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A non-homology boundary link with zero Alexander polynomial

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This paper presents a necessary condition for a ribbon link to be an homology boundary link and gives a consequent simple counter-example to the conjecture of Smythe that the vanishing of the first Alexander polynomial characterizes homology boundary links among all 2-component links.

It has been conjectured that if the first Alexander polynomial of a 2-component link vanishes, then the link is an homology boundary link; that is, its group maps onto the free group of rank 2 ([13]).

A simple counter-example is given below, as a consequence of the following theorems.

THEOREM 1. *If* G *is the group of a* μ*-component ribbon link, then* $G \to G/G_\omega$ *factors through a group of defect* μ .

THEOREM 2. *Let* H *have defect* μ *and* $H/H' = Z^\mu$. *If* H *maps onto* $F(\mu)/F(\mu)''$ *then* $E_\mu(H)$ *is principal.*

The ribbon group

DEFINITION. An oriented μ-component link $L : \coprod_{i=1}^{\mu} S_i^1 \to S^3$ is a ribbon link if L extends to an immersion $R : \coprod_{i=1}^{\mu} D_i^2 \looparrowright S^3$ with no

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triple points, and such that the components of the critical set either meet the boundary at both end points ("throughcut") or at neither ("slit"). In other words the singularities are as in Figure 1.

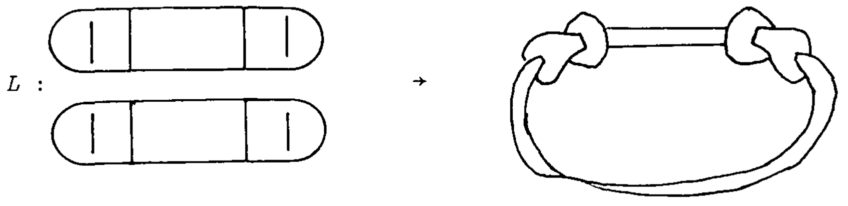

Figure 1.

It is easy to see that R may be deformed so that each component of the complement of the throughcuts is bounded by at most two throughcuts. (See Figure 2.) In what follows, R will always be so chosen.

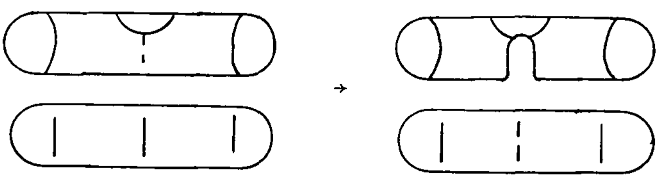

Figure 2.

It is also easy to see that every ribbon link is null concordant (a slice link in the strong sense). The converse remains an open conjecture, even for knots ([7]).

Each throughcut determines a conjugacy class $g(T) \subset G = \pi_1(S^3-L)$ represented by the image of the (oriented) boundary of a small disc neighbourhood of the corresponding slit.

DEFINITION. $H(R) = G/\langle\langle \cup\{g(T) \mid T \text{ a throughcut of } R\}\rangle\rangle$, where $\langle\langle S\rangle\rangle$ means the normal subgroup generated by $S \subset G$.

LEMMA 1. *The longitudes of* L *are in*

$$\langle\langle \cup\{g(T) \mid T \text{ a throughcut of } R\}\rangle\rangle .$$

Proof. Each longitude is represented (up to conjugacy) by a curve on and near the boundary of the corresponding disc, which is clearly homotopic

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to a product of (conjugates of) loops about the slits in that disc. //

LEMMA 2. *For all throughcuts* T , $g(T) \subset G_\omega$.

Proof. Certainly, for all T , $g(T) \subset G_2$.

Suppose all $g(T) \subset G_n$. Then their images $\overline{g(T)}$ are central in G/G_{n+1} . One shows then that $\overline{g(T)} = \overline{g(T')}$ where T' is a through-cut adjacent to T , and hence, moving along the ribbon, that all $\overline{g(T)} = \{1\}$; that is, that all $g(T) \subset G_{n+1}$. By induction, all $g(T) \subset G$. //

THEOREM 1. $G \to G/G_\omega$ *factors* $G \to G/\langle\langle \text{longitudes} \rangle\rangle \to H(R) \to G/G_\omega$ *and the Wirtinger presentation associated to a generic projection of* L *gives rise to a presentation of defect* μ *for* H *of the form*

$$\left\{X_{ij},\ 0 \le j \le J(i),\ 1 \le i \le \mu \ \middle|\ W_{ij}W_{ij-1}W_{ij}^{-1} = X_{ij},\right.$$
$$\left. 1 \le j \le T(i),\ 1 \le i \le \mu\right\},$$

where there is one generator X_{ij} *for each component of the complement of the throughcuts and one word* W_{ij} *of length one for each throughcut.*

Proof. The factorization of $G \to G/G_\omega$ follows from the lemmas. One may assume that in a generic projection of the ribbon there are no triple points. The Wirtinger generators of the link group corresponding to subarcs of the projection of the link which "lie under" a sequent of the ribbon may be deleted, and the two associated relations replaced by one stating that either adjacent generator is conjugate to the other loop around the overlying segment. One then proves that any loop about a segment of the ribbon is killed in $H(R)$, and hence that the remaining generators corresponding to subarcs of the boundary of a given component of the complement of the throughcuts coalesce. The relations deduced from the Wirtinger presentation after such deletions and identifications are complete because image $g(T) = \{1\}$ if and only if the pair of generators meeting the projection of T are identified. //

REMARKS. (1) Conversely any such presentation can be realized for some ribbon link.

