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WEIGHTED COMPOSITION OPERATORS ON ORLICZ–SOBOLEV SPACES

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Abstract

For an open subset Ω of the Euclidean space $\mathbf{R}^n$, a measurable non-singular transformation $T : \Omega \to \Omega$ and a real-valued measurable function u on $\mathbf{R}^n$, we study the weighted composition operator $uC_T : f \mapsto u \cdot (f \circ T)$ on the Orlicz–Sobolev space $W^{1,\varphi}(\Omega)$ consisting of those functions of the Orlicz space $L^\varphi(\Omega)$ whose distributional derivatives of the first order belong to $L^\varphi(\Omega)$. We also discuss a sufficient condition under which uC_T is compact.

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1. Introduction

Let $\varphi : [0,\infty) \to [0,\infty)$ be a *Young function* ([1, 6]). Thus φ is a continuous, convex, strictly increasing function satisfying $\varphi(0) = 0$, $\lim_{t\to\infty}\varphi(t) = \infty$, $\lim_{t\to 0+}\varphi(t)/t = 0$ and $\lim_{t\to+\infty}\varphi(t)/t = +\infty$. We say that φ satisfies the Δ_2-*condition* if there exist constants $k > 0$, $t_0 \geq 0$ such that $\varphi(2t) \leq k\varphi(t)$ for all $t \geq t_0$. Associated with φ, we have the *complementary Young function* $\psi : [0,\infty) \to [0,\infty)$ defined by $\psi(s) = \sup\{st - \varphi(t) : t \geq 0\}$.

Let $(\Omega, \mathscr{A}, \mu)$ be a measure space where Ω is an open subset of the Euclidean space $\mathbf{R}^n$ and $\mathscr{A}$ be the σ-algebra of Lebesgue measurable subsets of Ω and μ be the Lebesgue measure. The *Orlicz space* $L^\varphi(\Omega)$ is defined as the set of all (equivalence classes of) real-valued measurable functions f on Ω such that $\|f\|_\varphi < \infty$, where $\|\cdot\|_\varphi$ denotes the *Luxemberg norm* defined by

$$\|f\|_\varphi = \inf\left\{k > 0 : \int_\Omega \varphi\left(\frac{|f|}{k}\right) d\mu \leq 1\right\}.$$

$L^{\varphi}(\Omega)$ is a Banach space with respect to the above norm.

The *Orlicz–Sobolev space* $W^{1,\varphi}(\Omega)$ is defined as the set of all real-valued functions f in $L^{\varphi}(\Omega)$ whose weak partial derivatives $\partial f/\partial x_i$ (in the distributional sense) belong to $L^{\varphi}(\Omega)$, $i = 1, 2, \dots, n$. It is a Banach space with respect to the norm:

$$\|f\|_{1,\varphi} = \|f\|_{\varphi} + \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_{\varphi}$$

On the σ-finite measure space $(\Omega, \mathscr{A}, \mu)$, let $T : \Omega \to \Omega$ be a *measurable* (that is $T^{-1}(A) \in \mathscr{A}$ for every $A \in \mathscr{A}$) *non-singular transformation* (that is, $\mu(T^{-1}(A)) = 0$ whenever $\mu(A) = 0$). Let the function $f_T = d(\mu \circ T^{-1})/d\mu$ be the Radon–Nikodym derivative. Suppose u is a real-valued measurable function defined on $\mathbf{R}^n$. Then T induces a well-defined weighted composition linear transformation uC_T on $W^{1,\varphi}(\Omega)$ defined by

$$(uC_T f)(x) = u(x) f(T(x)), \quad x \in \Omega, \; f \in W^{1,\varphi}(\Omega).$$

If uC_T maps $W^{1,\varphi}(\Omega)$ into itself and is bounded then we call uC_T a *weighted composition operator* on $W^{1,\varphi}(\Omega)$ induced by T with weight u. If $u \equiv 1$ then C_T is called a *composition operator* induced by T.

Our present study of weighted composition operators on the Orlicz–Sobolev space $W^{1,\varphi}(\Omega)$ is motivated by the work of Kamowitz and Wortman ([4]). Other similar references include ([2, 3, 5] and [7]). In Section 2 we define the composition operator on $W^{1,\varphi}(\Omega)$ and Section 3 is devoted to the study of the compact weighted composition operator on $W^{1,\varphi}(\Omega)$.

