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ON DIFFERENCE OPERATORS AND THEIR FACTORIZATION

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1. Introduction. Throughout this paper A will be used to denote a given set and g a permutation of it. We shall assume that there is a subset $C \subseteq A$ so that

$$(1) \qquad A = \bigcup_{i \in \mathbf{Z}} g^i(C) \quad \text{and} \quad g^i(C) \cap g^j(C) = \emptyset, \, i \neq j.$$

Here $\mathbf{Z}$ denotes the set of integers. For $x \in A$ it now follows that there is an unique $\alpha(x) \in \mathbf{Z}$ so that

$$(2) \qquad g^{\alpha(x)}x \in C,$$

and then also

$$\alpha(gx) = \alpha(x) - 1.$$

In general we shall be concerned with solving the following equation for u

$$(3) \qquad \sum_{i=n}^{r} p_i(x)u(g^i x) = v(x), \quad x \in A,$$

where p_i, $n \leqq i \leqq r$, and v are given real valued functions on A and $p_n p_r$ does not vanish on A. For $B \subseteq A$, $F(B)$ will denote the set of all real valued functions defined on B. We let $E:F(A) \rightarrow F(A)$ be given by

$$Eu(x) = u(gx), \quad x \in A, u \in F(A).$$

A function $L:F(A) \rightarrow F(A)$ of the form

$$(4) \qquad Lu = \sum_{i=n}^{r} p_i E^i u, \quad u \in F(A),$$

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is called a *linear difference operator of order* $r - n$. We can then rewrite (3) as

$$Lu = v.$$

The difference operator $\Delta: F(A) \to F(A)$ is defined as

$$\Delta = E - I$$

where I is the identity operator, $Iu = u$.

Henceforth it will be assumed that L is a difference operator and that it is given by (4).

Difference operators and equations have been discussed extensively before [**1**, **5**, **6**, **7**, **8**] and results concerning existence and construction of solutions of (3) are known in various forms, [**5**, p. 147] and [**7**, p. 40]. Our approach follows most closely the treatment in [**5**]. However, because we assume L to be given by a summation of the form $\sum_{i=n}^{r}$ rather than $\sum_{i=0}^{r}$, the results in [**5**] are not sufficiently general and accordingly we have developed an appropriate version of these results in Sections 2 to 5. Our main purpose in using this approach is to give more symmetry to the theory of difference operators.

In Section 5, we introduce the one-sided Green's function of L. This enables us to write down an explicit solution of (3) once we know certain solutions of $Lu = 0$.

In Section 6 it is shown that there is a uniquely determined difference operator L^* so that for each $s \in \mathbf{Z}$ there is a unique bilinear form B_s in u, v satisfying

$$vLu - E^s(uL^*v) = \Delta(B_s(u, v)), \quad u, v \in F(A).$$

L^* is called the *adjoint* of L and it is of the form

$$L^* = \sum_{i=-r}^{-n} q_i E^i.$$

It is shown that $(L^*)^* = L$. Also, if H, H^* respectively denote the one-sided Green's functions of L, L^*, then $H(x, y) = -H^*(x, y)$ if $y = g^k x$ for some $k \in \mathbf{Z}$. These results have a more symmetrical form than those in [**6**, p. 49-50] with which they should be compared.

In Section 7 we introduce the idea of conjugate solutions of $Lu = 0$ and $L^*u = 0$ and we show how to construct such. The relevance of these ideas to the factorization of L as RQ or R^*VQ is discussed in Section 8.

The results obtained in this paper have been motivated by recent work on differential equations, [**2**, **4**, **9**, **10**], particularly the work of Zettl.

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It should be noted that in some applications it may be necessary to solve a difference equation of the form (3) where p_i, $n \leqq i \leqq r$, and v are defined only on a subset $B \subseteq A$. Such an equation may be reduced to an equivalent one on A by letting $p_i = 1$ on $A - B$ for $n \leqq i \leqq r$, or even by putting $p_n = p_r = 1$ on $A - B$ and letting p_i, $n \leqq i \leqq r$, be arbitrary on $A - B$. All that is necessary is to extend the definitions of p_i, $n \leqq i \leqq r$, to the whole of A so that $p_n p_r$ does not vanish on A. Hence, if only applications of the above type are considered, there is no loss in restricting our attention to the equation (3).

