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Cancellation Theorems for Projective Modules over a Two-Dimensional Ring and its Polynomial Extensions

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Abstract. We show that over polynomial extensions of normal affine domains of dimension two over perfect fields (char. $\neq 2$) of cohomological dimension $\leqslant 1$, all finitely generated projective modules are cancellative, thus answering a question of Weibel affirmatively in the case of polynomial extensions.

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1. Introduction

Let A be a commutative noetherian ring. A finitely generated projective A-module P is said to be *cancellative* if $A^n \oplus P \cong A^n \oplus Q$ implies $P \cong Q$. It is easy to see that projective A-modules of rank 1 are always cancellative.A classical result of Bass ([B 1]) asserts that every finitely generated projective A-module of rank $>$ dim A is cancellative. This is the best possible result in general, as is evidenced by the example of the tangent bundle of the real 2-sphere. However, if A is an affine algebra over an algebraically closed field, then a result of Suslin ([Su 3]) (proved earlier, in the special case of a two dimensional affine algebra, by Murthy and Swan ([M-S], Theorem 4)) says that projective A-modules of rank $=$ dim A are also cancellative. Subsequently, Mohan Kumar, Murthy and Roy proved a similar result in the case of finitely generated **Z**-algebras ([M-M-R], Corollary 2.5).

Now suppose that A is an affine algebra of dimension d over a perfect field k. Assume that cohomological dimension of $k \leqslant 1$ and $d! \in k^*$. In this set up, a result of Suslin implies that the free module A^d is cancellative (see ([Su 4], Theorem 2.4)). In view of this and above cancellation theorems,it is natural to ask:

Let A be an affine algebra of dimension d over a perfect field k. Assume that cohomological dimension of $k \leqslant 1$ and $d! \in k^*$. Are all projective A-modules of rank *d cancellative?*

Note that, in view of the above-mentioned result of Bass, an affirmative answer to this question for a two-dimensional affine algebra A would imply that all projective A-modules are cancellative.

In this paper, we settle this question affirmatively if A is the coordinate ring of a normal affine surface. More precisely, we prove:

THEOREM I (3.9). *Let A be an affine domain of dimension 2 over a field k such that* Spec(A) *is nonsingular in codimension* 1 *(for example k is perfect and A is normal). If $2 \in k^*$ and cohomological dimension of $k \leqslant 1$, then every projective A-module of* rank 2 *(and hence every projective A-module) is cancellative.*

We also prove the following extension of Theorem I answering a question of Weibel (see the introduction of [W]) affirmatively in the polynomial case.

THEOREM II (5.5). *Let k be a perfect field of characteristic $\neq 2$ and of cohomological dimension $\leqslant 1$. Let R be the coordinate ring of a normal affine surface over k. Then every projective $R[X_1, \cdots, X_r]$-module is cancellative.*

The above theorem is proved using the following 'Symplectic' cancellation theorem:

THEOREM III (4.8). *Let R be a Noetherian ring of dimension d. Let P, Q be two symplectic modules over $A = R[X_1, \ldots, X_r]$ such that $P \perp H(A) \simeq Q \perp H(A)$. If rank $(P) \geqslant d$, then $P \simeq Q$ (as symplectic modules).*

Theorem III can be regarded as a symplectic analogue of the cancellation theorem of Rao ([Ra 1], Theorem 2.5) which says that for a Noetherian ring R of dimension d, every projective $R[X_1, \ldots, X_r]$-module of rank $> d$ is cancellative.

We conclude the introduction by thanking the referee for his/her pertinent remark which enabled us to remove the earlier assumption $2 \in R^*$ in Theorem III.

2. Some Preliminary Results

In this section we state some results for later use. All rings considered in this paper are commutative and Noetherian. All modules considered, are assumed to be finitely generated. For a module M over a ring, $\mu(M)$ will denote minimal number of generators of M.

LEMMA 2.1. *Let B be a reduced ring and let M be a module over B. If for every prime ideal $\mathcal{P}$ of B $\mu(M_{\mathcal{P}}) = r$ then M is a projective B-module of rank r.*

Proof. Let $\mathcal{M}$ be a maximal ideal of B. To prove the lemma it is enough to show that $M_{\mathcal{M}} \cong {B_{\mathcal{M}}}^r$. So without loss of generality, we assume that B is local and $\mu(M) = r$. Let α: $B^r \twoheadrightarrow M$ be a surjection and let $N = \ker(\alpha)$. Let S denote the set of nonzero divisors of B. Since B is reduced, B_S is a finite product of fields. Moreover, since $\mu(M_{\mathcal{P}}) = r$ for every minimal prime ideal $\mathcal{P}$ of B, we see that M_S is a free B_S-module of rank r. Therefore α_S is an isomorphism showing that

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$N_S = 0$. Hence, as N is a submodule of B^r and S consists of nonzero divisors, $N = 0$. Thus $M \cong B^r$. □

LEMMA 2.2. *Let A be a ring and let $J \subset A$ be an ideal of height r. Let $\overline{P}, \overline{Q}$ be projective A/J-modules of rank r and let $\overline{\alpha}: \overline{P} \twoheadrightarrow J/J^2$ and $\overline{\beta}: \overline{Q} \twoheadrightarrow J/J^2$ be surjections. Let $\overline{\Psi}: \overline{P} \to \overline{Q}$ be a homomorphism such that $\overline{\beta}\overline{\Psi} = \overline{\alpha}$. Then $\overline{\Psi}$ is an isomorphism.*

Proof. Let K denote the radical of J. Then, since $\mathrm{ht}(J) = r$ and J/KJ is a surjective image of a projective A/K-module of rank r, by Lemma 2.1, J/KJ is a projective A/K-module of rank r. Therefore, the maps $\overline{\alpha} \otimes_{A/J} A/K$ and $\overline{\beta} \otimes_{A/J} A/K$ are isomorphisms. Hence $\overline{\Psi} \otimes_{A/J} A/K$ is an isomorphism. Now we are through, as $\overline{P}, \overline{Q}$ are projective A/J-modules and K/J is the nilradical of A/J. □

The following lemma is easy to prove.

LEMMA 2.3. *Let A be a ring and P a projective A-moduleof rank n. Suppose that we are given the following short exact sequence*

$$0 \to P_1 \to A \oplus P \xrightarrow{(b,-\alpha)} A \to 0.$$

Let $(a_0, p_0) \in A \oplus P$ be such that $a_0 b - \alpha(p_0) = 1$. Let $q_i = (a_i, p_i) \in P_1, 1 \leqslant i \leqslant n$. Then,

(i) *The map δ: $\wedge^n(P_1) \to \wedge^n(P)$ given by $\delta(q_1 \wedge \ldots \wedge q_n) = a_0(p_1 \wedge p_2 \wedge \ldots \wedge p_n) + \sum_{i=1}^{n}(-1)^i a_i(p_0 \wedge \ldots \wedge p_{i-1} \wedge p_{i+1} \wedge \ldots \wedge p_n)$ is an isomorphism.*
(ii) *$\delta(bq_1 \wedge \ldots \wedge q_n) = p_1 \wedge \ldots \wedge p_n$.*
(iii) *The map β: $P_1 \to A$ given by $\beta(q) = c$, where $q = (c, p)$, has the property that $\beta(P_1) = \alpha(P)$.*
(iv) *The map Φ: $P \to P_1$ given by $\Phi(p) = (\alpha(p), bp)$ has the property that $\beta\Phi = \alpha$ and $\delta \wedge^n(\Phi)$ (where δ is as in (i)) is scalar multiplication by b^{n-1}.*

The following lemma follows from the well-known Quillen splitting lemma ([Q], Lemma 1) and its proof is essentially contained in ([Q], Theorem 1).

LEMMA 2.4. *Let A be a ring and P a projective $A[T]$-module. Let $a, b \in A$ be such that $Aa + Ab = A$. Assume that the $A_{ab}[T]$-module P_{ab} is extended from A_{ab}. Let $\sigma(T)$ be an $A_{ab}[T]$-automorphism of P_{ab} such that $\sigma(0) = \mathrm{id}$. Then $\sigma(T) = \tau_a\theta_b$, where τ is an $A_b[T]$-automorphism of P_b such that $\tau = \mathrm{id}$ modulo the ideal (aT) and θ is an $A_a[T]$-automorphism of P_a such that $\theta = \mathrm{id}$ modulo the ideal (bT).*

LEMMA 2.5. *Let A be a ring and let J be a proper ideal of A. Let $J_1 \subset J$ and $J_2 \subset J^2$ be two ideals of A such that $J_1 + J_2 = J$. Then $J = J_1 + (e)$ for some $e \in J_2$ and $J_1 = J \cap J'$, where $J_2 + J' = A$.*

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Proof. Since J/J_1 is an idempotent ideal of a Noetherian ring A/J_1 and J_2 maps surjectively onto J/J_1, there exists an element $e \in J_2$ such that $J_1 + (e) = J$ and $e(1-e) \in J_1$. Therefore the result follows by taking $J' = J_1 + (1-e)$. □

Let A be a ring and let P be a projective A-module. Given an element $\phi \in P^*$ and an element $p \in P$, we define an endomorphism ϕ_p as the composite $P \overset{\phi}{\to} A \overset{p}{\to} P$. If $\phi(p) = 0$, then ${\phi_p}^2 = 0$ and $1 + \phi_p$ is an automorphism of P. By a *transvection*, we mean an automorphism of P of the type $1 + \phi_p$ where $\phi(p) = 0$ and either ϕ is unimodular in P^*or p is unimodular in P. We denote by $\mathrm{Um}(P)$ the set of all unimodular elements of P and by $E(P)$ the subgroup of $\mathrm{Aut}(P)$ generated by all transvections of P.

The following result is due to Bhatwadekar and Roy ([B-R 1], Proposition 4.1):

PROPOSITION 2.6. *Let A be a ring, I an ideal of A and P a projective A-module. Then any transvection of P/IP can be lifted to a (unipotent) automorphism of P. Moreover, if the map* $\mathrm{Um}(P)$ *to* $\mathrm{Um}(P/IP)$ *is surjective, then the map* $E(P)$ *to* $E(P/IP)$ *is surjective.*

We now quote a theorem of Eisenbud and Evans ([E-E]) as stated in ([P], p.1420) and deduce some consequences which will be used later.