2. Composition operator on $W^{1,\varphi}(\Omega)$

LEMMA 2.1. *Let $f_T, \partial T_k/\partial x_i \in L^{\infty}(\mu)$ with $\|\partial T_k/\partial x_i\|_{\infty} \le M$, $M > 0$ for $i, k = 1, 2, \dots, n$, where $T = (T_1, T_2, \dots, T_n)$ and $\partial T_k/\partial x_i$ denotes the first order partial derivative (in the classical sense). Then for each f in $W^{1,\varphi}(\Omega)$ we have $f \circ T \in L^{\varphi}(\Omega)$ and if the Young function φ satisfies the Δ_2-condition then the first order distributional derivatives of $(f \circ T)$, given by*

$$\frac{\partial}{\partial x_i}(f \circ T) = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial x_k} \circ T \right) \frac{\partial T_k}{\partial x_i} \tag{2.1}$$

for $i = 1, 2, \dots, n$, are in $L^{\varphi}(\Omega)$.

PROOF. For f in $W^{1,\varphi}(\Omega)$, we have

$$\begin{aligned}\|f\circ T\|_\varphi &= \inf\left\{k>0:\int_\Omega \varphi\left(\frac{1}{k}|f\circ T|\right)d\mu\le 1\right\}\\ &\le \inf\left\{k>0:\int_\Omega \varphi\left(\frac{|f|}{k}\right)f_T\,d\mu\le 1\right\}\\ &\le \inf\left\{k>0:\|f_T\|_\infty\int_\Omega \varphi\left(\frac{|f|}{k}\right)d\mu\le 1\right\}\\ &\le \inf\left\{k>0:\int_\Omega \varphi\left(\frac{|f|}{k}\right)d\mu\le 1\right\}=\|f\|_\varphi\end{aligned}$$

provided $\|f_T\|_\infty \le 1$. Therefore in this case $f\circ T\in L^\varphi(\Omega)$.

Now we assume that $1<\|f_T\|_\infty<\infty$. Then for $f\neq 0$ we have

$$\begin{aligned}\int_\Omega \varphi\left(\frac{|f\circ T|}{\|f_T\|_\infty\|f\|_\varphi}\right)d\mu &\le \int_\Omega \varphi\left(\frac{|f|}{\|f_T\|_\infty\|f\|_\varphi}\right)f_T\,d\mu\\ &\le \int_\Omega \frac{1}{\|f_T\|_\infty}\varphi\left(\frac{|f|}{\|f\|_\varphi}\right)\|f_T\|_\infty\,d\mu\le 1.\end{aligned}$$

Thus $\|f\circ T\|_\varphi\le\|f_T\|_\infty\|f\|_\varphi$ and hence $f\circ T\in L^\varphi(\Omega)$.

By the same arguments as were used to show that $\partial f/\partial x_k$ is in $L^\varphi(\Omega)$, it follows that $(\partial f/\partial x_k)\circ T\in L^\varphi(\Omega)$, for each $k=1,2,\dots,n$. Also $\partial T_k/\partial x_i\in L^\infty(\mu)$, therefore

$$\left(\frac{\partial f}{\partial x_k}\circ T\right)\frac{\partial T_k}{\partial x_i}\in L^\varphi(\Omega)\quad\text{for each } i,k=1,2,\dots,n.$$

Hence, by the triangle inequality, it follows that the function on the right hand side of (2.1) belongs to $L^\varphi(\Omega)$ for each $i=1,2,\dots,n$.

