sufficient to show that $Ia_1 + Ia_2 + \ldots + Ia_n$ is an ideal of $A(G)$. Now, if $x \in G$, we have

$$xa_\kappa = ha_\lambda \quad (h \in H, \lambda = \lambda(\kappa), 1 \leqq \lambda \leqq n),$$
$$a_\kappa x = h'a_\nu \quad (h' \in H, \nu = \nu(\kappa), 1 \leqq \nu \leqq n).$$

Thus

$$x(Ia_\kappa) = (xIx^{-1})xa_\kappa = Ixa_\kappa = Iha_\lambda \subseteq Ia_\lambda,$$
$$(Ia_\kappa)x = I(a_\kappa x) = Ih'a_\nu \subseteq Ia_\nu.$$

Hence $Ia_1 + Ia_2 + \ldots Ia_n$ is an ideal of $A(G)$. (ii) and (iii) follow from (i).

Since I is invariant under inner automorphisms of G, we may verify that if $\rho > 0$, then

$$J^\rho = I^\rho a_1 + I^\rho a_2 + \ldots + I^\rho a_n,$$

from which (iv) follows.

COROLLARY. (i) $N(H) \subseteq N(G)$.
(ii) $N(H) = N(G) \cap A(H)$.

Proof. (i) follows by observing that $N(H)$ is necessarily invariant under all automorphisms of $A(H)$. (ii) follows from (i).

LEMMA 3. *Let G be a group. Let* $x \in A(G)$ *be such that* $(e-g)x = 0$ *for all* $g \in G$. *Then*

$$x = \lambda \sum_{y \in G} y \quad (\lambda \in K).$$

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Proof. Let G have elements $g_1, g_2, \ldots, g_m$. Let $x = \sum_{\nu=1}^{m} \lambda_\nu g_\nu \quad (\lambda_\nu \in K,\ \nu = 1, 2, \ldots, m)$.

Then

$$\sum_{\nu=1}^{m} \lambda_\nu g_\nu = \sum_{\nu=1}^{m} \lambda_\nu g g_\nu \quad (g \in G).$$

Letting g run over the elements of G and comparing coefficients, we obtain the result.

LEMMA 4. *Let $G = PM$ be a group, where M is a normal subgroup and P is a p-Sylow subgroup of G such that $P \cap M = \{e\}$. Let I be the subspace of $A(G)$ spanned by elements of the form $\left(\sum_{x \in m} x\right)(e-s)$ $(s \in P)$. Then I is a nilpotent ideal of dimension $|P|-1$.*

Proof. A straightforward verification will show that I is an ideal of dimension $|P|-1$. We observe that in fact [**4**, p. 176]

$$I = \left(\sum_{x \in M} x\right) N(P)$$

and consequently I is nilpotent.

LEMMA 5. *Let J be the ideal of $A(G)$ generated by $\sum_{x \in G'} x$. Then $J \subseteq Z[A(G)]$.*

Proof. J is spanned, as a linear subspace of $A(G)$, by elements of the form $v\left(\sum_{x \in G'} x\right)$ $(v \in G)$. But, if $u \in G$, we have

$$u\left(v \sum_{x \in G'} x\right) = (uv)\left(\sum_{x \in G'} x\right) = \left(\sum_{x \in G'} x\right)(uv)$$
$$= \left(\sum_{x \in G'} x\right)(uvu^{-1}v^{-1})(vu) = \left(\sum_{x \in G'} x\right)(vu) = \left(v \sum_{x \in G'} x\right) u$$

and the lemma is proved.

We require also some results based on the modular representation theory of groups [**1**]. Let G be a group with a normal subgroup H of prime index q. Let R be an irreducible representation of H over K and let R^* be the representation of G induced by R. Either R^* is irreducible or R^* has, as irreducible constituents, q irreducible representations of G each of which when restricted to H is equivalent to R. If $q \neq p$, these q irreducible representations are inequivalent as representations of G, but if $q = p$, the p irreducible representations are equivalent representations of G. If we sum the squares of the degrees of the distinct irreducible representations of G and H we obtain finally:

LEMMA 6. *Let G be a group with a normal subgroup H of prime index q.*

(i) *If $q \neq p$, then* $\dim N(G) = q \dim N(H)$.

(ii) *If $q = p$, then* $\dim N(G) \geqq p \dim N(H) + (p-1)$.

We may combine these inequalities to obtain:

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LEMMA 7. *Let G be a group and H a normal subgroup such that G/H is soluble. Then*

$$\dim N(G) = |G:H| \dim N(H)$$

if and only if $|G:H|$ *is prime to p.*

LEMMA 8. *Let G be a group and H a normal subgroup such that G/H is soluble of order prime to p. Let J be the ideal of A(G) generated by N(H). Then J = N(G).*

Proof. By Lemma 2, J is nilpotent and $\dim J = |G:H| \dim N(H)$. By Lemma 7, $\dim N(G) = |G:H| \dim N(H)$. Hence $J = N(G)$.

3. Proof of the theorem.

Sufficiency. LEMMA 9. *Let G be a group such that G'P is a Frobenius group with G' as regular subgroup for the subgroup P, P being a p-Sylow subgroup of G. Then* $N(G) \subseteq Z[A(G)]$.

Proof. Since $G'P$ is a Frobenius group, [7, p. 128],

$$\dim N(G'P) = |P|-1.$$

Hence, by Lemma 4, $N(G'P)$ is spanned by elements of the form $\left(\sum_{x \in G'} x\right)(e-s)$ $(s \in P)$.