THEOREM 2.7. *Let A be a ring and M be an A-module. Let S be a subset of* Spec A *and $d: S \to \mathbf{N}$ be a generalized dimension function. Assume that $\mu(M_{\mathcal{P}}) \geqslant 1 + d(\mathcal{P})$ for all $\mathcal{P} \in S$. Let $(m, a) \in M \oplus A$ be basic at all prime ideals $\mathcal{P} \in S$. Then, there exists an element $m' \in M$ such that $m + am'$ is basic at all primes $\mathcal{P} \in S$.*

As a consequence of Theorem 2.7, we have the following result.

COROLLARY 2.8. *Let A be a ring and P be a projective A-module of rank n. Let $(\alpha, a) \in (P^* \oplus A)$. Then there exists an element $\beta \in P^*$ such that $\mathrm{ht}(I_a) \geqslant n$, where $I = (\alpha + a\beta)(P)$. In particular, if the ideal $(\alpha(P), a)$ has height $\geqslant n$ then $\mathrm{ht}\, I \geqslant n$. Further, if $(\alpha(P), a)$ is an ideal of height $\geqslant n$ and I is a proper ideal of A, then $\mathrm{ht}\, I = n$.*

Proof. Let S denote the subset of Spec A consisting of all prime ideals $\mathcal{Q}$ of A with the property: $a \notin \mathcal{Q}$ and height of $\mathcal{Q} \leqslant n-1$. Then by ([P], Example 1), there exists a generalized dimension function $d: S \to \mathbf{N}$ such that $d(\mathcal{Q}) \leqslant n-1$ for all $\mathcal{Q} \in S$. As $a \notin \mathcal{Q}$ for all $\mathcal{Q} \in S$, the element (α, a) of $P^* \oplus A$ is unimodular and, hence, basic at every member of S. Therefore, by Theorem 2.7, there exists an element $\beta \in P^*$ such that $\alpha + a\beta$ is basic and hence (as P^* is projective), is unimodular at all prime ideals $\mathcal{Q} \in S$.

Let $I = (\alpha + a\beta)(P)$. As $\alpha + a\beta$ is unimodular at all prime ideals $\mathcal{Q} \in S$, we have $I_{\mathcal{Q}} = A_{\mathcal{Q}}$ for every $\mathcal{Q} \in S$. Hence, $\mathrm{ht}(I_a) \geqslant n$. Since I is a surjective image of P, I

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is locally generated by n elements and, hence, if I is a proper ideal, then ht $I \leqslant n$. Therefore the rest of the conclusions follow. $\square$

As an application of Lemma 2.5 and Corollary 2.8 we have

COROLLARY 2.9. *Let A be a ring and let P be a projective A-module of rank n. Let $J \subset A$ be an ideal of height n and let $\overline{\alpha}: P/JP \twoheadrightarrow J/J^2$ be a surjection. Then there exists an ideal $J' \subset A$ and a surjection $\gamma: P \twoheadrightarrow J \cap J'$ such that*

(i) $J + J' = A$.
(ii) $\gamma \otimes A/J = \overline{\alpha}$.
(iii) *height* $(J') \geqslant n$.

We conclude this section by quoting the following result which is proved in ([B-RS 2], Theorem 3.3).

THEOREM 2.10 (Subtraction Principle). *Let A be a Noetherian ring with* $\dim A = n \geqslant 2$. *Let P and Q be projective A-modules of rank n and $n-1$ respectively such that $\wedge^n(P) \xrightarrow{\sim} \wedge^{n-1}(Q)$. Let $\chi: \wedge^n(P) \xrightarrow{\sim} \wedge^n(Q \oplus A)$ be an isomorphism. Let $J \subset A$ be an ideal of height $\geqslant n$ and J' be an ideal of height n which is comaximal with J. Let $\alpha: P \twoheadrightarrow J \cap J'$ and $\beta: Q \oplus A \twoheadrightarrow J'$ be surjections. Let bar denote reduction modulo J' and $\overline{\alpha}: \overline{P} \twoheadrightarrow J'/J'^2$, $\overline{\beta}: \overline{Q \oplus A} \twoheadrightarrow J'/J'^2$ be surjections induced from α and β respectively. Suppose that there exists an isomorphism $\delta: \overline{P} \xrightarrow{\sim} \overline{Q \oplus A}$ such that* (i) $\overline{\beta}\delta = \overline{\alpha}$, (ii) $\wedge^n(\delta) = \overline{\chi}$. *Then, there exists a surjection $\theta: P \twoheadrightarrow J$ such that $\theta \otimes A/J = \alpha \otimes A/J$.*

3. Cancellation over Two-Dimensional Rings

In this section we show that rank 2 (and, hence, all) projective modules over a normal affine surface over a perfect field k of cohomological dimension $\leqslant 1$ and of char. $\neq 2$ are cancellative (see Theorem 3.9).

Recall that all rings considered are commutative and Noetherian, all modules are finitely generated.

The proof of following lemma is implicit in the proof of ([M], Theorem 1.3).

LEMMA 3.1. *Let A be a ring and let P be a projective A-module of rank 2. Let $J \subset A$ be an ideal of height 2 such that there exists a surjection from P to J. Let J_1 be an ideal of A such that $J + J_1 = A$. Further assume that $J \cap J_1$ is a surjective image of a projective A-module Q of rank 2. Then there exists a projective A-module P_1 such that $P \oplus P_1 \cong Q \oplus (\wedge^2 P \oplus A)$ and J_1 is a surjective image of P_1.*

LEMMA 3.2. *Let A be a ring and let L be a projective A-module of rank 1. Let $J \subset A$ be an ideal of height 2 which is a surjective image of $L \oplus A$. Let*

θ': $L/JL \oplus A/J \twoheadrightarrow J/J^2$ be a surjection. Assume that $L \oplus A$ is cancellative. Then there exists a surjection γ: $L \oplus A \twoheadrightarrow J$ and an automorphism Θ' of $L/JL \oplus A/J$ of determinant 1 *such that* $\overline{\gamma}\Theta' = \theta'$, *where bar denotes reduction modulo* J. *Moreover, if* $\dim A = 2$ *then* θ' *can be lifted to a surjection from* $L \oplus A$ *to* J.

Proof. Let α: $L \oplus A \twoheadrightarrow J$ be a surjection. Let $\overline{\alpha}$: $L/JL \oplus A/J \twoheadrightarrow J/J^2$ be the surjection induced by α. Then, by Lemma 2.2, there exists an automorphism Ψ' of $L/JL \oplus A/J$ such that $\theta'\Psi' = \overline{\alpha}$. Let $\widetilde{\Psi} \in \mathrm{End}_A(L \oplus A)$ be a lift of Ψ' and let $a \in A$ be such that $\det(\widetilde{\Psi}) = a$. Since $J + Aa = A$, we get the following exact sequence:

$$0 \to P_1 \to A \oplus (L \oplus A) \overset{(a,-\alpha)}{\longrightarrow} A \to 0.$$

Therefore, by Lemma 2.3, we get an isomorphism δ: $\wedge^2(P_1) \overset{\cong}{\to} \wedge^2(L \oplus A)$, a homomorphism Φ: $L \oplus A \to P_1$ and a surjection β: $P_1 \twoheadrightarrow J$ such that, $\beta\Phi = \alpha$ and the endomorphism $\delta \wedge^2(\Phi)$ of $\wedge^2(L \oplus A)$ is a scalar multiplication by a.

Since $A \oplus P_1 \cong A \oplus (L \oplus A)$, by assumption, $P_1 \cong L \oplus A$. Hence, there exists an isomorphism Δ: $L \oplus A \overset{\cong}{\to} P_1$ such that $\wedge^2(\Delta) = \delta^{-1}$. Let $\gamma = \beta\Delta$ and $\Psi = \Delta^{-1}\Phi$. Then γ: $L \oplus A \twoheadrightarrow J$ is a surjection such that $\gamma\Psi = \alpha$. Moreover, the endomorphism $\wedge^2(\Psi)$ of $\wedge^2(L \oplus A)$ is a scalar multiplication by a.

Let bar denote reduction modulo J. Then we have $\overline{\gamma}\overline{\Psi} = \overline{\alpha} = \theta'\Psi'$. Moreover $\overline{\Psi}\Psi'^{-1}$ is an automorphism of $L/JL \oplus A/J$ of determinant 1. Therefore, putting $\Theta' = \overline{\Psi}\Psi'^{-1}$, we are through.

Now assume that $\dim A = 2$. Then, since $\dim A/J = 0$, we have $L/JL \oplus A/J = (A/J)^2$ and $\mathrm{SL}_2(A/J) = E_2(A/J)$. Therefore, by (2.6), Θ' can be lifted to an automorphism Θ of $L \oplus A$. Hence, θ' can be lifted to a surjection $\gamma\Theta$ from $L \oplus A$ to J. $\square$

LEMMA 3.3. *Let A be a ring and let P be a projective A-module of* rank 2. *Let* α: $P \twoheadrightarrow J$ *be a surjection where* J *is an ideal of* A. *Let* $\gamma_{(P,\alpha)}$: $\wedge^2(P) \to P$ *be a map defined by* $\gamma_{(P,\alpha)}(p \wedge q) = \alpha(q)p - \alpha(p)q$. *Then* $\mathrm{im}(\gamma_{(P,\alpha)}) \subset \ker(\alpha) \cap JP$. *Moreover, for* $a \in J$ *there exists a map* θ_P: $P \to \wedge^2(P)$ *such that the endomorphism* $\theta_P\gamma_{(P,\alpha)}$ *of* $\wedge^2(P)$ *is a scalar multiplication by* a.

Proof. It is obvious that $\mathrm{im}(\gamma_{(P,\alpha)}) \subset \ker(\alpha) \cap JP$.

Let Γ: $\mathrm{Hom}_A(P, \wedge^2(P)) \to \mathrm{End}_A(\wedge^2(P))$ be a map defined by $\Gamma(\phi) = \phi\gamma_{(P,\alpha)}$. Since $\wedge^2(P)$ is a projective A-module of rank 1, we have $\mathrm{End}_A(\wedge^2(P)) = A$. To complete the proof it is enough to prove the following claim.

CLAIM. $\Gamma(\mathrm{Hom}_A(P, \wedge^2(P))) = J$.

Proof of the claim. It is enough to prove the claim when A is local. So we assume that A is local. But then $P \cong A^2$. Let (e_1, e_2) be a basis of P and let $\alpha(e_i) = a_i$, $i = 1, 2$. Then $\gamma_{(P,\alpha)}(e_1 \wedge e_2) = a_2e_1 - a_1e_2$. Since $J = (a_1, a_2)$ the claim follows. $\square$

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The following lemma is easy to prove and, hence, we omit the proof.