Now $f\in W^{1,\varphi}(\Omega)$ and φ satisfies the Δ_2-*condition*, so there exists (by [1, Theorem 8.28(d) Page 247]) a sequence $\langle f_m\rangle$ in $C^\infty(\Omega)\cap W^{1,\varphi}(\Omega)$ such that $f_m\to f$ in $W^{1,\varphi}(\Omega)$. Hence $f_m\to f$ and for $i=1,2,\dots,n$ we have

$$\frac{\partial f_m}{\partial x_i}\to\frac{\partial f}{\partial x_i}\quad\text{in } L^\varphi(\Omega)$$

Let $g\in\mathscr{D}(\Omega)$ (*the space of all infinitely differentiable real-valued functions with compact support in* Ω). Then, by the ordinary chain rule for the smooth function f_m, we have

$$\text{(2.2)}\quad \int_\Omega (f_m\circ T)\frac{\partial g}{\partial x_i}d\mu=-\int_\Omega\frac{\partial}{\partial x_i}(f_m\circ T)g\,d\mu=\int_\Omega\sum_{k=1}^n\left(\frac{\partial f_m}{\partial x_k}\circ T\right)\frac{\partial T_k}{\partial x_i}g\,d\mu$$

for each $i=1,2,\dots,n$.

Now $g \in \mathscr{D}(\Omega)$ implies that $g, \partial g/\partial x_i \in L^{\psi}(\Omega)$ for each i, where ψ is the complementary Young function to φ. Therefore, by using the generalized version of *Hölder's inequality* in Orlicz spaces, we have

$$\begin{aligned}\int_\Omega |f_m \circ T - f \circ T| \left|\frac{\partial g}{\partial x_i}\right| d\mu &\le 2\|(f_m - f)\circ T\|_\varphi \left\|\frac{\partial g}{\partial x_i}\right\|_\psi \\ &\le 2\|f_T\|_\infty \|f_m - f\|_\varphi \left\|\frac{\partial g}{\partial x_i}\right\|_\psi \to 0.\end{aligned}$$

Therefore

$$\int_\Omega (f_m \circ T)\frac{\partial g}{\partial x_i} d\mu \quad \to \quad \int_\Omega (f \circ T)\frac{\partial g}{\partial x_i} d\mu$$

as $m \to \infty$ for $i = 1, 2, \dots, n$.

By similar arguments, using $\partial f_m/\partial x_k \to \partial f/\partial x_k$ in $L^\varphi(\Omega)$ and $\partial T_k/\partial x_i$ in $L^\infty(\mu)$, we deduce that

$$\sum_{k=1}^n \left(\frac{\partial f_m}{\partial x_k}\circ T\right)\frac{\partial T_k}{\partial x_i} \quad \to \quad \sum_{k=1}^n \left(\frac{\partial f}{\partial x_k}\circ T\right)\frac{\partial T_k}{\partial x_i}$$

in $L^\varphi(\Omega)$, for each $i = 1, 2, \dots, n$, and so by *Hölder's inequality* again we obtain as $m \to \infty$,

$$\int_\Omega \sum_{k=1}^n \left(\frac{\partial f_m}{\partial x_k}\circ T\right)\frac{\partial T_k}{\partial x_i} g\,d\mu \quad \to \quad \int_\Omega \sum_{k=1}^n \left(\frac{\partial f}{\partial x_k}\circ T\right)\frac{\partial T_k}{\partial x_i} g\,d\mu.$$

Hence, by taking limits on both the sides of (2.2) as $m \to \infty$, we obtain

$$\int_\Omega (f \circ T)\frac{\partial g}{\partial x_i} d\mu = -\int_\Omega \sum_{k=1}^n \left(\frac{\partial f}{\partial x_k}\circ T\right)\frac{\partial T_k}{\partial x_i} g\,d\mu.$$

Therefore

$$-\int_\Omega \frac{\partial}{\partial x_i}(f \circ T) g\,d\mu = -\int_\Omega \sum_{k=1}^n \left(\frac{\partial f}{\partial x_k}\circ T\right)\frac{\partial T_k}{\partial x_i} g\,d\mu.$$

for all $i = 1, 2, \dots, n$.