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(2) The two lemmas and hence the first part of the theorem hold for ribbon links in an arbitrary 3-monifold.

(3) In general $G/\langle\langle \text{longitudes} \rangle\rangle \neq H \neq G/G_\omega$, even for knots (for example the square knot). If one ribbon R_1 is obtained from another R_2 by knotting the ribbon or by inserting full twists, then $H(R_1) = H(R_2)$. Can $H(R)$ be characterised link- or group- theoretically? ($H(R)$ is the group of a link of μ S^2's in S^4, of which L is a slice, and where the longitudes clearly die.)

COROLLARY. $E_{\mu-1}(H(R)) = 0$; *hence* $E_{\mu-1}(G) = 0$.

This last is true of any null-concordant link. In fact, $\min\{k \mid E_k(G) \neq 0\}$ is invariant under concordance [8], as is the Murasugi nullity of L [9, 10]. Are they always equal?

In particular, 2-component ribbon links have $\Delta_1 = 0$.

Alexander ideals and defect

Let H be a group with a presentation of defect μ, and with abelianization $H/H' = Z^\mu$. Then $E_i(H) = 0$ for $i < \mu$ and $E_\mu(H) \equiv (1) \bmod \ker \epsilon$ (where ϵ is the augmentation homomorphism $\Lambda = Z[Z^\mu] \to Z$).

If $\mu = 1$ then H maps onto Z and $E_1(H)$ is principal, and $H/H'' = Z$ if and only if $E_1(H) = (1)$ [11].

THEOREM 2. *If* H *maps onto* $F(\mu)/F(\mu)''$, *then* $E_\mu(H)$ *is principal.*

Proof. Let $f : H \to F(\mu)/F(\mu)''$. Then f induces a surjection of H'/H'' onto $F(\mu)'/F(\mu)''$; hence the induced map of Alexander modules $\tilde{f} : A(H) \to \Lambda^\mu = A(F(\mu)/F(\mu)'')$ is onto (by the 5-lemma and functoriality of the sequence $0 \to H'/H'' \to A(H) \to \Lambda \xrightarrow{\epsilon} Z \to 0$, [5]). $A(H)$ has a presentation

$$0 \to \Lambda^N \xrightarrow{M} \Lambda^{N+\mu} \xrightarrow{P} A(H) \to 0 .$$

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(M is injective since $E_\mu(M) \neq 0$.) Let $q = \tilde{f} \circ p : \Lambda^{N+\mu} \to \Lambda^\mu$. Then $\Lambda^\mu \oplus \ker q \cong \Lambda^{N+\mu}$. The sequence

$$0 \to \Lambda^\mu \oplus \Lambda^N \xrightarrow{\mathrm{id} \oplus M} \Lambda^\mu \oplus \Lambda^{N+\mu} \xrightarrow{p \circ pr_2} A(H) \to 0$$

is also exact, and $\mathrm{Im}(\mathrm{id} \oplus M) \subset \ker(q \circ pr_2) = \Lambda^\mu \oplus \ker q$, which is a free direct summand of rank $N + \mu$, with free complement. Therefore $E_\mu(M) = E_\mu(\mathrm{id} \oplus M)$ is principal. //

REMARK. $E_\mu(H)$ principal does *not* imply $E_\mu(G)$ principal.

There is a partial converse.

THEOREM 3. *If* $\mu = 2$, *then* $E_2(H)$ *principal implies* H'/H'' *maps onto* $\Lambda \approx F(2)'/F(2)''$.

Proof. One may assume H has a ("pre-abelian") presentation $P = \{x, y, a_k, 1 \le k \le N \mid r_l, 1 \le l \le N\}^\phi$ where $\phi(a_k) \in H'$ and the images of x, y generate $H/H' = Z^2$. As in [6], the Alexander matrix of P has the form $M = \|(y-1)M', (1-x)M', M_2\|$ where M' is an $N \times 1$ column matrix and M_2 is $N \times N$, and $\tilde{M} = \|M', M_2\|$ is a presentation matrix for H'/H'' over Λ. Now

$$(x-1, y-1)E_1(\tilde{M}) \subset E_2(M) = (\det M_2, (x-1, y-1)E_1(\tilde{M}) \subset E_1(M)),$$