LEMMA 3.4. *Let A be a ring and let P, Q be projective A-modules of rank 2. Let $\alpha: P \twoheadrightarrow J$ and $\beta: Q \twoheadrightarrow J$ be surjections where J is an ideal of A. Let $\Psi: P \to Q$ be a homomorphism such that $\beta\Psi = \alpha$. Let $\gamma_{(P,\alpha)}: \wedge^2(P) \to P$ and $\gamma_{(Q,\beta)}: \wedge^2(Q) \to Q$ be homomorphisms as defined in Lemma 3.3. Then $\Psi\gamma_{(P,\alpha)} = \gamma_{(Q,\beta)} \wedge^2(\Psi)$.*

LEMMA 3.5. *Let A be a ring and let P and Q be projective A-modules of rank 2 such that $\wedge^2(P) \xrightarrow{\sim} \wedge^2(Q)$. Let $\chi: \wedge^2(P) \xrightarrow{\sim} \wedge^2(Q)$ be an isomorphism. Let $J \subset A$ be an ideal of height 2. Let $\alpha: P \twoheadrightarrow J$ and $\beta: Q \twoheadrightarrow J$ be surjections. Let bar denote reduction modulo J and $\overline{\alpha}: \overline{P} \twoheadrightarrow J/J^2$, $\overline{\beta}: \overline{Q} \twoheadrightarrow J/J^2$ be surjections induced from α and β respectively. Suppose that there exists an isomorphism $\delta: \overline{P} \xrightarrow{\sim} \overline{Q}$ such that*

(i) $\overline{\beta}\delta = \overline{\alpha}$
(ii) $\wedge^2(\delta) = \overline{\chi}$. Then, there exists an isomorphism $\Delta: P \xrightarrow{\sim} Q$ such that (1) $\beta\Delta = \alpha$, (2) Δ *is a lift of* δ *and* (3) $\wedge^2(\Delta) = \chi$.

Proof. First we show that δ can be lifted to a homomorphism $\widetilde{\Delta}: P \to Q$ such that $\beta\widetilde{\Delta} = \alpha$.

Let Δ' be a lift of δ. Then, since $\overline{\beta}\delta = \overline{\alpha}$, $(\beta\Delta' - \alpha)(P) \subset J^2$. Since $\beta(JQ) = J^2$, there exists a homomorphism $\eta: P \to JQ$ such that $\beta\eta = \beta\Delta' - \alpha$. Let $\widetilde{\Delta} = \Delta' - \eta$. Then $\beta\widetilde{\Delta} = \alpha$. Since $\eta(P) \subset JQ$, it follows that $\widetilde{\Delta}$ is a lift of δ.

Since $\wedge^2(\delta) = \overline{\chi}$ and $\widetilde{\Delta}$ is a lift of δ, the endomorphism $\chi^{-1} \wedge^2(\widetilde{\Delta})$ of $\wedge^2(P)$ is a scalar multiplication by $1 - a$ for some $a \in J$.

Let

$$\gamma_{(P,\alpha)}: \wedge^2(P) \to P \quad \text{and} \quad \gamma_{(Q,\beta)}: \wedge^2(Q) \to Q$$

be homomorphisms as defined in Lemma 3.3. Then, by Lemma 3.3, there exists a map $\theta_P: P \to \wedge^2(P)$ such that the endomorphism $\theta_P\gamma_{(P,\alpha)}$ of $\wedge^2(P)$ is a scalar multiplication by a. Let $\Delta = \widetilde{\Delta} + \gamma_{(Q,\beta)}\chi\theta_P$. Then, since $\beta\gamma_{(Q,\beta)} = 0$, we have $\beta\Delta = \beta\widetilde{\Delta} = \alpha$. Since $\gamma_{(Q,\beta)}(\wedge^2(Q)) \subset JQ$, it follows that Δ is a lift of δ. Moreover, a local computation shows that $\wedge^2(\Delta) = \chi$. Thus, the lemma is proved. □

Remark 3.6. Let $J \subset A$ be a local complete intersection ideal of height 2 and let $\alpha: P \twoheadrightarrow J$ be a surjection, where P is a projective A-module of rank 2. Then we get the short exact sequence

$$0 \to \wedge^2(P) \xrightarrow{\gamma_{(P,\alpha)}} P \xrightarrow{\alpha} J \to 0.$$

Therefore $\mathrm{Ext}^1(J, \wedge^2(P)) \cong A/J$ and the above short exact sequence corresponds to a generator (which we denote by $E(P,\alpha)$) of $\mathrm{Ext}^1(J, \wedge^2(P))$. Keeping this notation in mind, a classical result of Serre says that $E(P,\alpha) = E(Q,\beta)$ if and only if correspond-

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ing exact sequences are isomorphic. Lemma 3.5 is nothing but a reformulation of this result, where J is not necessarily local complete intersection.

PROPOSITION 3.7. *Let A be ring of dimension 2 and let P be a rank 2 projective A-module. If $\wedge^2(P) \oplus A$ is cancellative, then P is cancellative.*

Proof. Let P, Q be projective A-modules such that $A^n \oplus P \cong A^n \oplus Q$. We want to show that $P \cong Q$. Note that $\wedge^2(Q) \cong \wedge^2(P)$. Let $\chi\colon \wedge^2(P) \xrightarrow{\sim} \wedge^2(Q)$ be an isomorphism.

Since $\operatorname{rank}(P) = 2$, by Corollary 2.8, there exists a surjection $\alpha\colon P \twoheadrightarrow J$ where $J \subset A$ is an ideal of height $\geqslant 2$. If $\operatorname{ht}(J) > 2$ then $J = A$ and hence $P \cong \wedge^2(P) \oplus A$. Therefore, by the assumption, $P \cong Q$. So it is enough to consider the case $\operatorname{ht}(J) = 2$.

Since $\dim(A) = 2$, $\dim(A/J) = 0$ and therefore P/JP, Q/JQ are free A/J-modules. Hence there exists an isomorphism $\delta\colon P/JP \xrightarrow{\sim} Q/JQ$ such that $\wedge^2(\delta) = A/J \otimes_A \chi$. Let $\overline{\alpha}\colon P/JP \twoheadrightarrow J/J^2$ be the surjection induced by α and let $\overline{\beta} = \overline{\alpha}\delta^{-1}$.

CLAIM. *The surjection $\overline{\beta}\colon Q/JQ \twoheadrightarrow J/J^2$ can be lifted to a surjection $\beta\colon Q \twoheadrightarrow J$.*

Proof of the claim. By Corollary 2.9, there exists an ideal $J_1 \subset A$ and a surjection $\beta'\colon Q \twoheadrightarrow J \cap J_1$ such that (1) $J + J_1 = A$, (2) $\beta' \otimes A/J = \overline{\beta}$ and (3) $\operatorname{ht}(J_1) \geqslant 2$. If $\operatorname{ht}(J_1) > 2$, then $J_1 = A$ and the claim is proved. So we assume that $\operatorname{ht}(J_1) = 2$. Since $\beta'(Q) = J \cap J_1$ and $\alpha(P) = J$, by Lemma 3.1, there exists a projective A-module P_1 such that $P \oplus P_1 \cong Q \oplus (\wedge^2(P) \oplus A)$ and J_1 is a surjective image of P_1. Since P is stably isomorphic to Q, it follows that P_1 is stably isomorphic to $\wedge^2(P) \oplus A$. Hence, by the assumption, $P_1 \cong \wedge^2(P) \oplus A$. Thus, J_1 is a surjective image of $\wedge^2(P) \oplus A$.

The map β' gives rise to a surjection $A/J_1 \otimes_A \beta'\colon Q/J_1Q \twoheadrightarrow J_1/J_1{}^2$. Since $\dim A/J_1 = 0$, as before, we see that there exists an isomorphism $\delta_1\colon \wedge^2(P/J_1P) \oplus A/J_1 \xrightarrow{\sim} Q/J_1Q$ such that $\wedge^2(\delta_1) = A/J_1 \otimes_A \chi$. Let $\overline{\theta} = (A/J_1 \otimes_A \beta')\delta_1$. Since $\wedge^2(P) \oplus A$ is cancellative, by Lemma 3.2, the surjection $\overline{\theta}\colon \wedge^2(P/J_1P) \oplus A/J_1 \twoheadrightarrow J_1/J_1{}^2$ can be lifted to a surjection $\theta\colon \wedge^2(P) \oplus A \twoheadrightarrow J_1$.

Thus, we have surjections $\beta'\colon Q \twoheadrightarrow J \cap J_1$ and $\theta\colon \wedge^2(P) \oplus A \twoheadrightarrow J_1$ such that $(\theta \otimes_A A/J_1)\delta_1{}^{-1} = A/J_1 \otimes_A \beta'$ and $\wedge^2(\delta_1{}^{-1}) = A/J_1 \otimes_A \chi^{-1}$. Therefore, by Theorem 2.10, there exists a surjection $\beta\colon Q \twoheadrightarrow J$ such that $A/J \otimes_A \beta = A/J \otimes_A \beta'$. Since $\overline{\beta} = A/J \otimes_A \beta'$, β is a lift of $\overline{\beta}$ and thus the claim is proved.

Now we have surjections $\alpha\colon P \twoheadrightarrow J$, $\beta\colon Q \twoheadrightarrow J$ and an isomorphism $\delta\colon P/JP \xrightarrow{\sim} Q/JQ$ such that $\wedge^2(\delta) = A/J \otimes_A \chi$ and $\overline{\alpha} = \overline{\beta}\delta$. Therefore, by Lemma 3.5, there exists an isomorphism $\Delta\colon P \xrightarrow{\sim} Q$ such that $\beta\Delta = \alpha$, Δ is a lift of δ and $\wedge^2(\Delta) = \chi$. □

The proof of Proposition 3.7 yields the following corollary.