As g was chosen arbitrarily, Equation (2.1) follows for $i = 1, 2, \dots, n$. □

THEOREM 2.2. *Let $\Omega \subset \mathbf{R}^n$ be an open set and $T : \Omega \to \Omega$ a measurable non-singular transformation with $f_T = d(\mu \circ T^{-1})/d\mu$, $(\partial T_k/\partial x_i)$ in $L^\infty(\mu)$ and $\|\partial T_k/\partial x_i\|_\infty \le M$, $M > 0$, for $i, k = 1, 2, \dots n$, where $T = (T_1, T_2, \dots, T_n)$ and $\partial T_k/\partial x_i$ denotes the first order partial derivative in the classical sense. Then for a Young function φ satisfying the Δ_2-condition, the mapping C_T defined by $C_T(f) = f \circ T$ is a composition operator on the Orlicz–Sobolev space $W^{1,\varphi}(\Omega)$.*

PROOF. By the Lemma 2.1, we have $f \circ T \in W^{1,\varphi}(\Omega)$ and

$$\begin{aligned}\|f \circ T\|_{1,\varphi} &= \|f \circ T\|_{\varphi} + \sum_{i=1}^{n} \left\| \frac{\partial}{\partial x_i}(f \circ T) \right\|_{\varphi} \\ &= \|f \circ T\|_{\varphi} + \sum_{i=1}^{n} \left\| \sum_{k=1}^{n} \left(\frac{\partial f}{\partial x_k} \circ T \right) \frac{\partial T_k}{\partial x_i} \right\|_{\varphi} \\ &\le \|f_T\|_{\infty} \|f\|_{\varphi} + \sum_{i=1}^{n} \sum_{k=1}^{n} \|f_T\|_{\infty} \left\| \frac{\partial f}{\partial x_k} \right\|_{\varphi} \left\| \frac{\partial T_k}{\partial x_i} \right\|_{\infty} \\ &\le \|f_T\|_{\infty} \|f\|_{\varphi} + \|f_T\|_{\infty} M n \sum_{k=1}^{n} \left\| \frac{\partial f}{\partial x_k} \right\|_{\varphi} \le \|f_T\|_{\infty} (1 + nM) \|f\|_{1,\varphi}.\end{aligned}$$

The result follows. □

3. Compact weighted composition operator on $W^{1,\varphi}(\Omega)$

Suppose u is a real-valued measurable function on $\mathbf{R}^n$ and $T : \Omega \to \Omega$ is a measurable non-singular transformation and $(\Omega, \mathscr{A}, \mu)$ is the σ-finite measure space, where Ω an open subset of $\mathbf{R}^n$. On the same lines as in Lemma 2.1, we have the following.

LEMMA 3.1. *If all the conditions stated in Theorem 2.2 are satisfied and, in addition, $u \in L^{\infty}(\mu)$ is such that the first order classical partial derivatives $\partial u / \partial x_i$ satisfy $\|\partial u / \partial x_i\|_{\infty} \le M_1$, $M_1 > 0$, for $i = 1, 2, \ldots, n$, then the mapping uC_T defined by*

$$(uC_T) f = u \cdot (f \circ T)$$

is a weighted composition operator on the Orlicz–Sobolev space $W^{1,\varphi}(\Omega)$.

PROOF. By the same arguments as in Lemma 2.1, we find

$$\frac{\partial}{\partial x_i}(u \cdot (f \circ T)) = \frac{\partial u}{\partial x_i}(f \circ T) + u \sum_{k=1}^{n} \left(\frac{\partial f}{\partial x_k} \circ T \right) \frac{\partial T_k}{\partial x_i} \quad \text{for } i = 1, 2, \ldots, n.$$

Moreover

$$\|u \cdot (f \circ T)\|_{\varphi} \le \|u\|_{\infty} \|f_T\|_{\infty} \|f\|_{\varphi},$$

$$\left\| \frac{\partial u}{\partial x_i}(f \circ T) \right\|_{\varphi} \le M_1 \|f_T\|_{\infty} \|f\|_{\varphi} \quad \text{and}$$

$$\left\| u \sum_{k=1}^{n} \left(\frac{\partial f}{\partial x_k} \circ T \right) \frac{\partial T_k}{\partial x_i} \right\|_{\varphi} \le \|u\|_{\infty} M \|f_T\|_{\infty} \sum_{k=1}^{n} \left\| \frac{\partial f}{\partial x_k} \right\|_{\varphi}.$$