so $0 \to \Lambda^N \xrightarrow{\tilde{M}} \Lambda^{N+1} \to H'/H'' \to 0$ is exact (as above), and $E_2(H) = E_2(M)$ principal implies $E_1(\tilde{M}) = E_2(M) = (\Delta_2(H))$ is principal. (Conversely $E_1(\tilde{M})$ principal implies $E_2(H)$ is principal.) Let Δ_i be the ith $N \times N$ minor of $\tilde{M}$. Then $\Delta_2(H)$ divides each Δ_i. Let $q : \Lambda^{N+1} \to \Lambda$ map $(a_1, \ldots, a_{N+1})$ to $\sum_{i=1}^{N+1} (-1)^i(\Delta_i/\Delta_2(H))a_i$. Then q is onto (since $E_1(\tilde{M})$ is principal) and $q \circ \tilde{M} = 0$ so q induces $\bar{q} : H'/H'' \to \Lambda$. //

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If $H/H'' \approx F(\mu)/F(\mu)''$ then, clearly, $E_\mu(H) = (1)$. There is again a partial converse.

THEOREM 4. *If* $E_\mu(H) = (1)$, *then if* $\mu = 2$, $H'/H'' \approx \Lambda$, *and if* $\mu > 2$, $A(H) \approx \Lambda^\mu$.

Proof. First assume $\mu > 2$. Let P be a prime ideal of Λ. $E_\mu(M)_P = (1)$. Therefore at least one of the $N \times N$ sub determinants of M_P is a unit, so M_P splits. Hence M splits.
$(\mathrm{coker}(M^*)_P = \mathrm{coker}(M_P)^* = 0$ for all P.)

Therefore $A(H)$ is stably free of rank μ.

$$\Lambda = Z[Z^\mu] \approx Z[T, T^{-1}][x_1, \ldots, x_{\mu-1}], \text{ and } Z[T, T^{-1}]$$

is noetherian of dimension 2. Therefore, by a result of Souslin [2], $A(H)$ is free provided $\mu \geq 1 + \max\left(2, \frac{2+\mu-1}{2}\right)$.

If $\mu = 2$ we argue instead that $\tilde{M}$ splits; hence H'/H'' is stably free of rank 1, hence free. //

REMARK. $(\mu = 2)$. The extensions of Z^2 by Λ are classified by $H^2(Z^2, \Lambda) \approx Z$. In the above situation is H/H'' always isomorphic to $F(2)/F(2)''$?

It is probable that all projective modules over $Z[Z^\mu]$ are free (*cf.* [12]).

A counter-example

The ribbon link in Figure 1 (which has 2 *unknotted* components and 4 ribbon singularities) is a counter-example to Smythe's conjecture.

$$H(R) \approx \{x_1, x_2, x_3, y_1, y_2, y_3 \mid y_1^{-1}x_1y_1 = x_2,\ y_3x_2y_3^{-1} = x_3,\ x_1^{-1}y_1x_1 = y_2,\ x_3^{-1}y_1x_3 = y_3\}$$

$$\approx \{x, y, a \mid xy^{-1}xyx^{-1} = a^{-1}x^{-1}y^{-1}xayxa\}.$$

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Therefore $M = \|(y-1)(x^{-1}y^{-1}-xy),\ (1-x)(x^{-1}y^{-1}-xy),\ 1-y^{-1}-x\|$ and $\tilde{M} = \|x^{-1}y^{-1}-xy,\ 1-y^{-1}-x\|$. There is an exact sequence $0 \to \Lambda \xrightarrow{\tilde{M}} \Lambda^2 \xrightarrow{P} H'/H'' \to 0$. If there was a map f of G onto $Z * Z$ then there would be a map $\tilde{f}$ of H'/H'' onto Λ . Then $\Lambda^2 = \Lambda \oplus \ker(\tilde{f} \circ p)$ whence

$$\ker(\tilde{f} \circ p) = \ker(\tilde{f} \circ p) \oplus (\ker(\tilde{f} \circ p) \wedge \ker(\tilde{f} \circ p)) = \Lambda^2 \wedge \Lambda^2 = \Lambda$$

is free. Therefore $\tilde{M} = i \circ \delta$ where $\delta : \Lambda \to \Lambda$ and i extends to an automorphism of Λ^2 , so $E_1(\tilde{M}) = \delta . E_1(i) = (\delta)$ would be principal. But $E_1(\tilde{M}) = (x^2-1,\ 1-y+xy)$ is clearly not principal. //

REMARKS. (1) (Hence) this link is not a slice of trivial S^2's in S^4 ; that is, is not doubly null cobordant (*cf.* [7, 14]).

(2) H/H_ω is parafree but not free (*cf.* [3]).

(3) Any ribbon counterexample must have at least 4 ribbon singularities.

(4) One could also show this link is not an homology boundary link by computing $E_2(G)$ and using a recent result of Crowell and Brown (unpublished).

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