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COROLLARY 3.8. *Let A be ring of dimension 2 and let P be a rank 2 projective A-module. If $\wedge^2(P) \oplus A$ is cancellative, then the canonical map $\mathrm{Aut}_A(P)$ to $\mathrm{Aut}_A(\wedge^2(P)) = A^*$ is surjective.*

THEOREM 3.9. *Let A be an affine domain of dimension 2 over a field k such that* Spec(A) *is nonsingular in codimension* 1 *(for example, k is perfect and A is normal). If $2 \in k^*$ and cohomological dimension of $k \leqslant 1$, then every projective A-module is cancellative.*

Proof. Let P be a projective A-module. If rank$(P) \geqslant 3$ then, by a result of Bass ([B 1]), P is cancellative. Since projective modules of rank 1 are always cancellative, it remains to prove the result in the case where rank$(P) = 2$. If k is finite then the result follows from ([M-M-R], Corollary 2.5). So we assume that k is infinite.

In view of Proposition 3.7, it is enough to show that $\wedge^2(P) \oplus A$ is cancellative.

Suslin has shown that, under the assumptions of the theorem, A^2 is cancellative (see ([Su 4], Theorem 2.4)). His arguments can be used to show, that under the hypothesis of the theorem, $L \oplus A$ is cancellative for every projective A-module L of rank 1. We just indicate the salient points of his arguments.

In view of the above result of Bass, it is enough to show that if $(l, a, b) \in L \oplus A \oplus A$ is a unimodular element then there exists $\Psi \in \mathrm{Aut}(L \oplus A \oplus A)$ such that $\Psi(l, a, b) = (0, 0, 1)$.

Let $J \subset A$ be the ideal defining the singular locus of Spec(A). Then, since Spec(A) is nonsingular in codimension 1, $\dim(A/J) = 0$ and, hence, $L/JL \cong A/J$. Therefore, in view of Proposition 2.6, we can assume that $a, b \in J$. Now, using Swan's version of Bertini's theorem (see ([Sw 1], Theorems 1.3 and 1.4)), we can assume that if $I = l(L^{-1})$, then Spec(A/I) is a smooth affine curve. Therefore, using ([Su 4], Propositions 1.4, 1.7 and Lemma 2.1)), we can assume that $b = c^2, c \in A$. Now we are through by ([M-S], Theorem 4). □

Remark 3.10. Let A be a ring of dimension 2 and let L be a projective A-module of rank 1. In view of Proposition 3.7, it would be interesting to give conditions under which $L \oplus A$ is cancellative.

The group $\mathrm{SL}(L \oplus A^2)$ acts on $\mathrm{Um}(L \oplus A^2)$ and the orbit space can be identified with the set G of isomorphism classes of pairs (P, χ) where $P \oplus A \cong L \oplus A^2$ and $\chi\colon \wedge^2 (L \oplus A) \xrightarrow{\sim} \wedge^2(P)$ is an isomorphism. Note that, under this identification, the orbit of the unimodular element $(0, (0, 1))$ corresponds to the isomorphism class of $(L \oplus A, \mathrm{id.})$.

In ([B-RS 2], Sections 4 and 6), the notions of the *Euler class group* $E(A, L)$ with respect to L and that of the *weak Euler class group* $E_0(A, L)$ (a certain quotient of $E(A, L)$) are defined. Moreover, to the isomorphism class of a pair (P, χ) an element $e(P, \chi)$ of $E(A, L)$ is attached. This assignment is such that $e(L \oplus A, \mathrm{id.}) = 0$. Therefore we get a map from (the set) G to $E(A, L)$ which, by Lemma 3.5 and ([B-RS 2], (4.3)), is injective. Moreover, using ([B-RS 2], (5.1) and (6.2)), it can be shown that the image of G in $E(A, L)$ is precisely the kernel of the canonical map $E(A, L) \twoheadrightarrow E_0(A, L)$. Since $E(A, L)$ is Abelian, we have, thus, an induced Abelian group structure on the orbit space $\mathrm{Um}(L \oplus A^2)/\mathrm{SL}(L \oplus A^2)$. Setting $L = A$, we obtain a group structure on $\mathrm{Um}(A^3)/\mathrm{SL}_3(A)$. This coincides with the group structure defined by Vaserstein.

Now, since dim $A = 2$, $L \oplus A$ is cancellative if and only if the action of $\mathrm{SL}(L \oplus A^2)$ on $\mathrm{Um}(L \oplus A^2)$ is transitive i.e. the group $\mathrm{Um}(L \oplus A^2)/\mathrm{SL}(L \oplus A^2)$ is trivial.Thus $L \oplus A$ is cancellative if and only if $E(A, L) = E_0(A, L)$.

Now let $X = \mathrm{Spec}(A)$ be a smooth affine surface over the field $\mathbf{R}$ of real numbers.

If the topological space (with the Euclidean topology) $X(\mathbf{R})$ of real points of X has no compact connected component (for example $A = \mathbf{R}[X, Y, Z]/(X^n + Y^n + Z^n = 1)$, n odd), then, using ([Ro], Corollary 11.2 and Theorem 11.10), it can be shown that, for every projective A-module L of rank 1, $E(A, L) = E_0(A, L)$ and, hence, $L \oplus A$ is cancellative for every L.

On the other hand, if the topological space $X(\mathbf{R})$ has at least one compact connected component, then, by ([Ro], Corollary 12.9), $E(A, K_A) \neq E_0(A, K_A)$ where K_A denote the canonical module of A over $\mathbf{R}$. Therefore $K_A \oplus A$ is not cancellative.

We conclude this section by giving an example of a smooth real affine surface A such that A^2 is cancellative but $K_A \oplus A$ is not cancellative.

EXAMPLE 3.11. Let $X = \mathrm{Spec}\, A$ be an affine open subvariety of the projective 2-space $\mathbf{P}^2(\mathbf{R})$ which is the complement of $V(X^2 + Y^2 + Z^2)$. Then, by ([B-RS 1], Corollary 6.3 (ii) and Proposition 6.1), A^2 is cancellative.

Since $X(\mathbf{R})$ is compact, by the above discussion, one knows that $K_A \oplus A$ is not cancellative. But here we give an argument which (we feel) is more elementary.

Let $B = \mathbf{R}[T_1, T_2, T_3]/({T_1}^2 + {T_2}^2 + {T_3}^2 = 1)$ be the coordinate ring of real 2-sphere and let σ be an $\mathbf{R}$-algebra automorphism of B defined by $\sigma(t_i) = -t_i$ where t_i denotes the image of T_i in B. Then it is easy to see that $B^\sigma \cong A$ and B is a finite etale extension of A. Therefore $\Omega_{B/\mathbf{R}} = \Omega_{A/\mathbf{R}} \otimes_A B$.

Now the projective B-module $\Omega_{B/\mathbf{R}}$ does not have a B-linear automorphism of determinant -1 (see ([O], Proposition 2.11) or ([Sw 2], Corollary 7.5)) and, hence, the projective A-module $\Omega_{A/\mathbf{R}}$ does not have an A-linear automorphism of determinant -1. Thus, the canonical map from $\mathrm{Aut}_A(\Omega_{A/\mathbf{R}})$ to A^* is not surjective. Therefore, by Corollary 3.8, $K_A \oplus A$ is not cancellative.

4. Symplectic Cancellation over Polynomial Extensions

We first recall some preliminary facts about symplectic modules.

Let A be a ring and let P be a finitely generated projective A-module. A bilinear map $\langle\, ,\rangle\colon P \times P \to A$ is called alternating if $\langle p, p\rangle = 0\ \forall\, p \in P$. An alternating bilinear form $\langle\, ,\rangle$ induces a homomorphism $\alpha\colon P \to P^*(\mathrm{Hom}_A(P, A))$ (defined as $\alpha(p)(q) = \langle p, q\rangle$) such that $\alpha + \alpha^* = 0$. Conversely, if $2 \in A^*$, then a homomorphism $\alpha\colon P \to P^*$ with the property $\alpha + \alpha^* = 0$ gives rise to an alternating form on P.

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An alternating form $\langle\,,\rangle$ on P is called non-degenerate if the induced homomorphism from P to P^* is an isomorphism.

A symplectic A-module is a pair $(P, \langle\,,\rangle)$ where P is a finitely generated projective A-module and $\langle\,,\rangle\colon P \times P \to A$ is a nondegenerate alternating bilinear form. If $(P, \langle\,,\rangle)$ is a symplectic A-module then the rank of P is even and P has trivial determinant.

To make the notation simple, we will always denote a nondegenerate alternating bilinear form by $\langle\,,\rangle$ irrespective of the base module.

If $(P, \langle\,,\rangle)$ and $(Q, \langle\,,\rangle)$ are two symplectic modules then nondegenerate alternating bilinear forms on P and Q will give rise (in a canonical manner) to a nondegenerate alternating bilinear form on $P \oplus Q$ and we denote the symplectic module thus obtained by $(P \perp Q, \langle\,,\rangle)$. There is a unique (up to scalar multiplication by elements of A^*) nondegenerate alternating bilinear form $\langle\,,\rangle$ on the free module $A^2 = A \oplus A$, namely $\langle (a, b), (c, d)\rangle = ad - bc$ and hence in the sequel it will be understood that we are considering this form on A^2.

Two symplectic modules $(P, \langle\,,\rangle)$ and $(Q, \langle\,,\rangle)$ are isomorphic if there exists an isomorphism $\tau\colon P \overset{\cong}{\to} Q$ such that

$$\langle p_1, p_2\rangle = \langle \tau(p_1), \tau(p_2)\rangle, \quad p_1, p_2 \in P.$$

An isometry of the symplectic module $(P, \langle\,,\rangle)$ is an automorphism of $(P, \langle\,,\rangle)$. We denote by $\mathrm{Sp}(P, \langle\,,\rangle)$ the group of isometries of $(P, \langle\,,\rangle)$. $\mathrm{Sp}(P, \langle\,,\rangle)$ is a subgroup of $\mathrm{SL}(P)$ and it coincides with $\mathrm{SL}(P)$ when $\mathrm{rank}(P) = 2$. Therefore $\mathrm{SL}_2(A)$ can be identified with a subgroup of $\mathrm{Sp}(A^2 \perp P, \langle\,,\rangle)$.

Let $(P, \langle\,,\rangle)$ be a symplectic A-module and let $u, v \in P$ be such that $\langle u, v\rangle = 0$. Let $a \in A$ and let $\tau_{(a,u,v)}\colon P \to P$ be a map defined by

$$\tau_{(a,u,v)}(p) = p + \langle p, v\rangle u + \langle p, u\rangle v + a\langle p, u\rangle u.$$

Then $\tau_{(a,u,v)} \in \mathrm{Sp}(P, \langle\,,\rangle)$. Moreover

$$\tau_{(a,u,v)}{}^{-1} = \tau_{(-a,u,-v)} = \tau_{(-a,-u,v)} \quad \text{and} \quad \alpha\tau_{(a,u,v)}\alpha^{-1} = \tau_{(a,\alpha(u),\alpha(v))}$$

for an element α in $\mathrm{Sp}(P, \langle\,,\rangle)$.