2. Existence of solutions of $Lu = v$. In this section we are concerned with the existence and uniqueness of solutions to $Lu = v$ on A.

THEOREM 2.1. *Let* $v \in F(A)$ *be given and let* L *be given by* (4). *Then the equation* (3) *has a solution* $u \in F(A)$. *If* $r = n$ *this solution is unique, while in the case* $r - n \geqq 1$, *if*

$$B = \bigcup_{i=n}^{r-1} g^i(C)$$

and $u_0 \in F(B)$ *is given, there is a unique solution of* $Lu = v$ *on* A *so that* $u = u_0$ *on* B.

Proof. The result when $r = n$ is clear from (3). In the case $r - n \geqq 1$, we see from (3) that $Lu = v$ on A if, and only if,

$$u(gx) = \frac{v(g^{-r+1}x) - \sum_{i=n}^{r-1} p_i(g^{-r+1}x)u(g^{i+1-r}x)}{p_r(g^{-r+1}x)}, \quad x \in A,$$

or, equivalently,

$$u(g^{-1}x) = \frac{v(g^{-n-1}x) - \sum_{j=n+1}^{r} p_j(g^{-n-1}x)u(g^{j-n-1}x)}{p_n(g^{-n-1}x)}, \quad x \in A.$$

Now if $x \in g^{r-1}(C)$, then $g^{i+1-r}x \in B$ for $n \leqq i \leqq r-1$. Also, if $x \in g^n(C)$, then $g^{j-n-1}x \in B$ for $n + 1 \leqq j \leqq r$. Hence if we set $u = u_0$ on B, the above expressions for $u(gx)$ and $u(g^{-1}x)$ can be used to define u on $B \cup g^r(C) \cup g^{n-1}(C)$ so that the equation $Lu = v$ is satisfied on $C \cup g^{-1}(C)$. Now we repeat the argument to define u on $B \cup g^r(C) \cup g^{r+1}(C) \cup g^{n-1}(C) \cup g^{n-2}(C)$ with $Lu = v$ being satisfied on $C \cup g(C) \cup$

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$g^{-1}(C) \cup g^{-2}(C)$. The process may be continued until u is defined on all of A and $Lu = v$ on A with $u = u_0$ on B. This method produces a unique solution u on A of $Lu = v$ with $u = u_0$ on B.

3. The equation $\Delta u = v$. This equation is a special case of the equation $Lu = v$, where $L = \Delta$, $n = 0$, $r = 1$, $p_0 = -1$, $p_1 = 1$. Theorem 2.1 gives the existence of a unique solution of $\Delta u = v$ coinciding with a given function on C. In this case we can be more explicit.

THEOREM. 3.1. *Let* $v \in F(A)$, $u_0 \in F(C)$ *be given. Then the equation*

$$u(gx) - u(x) = v(x),\ x \in A,$$

has a unique solution u *satisfying* $u = u_0$ *on* C. *Also if* $\alpha(x)$ *is given by* (2), *then*

$$u(x) = u_0(g^{\alpha(x)}x) - (\operatorname{sign} \alpha(x)) \sum_{k \in I(x)} v(g^k x),$$

where

$$\begin{aligned} I(x) &= \{\alpha(x), \alpha(x)+1, \dots, -1\}, && \text{if } \alpha(x) \leqq -1, \\ &= \{0, 1, \dots, \alpha(x) - 1\}, && \text{if } \alpha(x) \geqq 1, \\ &= \emptyset, && \text{if } \alpha(x) = 0. \end{aligned}$$

We interpret an empty summation to be zero, which is equivalent to $u(x) = u_0(x)$, $x \in C$.