Now $G'P$ is normal in G and $G/G'P$ is abelian of order prime to p. Hence, by Lemma 8, $N(G)$ is generated by $N(G'P)$. This implies that $N(G)$ is contained in the ideal J of $A(G)$ generated by $\sum_{x \in G'} x$. By Lemma 5, $J \subseteq Z[A(G)]$. This completes the proof.

Necessity. In this section we assume that we have a group G for which $N(G) \subseteq Z[A(G)]$. We assume that p divides $|G|$ and that G is non-abelian. Let P be a p-Sylow subgroup of G. We show that $G'P$ is a Frobenius group with regular subgroup G'.

LEMMA 10. *Let* $x \in G'$. *Then* $(e-x)N(G) = \{0\}$.

Proof. Let $a, b \in G$. Let $w \in N(G)$. Then

$$(ab)w = a(bw) = a(wb) = (wb)a = w(ba) = (ba)w.$$

Hence

$$(e-b^{-1}a^{-1}ba)w = 0.$$

Let $x \in G'$. Then there exist commutators $c_1, c_2, \ldots, c_s$ such that $x = c_1 c_2 \ldots c_s$ and then

$$(e-x)w = (e-c_s)w + (e-c_1 c_2 \ldots c_{s-1})c_s w.$$

An obvious induction argument completes the proof of the lemma.

Let G' have index r in G and let

$$G = G'a_1 \cup G'a_2 \cup \ldots \cup G'a_r$$

be a coset decomposition. Let $w \in N(G)$. Then we may write

$$w = w_1 a_1 + w_2 a_2 + \ldots + w_r a_r \quad (w_\nu \in A(G'),\ \nu = 1, 2, \ldots, r).$$

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Let $x \in G'$. By Lemma 10 we have

$$\sum_{\nu=1}^{r} w_\nu a_\nu = \sum_{\nu=1}^{r} x w_\nu a_\nu.$$

Comparing linear combinations from distinct cosets we have

$$w_\nu a_\nu = x w_\nu a_\nu \quad (\nu = 1, 2, \ldots, r).$$

Hence

$$w_\nu = x w_\nu \quad (\nu = 1, 2, \ldots, r).$$

Thus, by Lemma 3,

$$w_\nu = \alpha_\nu \sum_{y \in G'} y \quad (\alpha_\nu \in K, \nu = 1, 2, \ldots, r).$$

Let I be the ideal of $A(G')$ spanned by $\sum_{x \in G'} y$. Then I is clearly invariant under automorphisms of G'. Let J be the ideal of $A(G)$ generated by I. Then we have shown that

$$N(G) \subseteq J.$$

We now consider two cases.

Case 1. Here we assume that $N(G) = J$. We wish to show that in fact this case does not arise.

It follows from Lemma 2 that I is nilpotent, and this implies that p divides $|G'|$. Further, by the corollary to Lemma 2, $N(G') = I$ and so $\dim N(G') = 1$. But if $|G'| = p^b m$, $(p, m) = 1$, then it is known [**7**, p. 128] that

$$\dim N(G') \geqq p^b - 1.$$

Hence we must have $p = 2$ and $b = 1$ and we must also have [**7**, p. 128] that G' is a Frobenius group with a subgroup of index 2 as regular subgroup. From our introductory remarks, G'' is the regular subgroup of G'.

If now $|G : G'|$ is divisible by 2 we would have, by Lemma 7,

$$\dim N(G) > |G : G'| \dim N(G') = |G : G'|.$$

Since this is false by Lemma 2 (ii), $|G : G'|$ is odd. We then obtain a contradiction on proving the following lemma.

LEMMA 11. *Let H be a group with a normal subgroup H_0 of order* 2. *If H/H_0 is abelian of odd order, then H is abelian.*

Proof. By Schur's Theorem [**8**, p. 162], there exists a subgroup M of index 2. M is normal in H and isomorphic to H/H_0. Clearly H is the direct product of the abelian groups H_0 and M.

The contradiction to the assumption that $N(G) = J$ arises on letting $H = G/G''$ and $H_0 = G'/G''$.

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Case 2. Here we assume that $N(G) \neq J$. Then, by Lemma 2, I is not nilpotent and so p does not divide $|G'|$. Thus p divides $|G:G'|$ and $P \cap G' = \{e\}$. We also have

$$\dim N(G) < \dim J = |G:G'|.$$

But, by Lemma 7, since $G'P$ is normal in G and $G/G'P$ is abelian of order prime to p,

$$\dim N(G'P) = \frac{1}{|G:G'P|}\dim N(G) < \frac{1}{|G:G'P|}|G:G'| = \frac{|G:G'P|\,|G'P:G'|}{|G:G'P|}$$
$$= |G'P:G'| = |P:P\cap G'| = |P|.$$

On the other hand [7, p. 128],

$$\dim N(G'P) \geqq |P| - 1.$$

Thus

$$\dim N(G'P) = |P| - 1.$$

This implies [7, p. 128] that $G'P$ is a Frobenius group with G' as regular subgroup and with the elements of P acting as a group of regular automorphisms on G'. This completes the proof of Lemma 1 and so proves the theorem.

4. Comments. Let G be a group such that $G'P$ is a Frobenius group as above. Since P is abelian, it follows that P is cyclic (Cf. Remarks in [7]). From the recent work of Thompson [5, 6] we know that G' is soluble and hence, by Higman [3, p. 322], G' is in fact nilpotent.

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