An isometry $\tau_{(a,u,v)}$ is called a *symplectic transvection* if either u or v is unimodular. We denote by $\mathrm{ESp}(P, \langle\,,\rangle)$ the subgroup of $\mathrm{Sp}(P, \langle\,,\rangle)$ generated by symplectic transvections. $\mathrm{ESp}(P, \langle\,,\rangle)$ is a normal subgroup of $\mathrm{Sp}(P, \langle\,,\rangle)$.

The above definition of symplectic transvection is slightly more restrictive than the one given by Bass.

Now we prove a few lemmas, some of which are well known and are included for the convenience of the reader. We begin with a result which is a symplectic version of Proposition 2.6.

LEMMA 4.1. *Let A be ring and let $(P, \langle\,,\rangle)$ be a symplectic A-module. Let I be an ideal of A. If the canonical map* $\mathrm{Um}(P) \to \mathrm{Um}(P/IP)$ *is surjective, then the canonical map* $\mathrm{ESp}(P, \langle\,,\rangle) \to \mathrm{ESp}(P/IP, \langle\,,\rangle)$ *is surjective.*

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Proof. Let $b \in A/I$ and let $x, y \in P/IP$ be such that $\langle x, y\rangle = 0$. Assume that $x \in \mathrm{Um}(P/IP)$. Let $u \in \mathrm{Um}(P)$ be a lift of x, $v' \in P$ be a lift of y and $a \in A$ be a lift of b. Then $\langle u, v'\rangle = c \in I$. Since u is a unimodular element of P, there exists $w \in P$ such that $\langle u, w\rangle = 1$. Let $v = v' - cw$. Then v is also a lift of y and $\langle u, v\rangle = 0$. Therefore, as u is unimodular in P, $\tau_{(a,u,v)}$ is a symplectic transvection of $(P, \langle\,,\rangle)$ which is a lift of the symplectic transvection $\tau_{(b,x,y)}$ of $(P/IP, \langle\,,\rangle)$. The case $y \in \mathrm{Um}(P/IP)$ can be proved similarly. Hence, the lemma follows. □

LEMMA 4.2. *Let A be ring and let $s \in A$ be a nonzero divisor. Let $(P, \langle\,,\rangle)$ be a symplectic A-module of rank $2n$. Let $e_1, \dots, e_n, f_1, \dots, f_n \in P$ be such that*

$$\langle e_i, e_j\rangle = 0 = \langle f_i, f_j\rangle, \quad 1 \leqslant i, j \leqslant n,$$

and

$$\langle e_i, f_i\rangle = s, \qquad \langle e_i, f_j\rangle = 0, \quad i \neq j.$$

Let $F = \sum_{i=1}^{n} Ae_i + \sum_{i=1}^{n} Af_i$ be a submodule of P. Then F is a free of A-module of rank $2n$ and $sP \subset F$.

Proof. Let $F^{\perp} = \{q \in P, \langle F, q\rangle = 0\}$. Then, since s is a nonzero divisor, F is a free A-module of rank $2n$ and $F \cap F^{\perp} = 0$. Let $p \in P$ and let $\langle e_i, p\rangle = b_i$, $\langle f_j, p\rangle = -a_j$.

Let $x = \sum_{j=1}^{n} a_j e_j + \sum_{i=1}^{n} b_i f_i \in F$. Then $\langle e_i, x\rangle = sb_i$, $\langle f_j, x\rangle = -sa_j$. Therefore $sp - x \in F^{\perp}$ and hence $sP \subset F \oplus F^{\perp} \subset P$. Since F and P have same rank and s is a nonzero divisor, it follows that $F^{\perp} = 0$ and, hence, the result follows. □

Let $(P, \langle\,,\rangle)$ be a symplectic A-module. Let $c, d \in A$, $q \in P$. If

$$u = (0, 1, 0) \quad \text{and} \quad v = (0, c, q) \in A \oplus A \oplus P,$$

then $\tau_{(-c,u,v,)}$ is a symplectic transvection of $(A^2 \perp P, \langle\,,\rangle)$ such that

$$\tau_{(-c,u,v,)}((a, b, p)) = (a, b + ca + \langle p, q\rangle, p + aq).$$

Similarly, if $u = (1, 0, 0)$ and $v = (-d, 0, -q)$, then

$$\tau_{(d,u,v)}((a, b, p)) = (a + bd + \langle q, p\rangle, b, p + bq).$$

In what follows we will use the following additional notation:

We denote by $\theta_{(c,q)}$ a symplectic transvection of $(A^2 \perp P, \langle\,,\rangle)$ defined as $(a, b, p) \to (a, b + ca + \langle p, q\rangle, p + aq)$. Similarly, $\sigma_{(d,q)}$ will denote a symplectic transvection $(a, b, p) \to (a + bd + \langle q, p\rangle, b, p + bq)$.

LEMMA 4.3. *Let A be a ring and let $(P, \langle\,,\rangle)$ be a symplectic A-module. Let $(a, b, p) \in A^2 \oplus P$. Let $t = ca^2 + db^2 \in A$ and $q \in P$. Then, there exists $\psi \in \mathrm{ESp}(A^2 \perp P, \langle\,,\rangle)$ such that $\psi(a, b, p) = (a, b, p + tq)$.*

Proof. Let $q_1 = caq$, $q_2 = dbq$, and $\langle p, q\rangle = s$. Then $\langle p, q_1\rangle = acs$, $\langle p, q_2\rangle = bds$. Let $\psi = \sigma_{(ds,q_2)}\theta_{(-cs,q_1)}$. Then $\psi \in \mathrm{ESp}(A^2 \perp P, \langle\,,\rangle)$ and $\psi(a, b, p) = (a, b, p + tq)$. □

LEMMA 4.4. *Let A be a ring and let $(P, \langle\, , \rangle)$ be a symplectic A-module. Let $(a, b, p) \in A^2 \oplus P$ be such that either (a, p) or $(b, p) \in Um(A \oplus P)$ Then, there exists an element ϕ of* $\mathrm{ESp}(A^2 \perp P, \langle\, , \rangle)$ *such that $\phi(a, b, p) = (1, 0, 0)$.*

Proof. First assume that $(b, p) \in Um(A \oplus P)$. Then, since the alternating form $\langle\, , \rangle$ on P is nondegenerate, there exist $q \in P$ and $d \in A$ such that $1 - a = db + \langle q, p\rangle$. Therefore

$$\sigma_{(d,q)}(a, b, p) = (a + db + \langle q, p\rangle, b, p + bq) = (1, b, p + bq).$$

Now $\theta_{(-b,-p-bq)}(1, b, p + bq) = (1, 0, 0)$. Hence, by putting $\phi = \theta_{(-b,-p-bq)}\sigma_{(d,q)}$, we are through. In the case where $(a, p) \in \mathrm{Um}(A \oplus P)$ the proof is similar. □

Let B be a ring and let $A = B[X]$. Let F be a free A-module with a basis $(e_1, \dots, e_r)$. Let $p(X) = \sum_{i=1}^{r} \gamma_i(X)e_i \in F$. Then, for $b \in B$, we denote by $p(bX)$ the element $\sum_{i=1}^{r} \gamma_i(bX)e_i$ of F. Keeping this convention in mind, we state the following lemmas.

LEMMA 4.5. *Let B be a ring and let $s \in B$ be a nonzero divisor. Let $A = B[X]$ and let $(P, \langle\, , \rangle)$ be a symplectic A-module of rank $2n$. Let $e_1, \dots, e_n, f_1, \dots, f_n \in P$ be such that $\langle e_i, e_j\rangle = 0 = \langle f_i, f_j\rangle$, $1 \leqslant i, j \leqslant n$, and $\langle e_i, f_i\rangle = s$, $\langle e_i, f_j\rangle = 0$, $i \neq j$. Let $F = \sum_{i=1}^{n} Ae_i + \sum_{i=1}^{n} Af_i$. Let $(\alpha(X), \beta(X), p(X)) \in \mathrm{Um}(A^2 \oplus F)$ be such that $\alpha(X) \equiv 1$ modulo (sX). Then, for $b \in B$,*

$$(\alpha(bX), \beta(bX), p(bX)) \in \mathrm{Um}(A^2 \oplus P).$$

Proof. Since $(\alpha(X), \beta(X), p(X)) \in \mathrm{Um}(A^2 \oplus F)$,

$$(\alpha(bX), \beta(bX), p(bX)) \in \mathrm{Um}(A^2 \oplus F).$$

Therefore, since, by Lemma 4.2, $F_s = P_s$, $(\alpha(bX), \beta(bX), p(bX))$ is a unimodular element of $(A^2 \oplus P)_s$. Hence, as $\alpha(bX) \equiv 1$ (modulo (sX)), $(\alpha(bX), \beta(bX), p(bX))$ is a unimodular element of $A^2 \oplus P$. □

The following lemma is due to Suslin ([Su 2], Lemma 2.1).

LEMMA 4.6. *Let B be a ring and let $\alpha(X), \beta(X) \in B[X]$. Let $c \in B \cap (\alpha(X), \beta(X))$. Then for any ideal I of $B[X]$ and $g(X), h(X) \in B[X]$ with $h(X) - g(X) \in cI$, there exists $\Delta \in SL_2(B[X], I)$ such that $[\alpha(h(X)), \beta(h(X))]\Delta = [\alpha(g(X)), \beta(g(X))]$.*

PROPOSITION 4.7. *Let the hypothesis and notation be as in Lemma 4.5. Further assume that $\beta(X)$ is a monic polynomial. Let $b, b' \in B$ be such that $b - b' \in sB$. Then there exists* $\Psi \in \mathrm{SL}_2(A, (sX))\mathrm{ESp}(A^2 \perp P, \langle\, , \rangle)$ *such that*

$$\Psi(\alpha(bX), \beta(bX), p(bX)) = (\alpha(b'X), \beta(b'X), p(b'X)).$$

Proof. Note that, since $\mathrm{ESp}(A^2 \perp P, \langle\, , \rangle)$ is a normal subgroup of $\mathrm{Sp}(A^2 \perp P, \langle\, , \rangle)$, $\mathrm{SL}_2(A, (sX))\mathrm{ESp}(A^2 \perp P, \langle\, , \rangle)$ is a group.