Proof. Let $\Delta u = v$ on A, $u = u_0$ on C and consider $\alpha(x) \geqq 1$. Then for $k = 1, 2, \dots, \alpha(x)$,

$$u(g^k x) - u(g^{k-1}x) = v(g^{k-1}x).$$

We add these equations to obtain

$$u(g^{\alpha(x)}x) - u(x) = \sum_{k=0}^{\alpha(x)-1} v(g^k x) = \sum_{k \in I(x)} v(g^k x).$$

When $\alpha(x) \leqq -1$, we have

$$u(g^{-k}x - u(g^{-k-1}x) = v(g^{-k-1}x), \quad k = 0, 1, \dots, -\alpha(x) - 1,$$

and adding, we see that

$$u(x) - u(g^{\alpha(x)}x) = \sum_{k=\alpha(x)}^{-1} v(g^k x) = \sum_{k \in I(x)} v(g^k x).$$

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Note that $g^{\alpha(x)}x \in C$; the result now follows immediately.

4. Solutions of $Lu = 0$. We need some preliminary definitions. A function $f \in F(A)$ is said to be *g-invariant* if $f(g(x)) = f(x)$ for all $x \in A$. It is clear from (1) that a function $f \in F(g^q C)$, for some $q \in \mathbf{Z}$, can be extended uniquely to A so as to produce a g-invariant function on all of A.

A collection of functions $f_i \in F(A)$, $1 \leqq i \leqq p$, is said to be *g-independent* if whenever $h_1, \ldots, h_p$ are g-invariant in $F(A)$ and

$$\sum_{i=1}^{p} h_i f_i = 0 \quad \text{on } A,$$

then $h_i = 0$ on A, $1 \leqq i \leqq p$.

Now let L be given by (4) where $r - n \geqq 1$, let

$$B = \bigcup_{i=n}^{r-1} g^i(C),$$

and define functions u_{0i}, $n \leqq i \leqq r-1$, on B as follows:

$$\begin{aligned} u_{0i}(x) &= 1, \quad \text{if } x \in g^i(C), \\ &= 0, \quad \text{if } x \in g^j(C), j \neq i. \end{aligned}$$

By Theorem 2.1 we may find solutions u_i, $n \leqq i \leqq r - 1$, satisfying

(5) $\qquad Lu_i = 0$ on A, $u_i = u_{0i}$ on B.

THEOREM 4.1. *The functions* u_i, $n \leqq i \leqq r-1$, *are g-independent in* $F(A)$ *and* $Lu_i = 0$ *on* A *for each* i.

Proof. We need only establish g-independence. Let h_i, $n \leqq i \leqq r - 1$, be g-invariant and let

$$\sum_{i=n}^{r-1} h_i u_i = 0 \quad \text{on } A.$$

Because $u_i = u_{0i}$ on B we deduce that $h_j(x) = 0$ if $x \in g^j(C)$, $n \leqq j \leqq r - 1$. The g-invariance of h_j yields $h_j = 0$ on A, $n \leqq j \leqq r-1$, as required.

Now suppose that w_j, $n < j \leqq r - 1$, are $r - n$ functions in $F(A)$. The following determinant, known as the *Casorati* of $w_n, \ldots, w_{r-1}$ will occur frequently in the sequel. It is given by

$$C(w_n, \ldots, w_{r-1})(x) = \det (w_j(g^i x))_{n \leqq i,j \leqq r-1}, \quad x \in A.$$

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This quantity seems to have been introduced by Casorati, [**3**, p. 19]. Its rôle in the theory of difference equations is analogous to that of the Wronskian in the theory of differential equations.

THEOREM 4.2. *Let* $Lw_j = 0$ *on* A, $n \leqq j \leqq r - 1$. *Then*

$$C(w_n, \dots, w_{r-1})(gx) = \frac{(-1)^{r-n} p_n(x) C(w_n, \dots, w_{r-1})(x)}{p_r(x)}, \quad x \in A.$$

Proof. Since $Lw_j = 0$ on A, we see that the entry $w_j(g^r x)$ in the last row of $C(w_n, \dots, w_{r-1})(gx)$ may be replaced by

$$-\sum_{k=n}^{r-1} p_k(x) w_j(g^k x)/p_r(x), \quad n \leqq j \leqq r - 1.$$

The result now follows easily.