Hence it follows that

$$
\begin{aligned}
&\|(uC_T)f\|_{1,\varphi} \\
&\quad = \|u\cdot(f\circ T)\|_{1,\varphi} = \|u\cdot(f\circ T)\|_{\varphi} + \sum_{i=1}^{n}\left\|\frac{\partial}{\partial x_i}\big(u\cdot(f\circ T)\big)\right\|_{\varphi} \\
&\quad \le \|u\|_\infty\|f_T\|_\infty\|f\|_\varphi + \sum_{i=1}^{n}\left\|\frac{\partial u}{\partial x_i}(f\circ T)\right\|_{\varphi} + \sum_{i=1}^{n}\left\|u\sum_{k=1}^{n}\left(\frac{\partial f}{\partial x_k}\circ T\right)\frac{\partial T_k}{\partial x_i}\right\|_{\varphi} \\
&\quad \le \|u\|_\infty\|f_T\|_\infty\|f\|_{1,\varphi} + M_1\|f_T\|_\varphi\|f\|_{1,\varphi} + nM\|u\|_\infty\|f_T\|_\infty\|f\|_{1,\varphi}.
\end{aligned}
$$

Thus $\|(uC_T)f\|_{1,\varphi} \le K\|f\|_{1,\varphi}$ for some $K > 0$. □

We now give some additional conditions on Ω, u, T and φ to obtain sufficient conditions for uC_T to be compact.

THEOREM 3.2. *With all the conditions stated in Lemma* 3.1, *let* $\Omega \subset \mathbf{R}^n$ *($n \ge 2$) be an open set having the cone property ([1, Definition 4.3 Page 66]) with* $\mu(\Omega) < \infty$. *Let* u, $\partial u/\partial x_i$ *be continuous,* $uT' = 0$ *and* $(\partial u/\partial x_i)T' = 0$, $i = 1, 2, \dots, n$, *where* T' *denotes the* n^{th} *order Jacobian matrix of the first order classical partial derivatives* $\partial T_k/\partial x_i$. *If the Young function* φ *also satisfies* $\int_1^\infty (\varphi^{-1}(t)/t^{1+1/n})dt < \infty$, *where* $\varphi^{-1}(t) = \inf\{s > 0 : \varphi(s) > t\}$ *is the right continuous inverse of* φ, *then the weighted composition operator* $uC_T : f \mapsto u\cdot(f\circ T)$ *is compact on the Orlicz–Sobolev space* $W^{1,\varphi}(\Omega)$.

PROOF. Let $\langle f_m\rangle$ be a sequence in $W^{1,\varphi}(\Omega)$ with $\|f_m\|_{1,\varphi} \le 1$. We prove that there exists an element $g \in W^{1,\varphi}(\Omega)$ and a subsequence $\langle f_{m_k}\rangle$ with $(uC_T)(f_{m_k}) \to g$ in $W^{1,\varphi}(\Omega)$ as $k \to \infty$. Equivalently, it suffices to show that $uC_T(f_{m_k}) \to g$ in $L^\varphi(\Omega)$ and $\partial(uC_T f_{m_k})/\partial x_i$ is bounded in $L^\varphi(\Omega)$. Let

$$E = \bigcup_{i=0}^{n}\left\{x \in \Omega : \frac{\partial u(x)}{\partial x_i} \ne 0\right\}, \qquad \frac{\partial u(x)}{\partial x_0} \equiv u.$$

Then, since u and $\partial u/\partial x_i$ are continuous, E becomes an open subset of Ω. Let $E = \bigcup_{i=1}^{\infty}\Omega_i$ where the Ω_is are closed cubes with disjoint interiors in Ω [8, Theorem 1.11, Page 8]. Thus for all x in Ω_i we have $T(x) = C_i$ for some $C_i \in \mathbf{R}^n$.