Let $G = \mathrm{SL}_2(A, (sX))\mathrm{ESp}(A^2 \perp P, \langle\,,\rangle)$ and let J be a set of elements $c \in B$ having the following property:

$$b - b' \in csB \Rightarrow \exists \Phi \in G$$

such that

$$\Phi(\alpha(bX), \beta(bX), p(bX)) = (\alpha(b'X), \beta(b'X), p(b'X)).$$

It is easy to see that, since G is a group, J is an ideal of B. We shall prove that $J = B$.

Let $t \in B \cap (\alpha(X)^2, \beta(X)^2)$ and let $b, b' \in B$ be such that $b - b' \in tsB$. Then $p(bX) = p(b'X) - tsq(X)$. Since $t \in B$, it also belongs to the ideal of $B[X]$ generated by $\alpha(bX)^2, \beta(bX)^2$. Therefore, by Lemma 4.3, there exists $\psi \in \mathrm{ESp}(A^2 \perp P, \langle\,,\rangle)$ such that $\psi(\alpha(bX), \beta(bX), p(bX)) = (\alpha(bX), \beta(bX), p(b'X))$. Since $bX - b'X \in (tsX)$, by Lemma 4.6, the element $(\alpha(bX), \beta(bX), p(b'X))$ can be transformed to $(\alpha(b'X), \beta(b'X), p(b'X))$ by an element of $\mathrm{SL}_2(A, (sX))$. Hence, $t \in J$.

Since $\beta(X)$ is monic and $\alpha(X) \equiv 1$ (modulo (sX)), $B \cap (\alpha(X)^2, \beta(X)^2) + sB = B$. Therefore the above argument shows that $J + sB = B$.

Let m be a maximal ideal of B. If $s \in m$ then $m + J = B$. Now assume that $s \notin m$. To complete the proof, it is enough to show that $m + J = B$.

Since

$$\alpha(X) \equiv 1(modulo(sX)) \quad \text{and} \quad (\alpha(X), \beta(X), p(X)) \in \mathrm{Um}(A^2 \oplus P),$$

$(\alpha(X), \beta(X), sXp(X)) \in \mathrm{Um}(A^2 \oplus P)$. Therefore, there exists $p' \in P$ such that $mB[X]$ and the ideal $(\alpha(X) + \langle p', sXp(X)\rangle, \beta(X))$ are comaximal. Since $sP \subset F$, $sXp' \in F$. Let

$$q(X) = sXp', \qquad p_1(X) = p(X) + \beta(X)q(X)$$

and

$$\eta(X) = \alpha(X) + \langle q(X), p(X)\rangle.$$

Then

$$\eta(X) \equiv 1(modulo(sX)) \quad \text{and} \quad mB[X] + (\eta(X), \beta(X)) = B[X].$$

Moreover, for $b \in B$,

$$\sigma_{(0,q(bX))}(\alpha(bX), \beta(bX), p(bX)) = (\eta(bX), \beta(bX), p_1(bX)).$$

Let $J_1 = B \cap (\eta(X)^2, \beta(X)^2)$ Then, since $mB[X] + (\eta(X), \beta(X)) = B[X]$ and $\beta(X)$ is monic, $m + J_1 = B$.

Let $t_1 \in J_1$. As before, we see that if $b, b' \in B$ such that $b - b' \in st_1B$, then there exists $\Phi \in G$ such that

$$\Phi(\eta(bX), \beta(bX), p_1(bX)) = (\eta(b'X), \beta(b'X), p_1(b'X)).$$

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Therefore

$$\sigma_{(0,q(b'X))}{}^{-1}\Phi\sigma_{(0,q(bX))}(\alpha(bX), \beta(bX), p(bX)) = (\alpha(b'X), \beta(b'X), p(b'X)).$$

Hence, $J_1 \subset J$ and thus $m + J = B$. Therefore $J = B$ and we are through. □

THEOREM 4.8. *Let R be a ring of dimension d. Let A be a polynomial ring in $r(\geqslant 0)$ variables over R. Let $(P, \langle\,,\rangle)$ be a symplectic A-module of rank $2n > 0$. If $2n \geqslant d$ then $\mathrm{ESp}(A^2 \perp P, \langle\,,\rangle)$ acts transitively on $\mathrm{Um}(A^2 \oplus P)$.*

Proof. Let $(g_1, g_2, p) \in \mathrm{Um}(A^2 \oplus P)$. We want to show that there exists $\Phi \in \mathrm{ESp}(A^2 \perp P, \langle\,,\rangle)$ such that $\Phi(g_1, g_2, p) = (1, 0, 0)$. We prove the result by induction on r.

If $r = 0$ (i.e. $A = R$), then this is a result of Bass. We give, however, the proof for the sake of completeness.

Since $\mathrm{rank}(A \oplus P) > d$, by a theorem of Bass ([B 1]), there exist $h \in A$, $q \in P$ such that $(g_2 + g_1h, p + g_1q)$ is a unimodular element of $A \oplus P$. Therefore $(g_2 + g_1h + \langle p, q\rangle, p + g_1q)$ is also unimodular. Therefore, since

$$\theta_{(h,q)}(g_1, g_2, p) = (g_1, g_2 + g_1h + \langle p, q\rangle, p + g_1q),$$

by Lemma 4.4, we are through.

Let $A = R[X_1, \ldots, X_r]$, $r \geqslant 1$. With out loss of generality we can assume that R is reduced. Let S be a set of nonzero divisors of R. Then R_S is a finite direct product of fields and therefore, by the Quillen–Suslin theorem ([Q], Theorem 4, [Su 1], Theorem 3), every projective A_S-module is free. Hence, we can find a basis $\widetilde{p}_1, \ldots, \widetilde{p}_n$, $\widetilde{q}_1, \ldots, \widetilde{q}_n$ of P_S such that $\langle\widetilde{p}_i, \widetilde{p}_j\rangle = 0 = \langle\widetilde{q}_i, \widetilde{q}_j\rangle$, $1 \leqslant i, j \leqslant n$, and $\langle\widetilde{p}_i, \widetilde{q}_i\rangle = 1$, $\langle\widetilde{p}_i, \widetilde{q}_j\rangle = 0$, $i \neq j$.

Let $\widetilde{p}_i = e_i/t, \widetilde{q}_i = f_i/t$, $t \in S, e_i, f_i \in P$, $1 \leqslant i \leqslant n$ and let $s = t^2$. Let $F = \sum_{i=1}^n Ae_i + \sum_{i=1}^n Af_i$. Then, by Lemma 4.2, F is a free-submodule of P of rank $2n$ and $sP \subset F$.

Since $s \in R$ is a nonzero divisor, $\overline{R} = R[X_r]/(sX_r)$ is a ring of dimension d and $\overline{A} = A/(sX_r)$ is a polynomial ring in $r - 1$ variables over $\overline{R}$. Moreover, since $\mathrm{rank}(P) \geqslant d$, by ([L], Lemma 1.11), the map $\mathrm{Um}(A^2 \oplus P) \to \mathrm{Um}((\overline{A})^2 \oplus P/sX_rP)$ is surjective. Therefore, by (4.1) and the induction hypothesis, there exists $\psi \in \mathrm{ESp}(A^2 \perp P, \langle\,,\rangle)$ such that $\psi(g_1, g_2, p) \equiv (1, 0, 0)$ (modulo (sX_r).)

Let $\psi(g_1, g_2, p) = (g_1', g_2', p')$. Then, by Corollary 2.8, there exist $h' \in A$ and $p_1 \in P$ such that $\mathrm{ht}(Ag_3 + I) \geqslant \mathrm{rank}(A \oplus P) > d$, where

$$g_3 = g_1' + h'g_2', \qquad p_2 = p' + g_2'p_1, \quad \text{and} \quad I = p_2(P^*) = \langle P, p_2\rangle.$$

Put $\alpha(X_r) = g_3 + \langle p_1, p'\rangle$ then $\sigma_{(h',p_1)}(g_1', g_2', p') = (\alpha(X_r), g_2', p_2)$.

Note that, since $g_2' \in (sX_r)$ and $p' \in sX_rP$, $\alpha(X_r) \equiv 1$ (modulo (sX_r)). Moreover, since $\langle p_1, p_2\rangle = \langle p_1, p'\rangle \in I$ and $\alpha(X_r) \equiv 1$ (modulo (sX_r)), $Ag_3 + I = A\alpha(X_r) + I = A\alpha(X_r) + sX_rI$.

Since $A\alpha(X_r) + sX_rI$ is an ideal of $A = R[X_1, \ldots, X_r]$ of height $> d = \dim(R)$, by ([B-2], Lemma 3 of Section 4), there exist positive integers $l_1, \ldots, l_{r-1}$ such that, denoting $X_j + X_r^{l_j}$ by Y_j $(1 \leqslant j \leqslant r-1)$, $A\alpha(X_r) + sX_rI$ contains a monic polynomial $\gamma(X_r)$ in X_r with coefficients in $R[Y_1, \ldots, Y_{r-1}]$.

Let $B = R[Y_1, \ldots, Y_{r-1}]$. We denote X_r by X. Then $A = B[X]$. Without loss of generality we can assume that X-degree of $\gamma(X) > X$-degree of the element $g_2{}' \in B[X]$. Let $\gamma(X) = \mu(X)\alpha(X) + \nu(X)$ where $\nu(X) \in sXI$. Since $I = \langle P, p_2\rangle$ there exists $p_3 \in sXP$ such that $\nu(X) = \langle -p_3, p_2\rangle = \langle p_2, p_3\rangle$. Put $\beta(X) = g_2{}' + \gamma(X)$ and $p_4 = p_2 + \alpha(X)p_3$. Then $\theta_{(\mu(X),p_3)}(\alpha(X), g_2{}', p_2) = (\alpha(X), \beta(X), p_4)$. Note that, by construction, $\alpha(X) \equiv 1$ (modulo (sX)), $\beta(X)$ monic and $p_4 \in sXP$. Therefore, by Lemma 4.2, $p_4 \in XF$. Hence, we write p_4 as $p_4(X)$.