COROLLARY 4.3. *Let* u_i, $n \leqq i \leqq r - 1$, *be the* $r - n$ *solutions of* $Lu = 0$ *given in* (5). *Then* $C(u_n, \dots, u_{r-1})$ *does not vanish on* A.

Proof. If $x \in C$ we observe that $C(u_n, \dots, u_{r-1})(x) = 1$. The result is now immediate from (1) and Theorem 4.2.

THEOREM 4.4. *Let* $Lw_i = 0$ *on* A *for* $n \leqq i \leqq r - 1$. *Then the following are equivalent.*

(i) $C(w_n, \dots, w_{r-1})$ *does not vanish on* A,

(ii) $w_n, \dots, w_{r-1}$ *are* g*-independent on* A,

(iii) *There is* $p \in \mathbf{Z}$ *so that* $w_n, \dots, w_{r-1}$ *are* g*-independent over* $g^p(C)$,

(iv) *If* $Lu = 0$ *on* A, *there are* g*-invariant functions* f_i, $n \leqq i \leqq r - 1$, *so that*

$$u = \sum_{i=n}^{r-1} f_i w_i \text{ on } A.$$

Proof. Let (i) hold and suppose h_i, $n \leqq i \leqq r-1$, are g-invariant so that

$$\sum_{i=n}^{r-1} h_i w_i = 0 \text{ on } A.$$

Then for $x \in A$ and $n \leqq j \leqq r - 1$,

$$\sum_{i=n}^{r-1} h_i(x) w_i(g^j x) = 0.$$

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Since det $(w_i(g^j x)) \neq 0$, we deduce that $h_i(x) = 0$ and (ii) holds.

That (ii) implies (iii) is obvious. Now let (iii) hold. Assume that

$$\sum_{i=n}^{r-1} h_i w_i = 0 \text{ on } A$$

where each h_i is g-invariant. By (iii), $h_i = 0$ on $g^p(C)$ and thus by g-invariance, $h_i = 0$ on A. Hence (iii) implies (ii).

Now let (ii) hold and suppose that $C(w_n, \dots, w_{r-1})(x) = 0$. By Theorem 4.2. we may assume that $x \in C$. Choose $h_i(x)$, $n \leqq i \leqq r - 1$, not all zero, so that

$$\sum_{j=n}^{r-1} h_j(x) w_j(g^i x) = 0, \quad n \leqq i \leqq r - 1.$$

If $x \in C$ and $C(w_n, \dots, w_{r-1})(x) \neq 0$, set $h_i(x) = 0$, $n \leqq i \leqq r - 1$. Now each h_j can be extended from C to the whole of A to give a g-invariant function, also denoted by h_j, on A. Then

$$\sum_{i=n}^{r-1} h_i w_i = 0 \text{ on } B = \bigcup_{i=n}^{r-1} g^i(C)$$

and is a solution of $Lu = 0$ on A. By Theorem 2.1,

$$\sum_{i=n}^{r-1} h_i w_i = 0 \text{ on } A,$$

which contradicts (ii). Hence (ii) implies (i).

Now let (i) hold and let $Lu = 0$ on A. Let $f_i \in F(A)$, $n \leqq i \leqq r - 1$, be g-invariant and such that

$$u(g^j x) = \sum_{i=n}^{r-1} f_i(x) w_i(g^j x), \quad x \in C, n \leqq j \leqq r - 1.$$

Then

$$Lu = 0, \ L\left(\sum_{i=n}^{r-1} f_i w_i\right) = 0 \quad \text{and}$$

$$u = \sum_{i=n}^{r-1} f_i w_i \text{ on } B = \bigcup_{i=n}^{r-1} g^i(C).$$

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By Theorem 2.1,

$$u = \sum_{i=n}^{r-1} f_i w_i.$$

Hence (iv) holds.