Now, from [1, Theorem 8.35, Page 252] it follows that $W^{1,\varphi}(\Omega)$ can be embedded in $C(\Omega) \cap L^\infty(\Omega)$. Therefore we can consider the sequence $\langle f_m\rangle$ in $W^{1,\varphi}(\Omega)$ as a bounded sequence of continuous functions on Ω. For $x \in \Omega_1$, we have

$$\langle (f_m\circ T)(x)\rangle = \langle f_m(T(x))\rangle = \langle f_m(C_1)\rangle$$

is a bounded sequence of real numbers and so there exists a subsequence $\langle f_{1,m}\rangle$ of $\langle f_m\rangle$ and $A_1 \in \mathbf{R}$ such that

$$f_{1,m}(C_1) \quad \to \quad A_1.$$

Similarly, for x in Ω_2, we can find a subsequence $\langle f_{2,m}\rangle$ of $\langle f_{1,m}\rangle$ and $A_2 \in \mathbf{R}$ such that

$$f_{2,m}(C_2) \quad \to \quad A_2.$$

Continuing in this way, by induction we obtain for each positive integer i, a real number A_i and a subsequence $\langle f_{i,m}\rangle$ of $\langle f_{i-1,m}\rangle$ with

$$f_{i,m}(C_i) \quad \to \quad A_i \quad \text{as } m \to \infty.$$

For each positive integer k, take $f_{m_k} = f_{k,k}$. Thus by the above construction we obtain that for each i

$$f_{m_k}(C_i) \quad \to \quad A_i \quad \text{as } k \to \infty.$$

Therefore for $\epsilon > 0$ we have, for sufficiently large m_k,

$$|f_{m_k}(C_i) - A_i| < \frac{\epsilon}{2^i}.$$

Let

$$g(x) = \begin{cases} A_i u(x) & \text{if } x \in \Omega_i \\ 0 & \text{if } x \notin E = \bigcup_{i=1}^{\infty} \Omega_i. \end{cases}$$

This means that if u and $\partial u/\partial x_i$ $(i = 1, 2, \ldots, n)$ do not vanish then we put

$$g(x) = A_i u(x),$$

while $g(x) = 0$ if $u(x) = 0 = \partial u/\partial x_i$ for $i = 1, 2, \ldots, n$. Thus we have $g \in L^{\varphi}(\Omega)$.

We now show that $u \cdot (f_{m_k} \circ T) \to g$ in $L^{\varphi}(\Omega)$ as $k \to \infty$. For sufficiently large m_k, consider

$$\begin{aligned}
\int_{\Omega} \varphi\left(\frac{|u \cdot (f_{m_k} \circ T) - g|}{\epsilon\|u\|_\infty[\varphi^{-1}(\frac{1}{\mu(E)})]^{-1}}\right) d\mu &= \int_E \varphi\left(\frac{|u(x) f_{m_k}(T(x)) - g(x)|}{\epsilon\|u\|_\infty[\varphi^{-1}(\frac{1}{\mu(E)})]^{-1}}\right) d\mu \\
&= \sum_{i=1}^{\infty} \int_{\Omega_i} \varphi\left(\frac{|u(x)|\,|f_{m_k}(C_i) - A_i|}{\epsilon\|u\|_\infty[\varphi^{-1}(\frac{1}{\mu(E)})]^{-1}}\right) d\mu \\
&\le \sum_{i=1}^{\infty} \int_{\Omega_i} \varphi\left(\frac{1}{2^i}\left(\varphi^{-1}\left(\frac{1}{\mu(E)}\right)\right)\right) d\mu \\
&\le \sum_{i=1}^{\infty} \frac{1}{2^i} \int_{\Omega_i} \varphi\left(\varphi^{-1}\left(\frac{1}{\mu(E)}\right)\right) d\mu \\
&\le 1.
\end{aligned}$$

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Therefore, by the definition of infimum, we have

$$\|u \cdot (f_{m_k} \circ T) - g\|_\varphi \leq \epsilon \|u\|_\infty \left[\varphi^{-1}\left(\frac{1}{\mu(E)}\right)\right]^{-1}$$

Hence $(uC_T)(f_{m_k}) \to g$ in $L^\varphi(\Omega)$ as $k \to \infty$.

Now $\|f_{m_k}\|_{1,\varphi} \leq 1$, therefore, by using Lemma 3.1, we have

$$\left\|\frac{\partial}{\partial x_i}\left(u \cdot (f_{m_k} \circ T)\right)\right\|_\varphi \leq \left\|\frac{\partial}{\partial x_i}\left(u \cdot (f_{m_k} \circ T)\right)\right\|_{1,\varphi} \leq K\|f_{m_k}\|_{1,\varphi} \leq K.$$

The result follows. □

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