Since $(\alpha(X)) + sB[X] = B[X]$ and $\beta(X)$ is monic, there exists $c \in B$ such that $1 - cs \in B \cap (\alpha(X), \beta(X))$. Then, writing $b = 1, b' = 1 - sc$ and applying Proposition 4.7, we see that there exists $\Psi \in \mathrm{SL}_2(A, (sX))\mathrm{ESp}(A^2 \perp P, \langle\, , \rangle)$ such that $\Psi(\alpha(X), \beta(X),\ p_4(X)) = (\alpha(b'X), \beta(b'X), p_4(b'X))$. Since $\alpha(X) \equiv 1$ (modulo (sX)) $\alpha(b'X) \equiv 1$ (modulo $(sb'X)$). Moreover $b' = 1 - cs \in B \cap (\alpha(b'X), \beta(b'X))$. Therefore $[\alpha(b'X), \beta(b'X)]$ is a unimodular row.

Let $\Psi = \Delta^{-1}\phi$ where $\Delta \in \mathrm{SL}_2(A, (sX))$ and $\phi \in \mathrm{ESp}(A^2 \perp P, \langle\, , \rangle)$. Let

$$[\alpha(b'X), \beta(b'X)]\Delta = [\alpha_1(X), \beta_1(X)].$$

Then

$$\phi(\alpha(X), \beta(X), p_4(X)) = (\alpha_1(X), \beta_1(X), p_4(b'X)).$$

Since $\Delta \in \mathrm{SL}_2(A, (sX))$, $\alpha_1(X) \equiv 1$ (modulo (sX)) and $[\alpha_1(X), \beta_1(X)]$ is a unimodular row. Therefore, by (4.3), there exists $\phi_1 \in \mathrm{ESp}(A^2 \perp P, \langle\, , \rangle)$ such that

$$\phi_1(\alpha_1(X), \beta_1(X), p_4(b'X)) = (\alpha_1(X), \beta_1(X), e_1).$$

Since $\langle e_1, f_1\rangle = s$, $(\alpha_1(X), e_1)$ is an element of $\mathrm{Um}(A \oplus P)$. Therefore, by Lemma 4.4, there exists $\phi_2 \in \mathrm{ESp}(A^2 \perp P, \langle\, , \rangle)$ such that $\phi_2(\alpha_1(X), \beta_1(X), e_1) = (1, 0, 0)$.

Let $\Phi = \phi_2\phi_1\phi\theta_{(\mu(X),p_3)}\sigma_{(h',p_1)}\psi$. Then $\Phi(g_1, g_2, p) = (1, 0, 0)$. □

Remark 4.9. Let R be a ring of dimension d and let A be a polynomial ring over R. Suslin ([Su 2], Theorem 2.6) has shown that if $n \geqslant \max(1, d)$, then $E_{n+2}(A)$ acts transitively on the set $\mathrm{Um}_{n+2}(A)$ of unimodular rows of length $n+2$. Lindel ([L], Theorem 2.6) has generalised this result of Suslin by proving that, for a projective A-module P of rank $n \geqslant \max(1, d)$, $E(A^2 \oplus P)$ acts transitively on $\mathrm{Um}(A^2 \oplus P)$. Theorem 4.8 is a symplectic analogue of this result of Lindel and our proof of Theorem 4.8 is a simple adaptation of the proof given by Lindel.

5. Projective Modules over Polynomial Extensions of Two-Dimensional Rings

Let S be a two-dimensional local ring with $2 \in S^*$. Then every unimodular row $[f_1, f_2, f_3]$ over $S[X]$ is completable (see ([Ra 2], Lemma 2.9) for a proof). The following lemma is a consequence of this result.

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LEMMA 5.1. *Let R be a two dimensional ring with $2 \in R^*$. Let $A = R[X_1, \dots, X_r]$ and let P, Q be two projective A-modules of rank 2 such that $A^n \oplus P \cong A^n \oplus Q$. If Q is extended from R, then P is also extended from R.*

Proof. Since, by ([B-R 2], Theorem 3.1), every projective A-module of rank > 2 is cancellative, we can assume that $n = 1$.

We prove the result by induction on r. If $r = 1$, then the result follows from above result and the Quillen localization theorem ([Q], Theorem 1).

Let $\widetilde{P} = P/(X_1, \dots, X_r)P$ and $\overline{P} = P/(X_2, \dots, X_r)P$. Then, by the case $r = 1$, $\overline{P} = \widetilde{P} \otimes_R R[X_1]$. Let $R(X_1)$ denote the ring obtained from $R[X_1]$ by inverting monic polynomials in X_1. Let $B = R(X_1)[X_2, \dots, X_r]$. Then, by the induction hypothesis, $P \otimes_A B \cong \overline{P} \otimes_{R[X_1]} B = \widetilde{P} \otimes_R B$. Therefore, by the monic inversion theorem of Quillen and Suslin ([Q], [Su 1]), P is extended from $R[X_2, \dots, X_r]$. Hence, applying the induction hypothesis again, we see that P is extended from R. □

LEMMA 5.2. *Let S be a reduced semilocal ring of dimension 2 with $2 \in S^*$ and let $B = S[X_1, \dots, X_r]$. Let P, Q be two projective B-modules of rank 2 having trivial determinant such that $B^n \oplus P \cong B^n \oplus Q$. Let ω_P, ω_Q be generators of $\wedge^2(P)$ and $\wedge^2(Q)$ respectively. Then there exist comaximal ideals I, I_1 of B with $\mathrm{ht}(I) = 2$, $\mathrm{ht}(I_1) = 2$ or $I_1 = B$, surjections $\alpha: P \twoheadrightarrow I$, $\beta: Q \twoheadrightarrow I \cap I_1$, $\theta: B^2 \twoheadrightarrow I_1$ and isomorphisms $\delta: Q/IQ \xrightarrow{\cong} P/IP$, $\delta_1: Q/I_1Q \xrightarrow{\cong} (B/I_1)^2$ such that*

(i) $(\alpha \otimes B/I)\delta = \beta \otimes B/I$.
(ii) $(\theta \otimes B/I_1)\delta_1 = \beta \otimes B/I_1$.
(iii) $\wedge^2(\delta)(\omega_Q \otimes B/I) = \omega_P \otimes B/I$.
(iv) $\wedge^2(\delta_1)(\omega_Q \otimes B/I_1) = (e_1 \wedge e_2) \otimes B/I_1$ *where* $e_1 = (1, 0), e_2 = (0, 1)$.

Proof. If P is free then, by Lemma 5.1, Q is free. In this case the proof is easy. So we assume that P is not free.

By the Quillen–Suslin theorem there exists a nonzero divisor $s \in S$ such that P_s, Q_s are free. Since $\dim(S/sS) \leqslant 1$, by ([B-R 1], Theorem 3.1), there exists $\alpha_1 \in P^*$ such that $(\alpha_1, s) \in P^* \oplus B$ is unimodular. Therefore, by Corollary 2.8, there exists $\alpha_2 \in P^*$ such that $\mathrm{ht}(I) = 2$ where $\alpha = \alpha_1 + s\alpha_2$ and $I = \alpha(P)$.

The surjection $\alpha: P \twoheadrightarrow I$ induces a surjection $\overline{\alpha}: P/IP \twoheadrightarrow I/I^2$. Since $I + sB = B$ and P_s, Q_s are free, Q/IQ and P/IP are free. Therefore, there exists an isomorphism $\delta: Q/IQ \xrightarrow{\cong} P/IP$ such that $\wedge^2(\delta)(\omega_Q \otimes B/I) = \omega_P \otimes B/I$.

Let $\beta' = \overline{\alpha}\delta$. Then $\beta': Q/IQ \twoheadrightarrow I/I^2$ is a surjection. Since, by ([B-R 1], Theorem 3.1), Q/sQ has a unimodular element, using the fact $I + sB = B$, it is easy to see that there exists a lift $\widetilde{\beta}: Q \to I$ of β' such that $\widetilde{\beta}(Q) + sB = B$. Let $I_2 = \widetilde{\beta}(Q)$. Then $I_2 + I^2 = I, I_2 + sB = B$. Therefore, $I_2 + sI^2 = I$ and, hence, by Lemma 2.5, there exists $g \in sI^2$ such that $I_2 + (g) = I$. Now applying Corollary 2.8 to the pair $(\widetilde{\beta}, g)$, we see that there exists $\eta \in Q^*$ such that $(\widetilde{\beta} + g\eta)(Q)$ is an ideal of height two. Let $\beta = \widetilde{\beta} + g\eta$. Then, by construction, $\beta(Q) = I \cap I_1$ where either $I_1 = B$ or I_1 is an ideal of height two comaximal with

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(g). Note that $(\alpha \otimes B/I)\delta = \beta' = \beta \otimes B/I$. Therefore if $I_1 = B$ we are through. So assume that I_1 is an ideal of height two.

By Lemma 3.1, there exists a projective B-module P' of rank two such that $P \oplus P' \cong Q \oplus (\wedge^2(P) \oplus B)$ and I_1 is a surjective image of P'. Since $\wedge^2(P) \cong B$, $B^n \oplus P \cong B^n \oplus Q, P'$ is stably isomorphic to B^2. Since S is semilocal, by (5.1), B^2 is cancellative. Therefore $P' \cong B^2$. Hence $I_1$is a surjective image of B^2.

Since $I_1 + Bg = B$, g is a multiple of s and Q_s is free, Q/I_1Q is free. Therefore there exists an isomorphism $\widetilde{\delta}: Q/I_1Q \stackrel{\cong}{\to} (B/I_1)^2$ such that $\wedge^2(\widetilde{\delta})(\omega_Q \otimes B/I_1) = e_1 \wedge e_2 \otimes B/I_1$, where $e_1 = (1, 0), e_2 = (0, 1)$.

Let $\theta': (B/I_1)^2 \twoheadrightarrow I_1/I_1{}^2$ be a surjection defined by $\theta' = (\beta \otimes B/I_1)(\widetilde{\delta})^{-1}$. Then, by (3.2), there exists a surjection $\theta: B^2 \twoheadrightarrow I_1$ and an element Θ' of $\mathrm{SL}_2(B/I_1)$ such that $(\theta \otimes B/I_1)\Theta' = \theta' = (\beta \otimes B/I_1)(\widetilde{\delta})^{-1}$. Now the proof is complete by setting $\delta_1 = \Theta'\widetilde{\delta}$. □

Let B be a ring and let P be a projective B-module of rank 2 with trivial determinant. Then, having a nondegenerate alternating form $\langle\, , \rangle$ on P is equivalent to giving an isomorphism $\lambda: \wedge^2(P) \stackrel{\sim}{\to} B$. Thus, we can identify the symplectic module $(P, \langle\, , \rangle)$ with (P, ω), where ω is the generator of $\wedge^2(P)$ given by $\lambda^{-1}(1)$. It is easy to see, that the isometry classes of $(P, \langle\, , \rangle)$ coincide with the isomorphism classes of (P, ω). In what follows, for convenience of notation, we will denote by (P, ω) a corresponding symplectic module.

Now we state a result, a proof of which is implicit in the proof of Theorem 7.2 of ([B-RS 2]).

PROPOSITION 5.3. *Let B be a ring and let P be a projective B-module of rank 2 with trivial determinant. Let ω_P be a generator of $\wedge^2(P)$. Let $\alpha: P \twoheadrightarrow I$ and $\theta: B^2 \twoheadrightarrow I_1$ be surjections, where I and I_1 are ideals of height 2 which are comaximal. Then there exists a projective B-module P_1 of rank 2 with trivial determinant, a generator ω_1 of $\wedge^2(P_1)$, a surjection $\beta_1: P_1 \twoheadrightarrow I \cap I_1$ and maps $\lambda_1: P_1 \to P$, $\lambda_2: P_1 \to B^2$ such that*

(i) $\alpha\lambda_1 = \beta_1 = \theta\lambda_2$.
(ii) $\wedge^2\lambda_1(\omega_1) = u\omega_P$, $u - 1 \in I$, $\wedge^2\lambda_2(\omega_1) = v(e_1 \wedge e_2)$, $v - 1 \in I_1$ *where* $e_1 = (1, 0)$, $e_2 = (0, 1)$.
(iii) $(B^2, (e_1 \wedge e_2)) \perp (P, \omega_P) \cong (B^2, e_1 \wedge e_2) \perp (P_1, \omega_1)$.

The following proposition is an extension of Proposition 3.7 to the polynomial situation.

PROPOSITION 5.4. *Let R be a ring of dimension two with $2 \in R^*$ Let $A = R[X_1, \ldots, X_r]$ and let P, Q be two projective A-modules of rank 2 such that $A^n \oplus P \cong A^n \oplus Q$. Let L be a rank one projective R-module such that $L \otimes_R A \cong \wedge^2(P) \cong \wedge^2(Q)$. Let $\chi: \wedge^2(P) \stackrel{\cong}{\to} \wedge^2(Q)$ be an isomorphism . If $L \oplus R$ is cancellative, then there exists an isomorphism $\Delta: P \stackrel{\cong}{\to} Q$ such that $\wedge^2(\Delta) = \chi$.*

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Proof. Let N denote the nilradical of R. Then, since $2 \in A^*$, every element of A^* which is congruent to 1 modulo NA has a square root in A^*. Therefore, it is enough to prove the result in the case R is reduced.

Let bar denote reduction modulo $(X_1, \dots, X_r)$. Then $R^n \oplus \overline{P} \cong R^n \oplus \overline{Q}$. Since $L \cong \wedge^2(\overline{P})$ and $L \oplus R$ is cancellative, by Proposition 3.7, there exists an isomorphism $\widetilde{\Delta}$: $\overline{P} \overset{\cong}{\to} \overline{Q}$ such that $\wedge^2(\widetilde{\Delta}) = \overline{\chi}$.

If Q is extended from R, then, by Lemma 5.1, P is extended from R. Therefore $\widetilde{\Delta}$ can be lifted to an isomorphism Δ: $P \overset{\cong}{\to} Q$. Since R is reduced and $\wedge^2(\widetilde{\Delta}) = \overline{\chi}$, $\wedge^2(\Delta) = \chi$.So we assume that Q is not extended from R.

Let $J = \{a \in R$: Q_a *extended from* R_a.$\}$ Then,by ([Q], Theorem 1), J is an ideal of R and since $\wedge^2(Q)$ is extended from R, by ([B-R 1], Theorem 3.1), $\mathrm{ht}(J) = 2$.

Let $S = R_{1+J}$, $B = A_{1+J} = S[X_1, \dots, X_r]$. Then S is a semilocal (reduced) ring. Hence $L_{1+J} \cong S$. Therefore P_{1+J} and Q_{1+J} have trivial determinant. Let ω_P, ω_Q be generators of $\wedge^2(P_{1+J}), \wedge^2(Q_{1+J})$ respectively such that $\chi_{1+J}(\omega_P) = \omega_Q$.

By Lemma 5.2, there exist comaximal ideals I, I_1 of B with $\mathrm{ht}(I) = 2 = \mathrm{ht}(I_1)$, surjections α: $P_{1+J} \twoheadrightarrow I$, β: $Q_{1+J} \twoheadrightarrow I \cap I_1$, θ: $B^2 \twoheadrightarrow I_1$ and isomorphisms δ: $Q_{1+J}/IQ_{1+J} \overset{\cong}{\to} P_{1+J}/IP_{1+J}$, δ_1: $Q_{1+J}/I_1Q_{1+J} \overset{\cong}{\to} (B/I_1)^2$ such that

(i) $(\alpha \otimes B/I)\delta = \beta \otimes B/I$.
(ii) $(\theta \otimes B/I_1)\delta_1 = \beta \otimes B/I_1$.
(iii) $\wedge^2(\delta)(\omega_Q \otimes B/I) = \omega_P \otimes B/I$.
(iv) $\wedge^2(\delta_1)(\omega_Q \otimes B/I_1) = (e_1 \wedge e_2) \otimes B/I$ *where* $e_1 = (1, 0), e_2 = (0, 1)$.

Therefore, by Proposition 5.3, there exists a projective B-module P_1 of rank 2 having trivial determinant, a generator ω_1 of $\wedge^2(P_1)$, a surjection β_1: $P_1 \twoheadrightarrow I \cap I_1$ and maps λ_1: $P_1 \to P_{1+J}$, λ_2: $P_1 \to B^2$ such that

(i) $\lambda_1 = \beta_1 = \theta\lambda_2$,
(ii) $\wedge^2\lambda_1(\omega_1) = u\omega_P$, $u - 1 \in I$, $\wedge^2\lambda_2(\omega_1) = ve_1 \wedge e_2$, $v - 1 \in I_1$,

where $e_1 = (1, 0), e_2 = (0, 1)$ and

(iii) $(B^2, e_1 \wedge e_2) \perp (P_{1+J}, \omega_P) \cong (B^2, e_1 \wedge e_2) \perp (P_1, \omega_1)$.

But then, by (4.8), the symplectic modules (P_{1+J}, ω_P) and (P_1, ω_1) are isomorphic.

Let $K = I \cap I_1$. Then we have surjections β_1: $P_1 \twoheadrightarrow K$ β: $Q_{1+J} \twoheadrightarrow K$, Let $\widetilde{\Gamma}$: $P_1/KP_1 \overset{\cong}{\to} Q_{1+J}/KQ_{1+J}$ be an isomorphism defined as $\widetilde{\Gamma} \otimes B/I = \delta^{-1}(\lambda_1 \otimes B/I)$ and $\widetilde{\Gamma} \otimes B/I_1 = {\delta_1}^{-1}(\lambda_2 \otimes B/I_2)$. Then it is easy to see that $\wedge^2(\widetilde{\Gamma})(\omega_1 \otimes B/K) = \omega_Q \otimes B/K$ and $(\beta \otimes B/K)\widetilde{\Gamma} = \beta_1 \otimes B/K$. Therefore, by Lemma 3.5, the symplectic modules (P_1, ω_1) and (Q_{1+J}, ω_Q) are isomorphic. Hence, there exists an isomorphism Δ_1: $P_{1+J} \overset{\cong}{\to} Q_{1+J}$ such that $\wedge^2(\Delta_1)(\omega_P) = \omega_Q$.

Since R is semilocal, $\overline{P}_{1+J}$ is free and $\mathrm{SL}_2(R) = E_2(R)$. Therefore, in view of Proposition 2.6, we can assume that Δ_1 is a lift of $\widetilde{\Delta}_{1+J}$. Therefore there exists $a \in J$ such that Δ_1 is an isomorphism from P_b to Q_b lifting $\widetilde{\Delta}_b$, where $b = 1 + a$.

Since P_a and Q_a are extended from R_a, there exists an isomorphism Δ_2: $P_a \overset{\cong}{\to} Q_a$ which is a lift of $\widetilde{\Delta}_a$.

Since the automorphism $(\Delta_2)_b^{-1}(\Delta_1)_a$ of the extended module P_{ab} is identity modulo $(X_1, \ldots, X_r)$, by Lemma 2.4, Δ_1 and Δ_2 patch up and give rise to an isomorphism Δ: $P \overset{\cong}{\to} Q$ which is a lift of $\widetilde{\Delta}$. Since R is reduced and $\wedge^2(\widetilde{\Delta}) = \overline{\chi}$, it follows that $\wedge^2(\Delta) = \chi$. □

THEOREM 5.5. *Let k be a perfect field of characteristic $\neq 2$ and of cohomological dimension $\leqslant 1$. Let R be a normal affine surface over k. Then every projective $R[X_1, \ldots, X_r]$-module is cancellative.*

Proof. Let P be a projective $R[X_1, \ldots, X_r]$-module. If rank $P = 1$ then obviously P is cancellative. So we assume that rank $P \geqslant 2$.

If rank $P > 2$, then, by ([B-R 2], Theorem 3.1), P is cancellative. If rank $P = 2$, then, since R is normal, there exists $L \in \mathrm{Pic}(R)$ such that $L \otimes_R R[X_1, \ldots, X_r] \cong \wedge^2(P)$. Since, by Therem 3.9, $L \oplus R$ is cancellative, by Proposition 5.4, P is cancellative. □

Remark 5.6. Let k be a field with $2 \in k^*$. Assume that either k is algebraically closed or finite. Let R be a reduced two-dimensional affine k algebra. Then, by ([M-S], Theorem 4) (if k is algebraically closed) or by ([M-M-R], Corollary 2.5) (if k is finite), any projective R-module is cancellative. If R is seminormal, then Pic $R[X_1, \ldots, X_r] = \mathrm{Pic}(R)$ and hence, by Proposition 5.4 and ([B-R 2], Theorem 3.1), all projective $R[X_1, \ldots, X_r]$-modules are cancellative. Even if R is not seminormal, a similar result holds. The proof is a bit technical and involves a conductor diagram between R and its seminormalisation.

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