Finally, let (iv) hold. Then we may write the functions u_i in (5) in the form

$$u_i = \sum_{k=n}^{r-1} f_{ik} w_k, \quad n \leqq i \leqq r - 1,$$

where the f_{ik} are g-invariant in $F(A)$. We then have

$$u_i(g^j x) = \sum_{k=n}^{r-1} f_{ik}(x) w_k(g^j x), \quad n \leqq i, j \leqq r - 1.$$

From Corollary 4.3 we deduce that $C(w_n, \ldots, w_{r-1})$ does not vanish on A and thus (i) holds.

Bearing this result in mind, we call a set of solutions $w_n, \ldots, w_{r-1}$ of $Lu = 0$ having the properties (i) to (iv) a *fundamental set of solutions*.

5. The equation $Lu = v$ and the Green's function. We now show how the equation $Lu = v$ may be solved for u given a fundamental set of solutions of $Lu = 0$. We shall let L be given by (4) with $r - n \geqq 1$. The method parallels variation of parameters used in the study of differential equations and leads to the concept of the Green's function of the difference operator L. The approach is similar to that in [**5**, pp. 133-149].

Throughout this section, w_i, $n \leqq i \leqq r - 1$, will be a fundamental set of solutions of $Lu = 0$. Hence $C(w_n, \ldots, w_{r-1})$ never vanishes. In trying to solve $Lu = v$ we seek solutions of the form

$$(6) \qquad u = \sum_{j=n}^{r-1} v_j w_j$$

where $v_n, \ldots, v_{r-1} \in F(A)$ are to be determined.

LEMMA 5.1. *Let $q \in \mathbf{Z}$ be given, where $n \leqq q \leqq r - 1$. Let v_j, $n \leqq j \leqq r - 1$, be functions in $F(A)$ so that*

$$(7) \qquad \sum_{j=n}^{r-1} \Delta(E^q v_j) E^k w_j = 0, \quad n+1 \leqq k \leqq r - 1,$$

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and

$$(8)\qquad \sum_{j=n}^{r-1} \Delta(E^q v_j)E^r w_j = v/p_r.$$

If $r - n = 1$, *only equation* (8) *is considered. Then if* u *is given by* (6), $Lu = v$ *on* A.

Proof. Consider the statement

$$(9)\qquad E^k u = \sum_{j=n}^{r-1} E^q v_j E^k w_j.$$

Since u is given by (6), this is true for $k = q$. Suppose now that (9) holds for some k, where $q \leqq k \leqq r - 2$. Then we have

$$E^{k+1}u = E(E^k u) = \sum_{j=n}^{r-1} E^{q+1} v_j E^{k+1} w_j, \quad \text{by (9)},$$

$$= \sum_{j=n}^{r-1} \Delta(E^q v_j)E^{k+1} w_j + \sum_{j=n}^{r-1} E^q v_j E^{k+1} w_j$$

$$= \sum_{j=n}^{r-1} E^q v_j E^{k+1} w_j, \quad \text{by (7)},$$

as $n + 1 \leqq k + 1 \leqq r - 1$. Hence (9) holds with $k + 1$ in place of k.

Also, if (9) holds for some k where $n+1 \leqq k \leqq q$, then

$$E^{k-1}u = E^{-1}(E^k u) = \sum_{j=n}^{r-1} E^{q-1} v_j E^{k-1} w_j, \quad \text{by (9)},$$

$$= \sum_{j=n}^{r-1} E^q v_j E^{k-1} w_j - E^{-1}\left(\sum_{j=n}^{r-1} \Delta(E^q v_j)E^k w_j\right)$$

$$= \sum_{j=n}^{r-1} E^q v_j E^{k-1} w_j, \quad \text{by (7)}.$$

Hence (9) holds with $k - 1$ in place of k. We now deduce by induction that (9) holds for $n \leqq k \leqq r - 1$.

To prove that $Lu = v$, observe that

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$$\begin{aligned} Lu &= \sum_{k=n}^{r} p_k E^k u, \quad \text{by (4)}, \\ &= p_r E^r u + \sum_{k=n}^{r-1} p_k E^k u \\ &= p_r E^r u + \sum_{j=n}^{r-1} \sum_{k=n}^{r-1} p_k E^q v_j E^k w_j, \quad \text{by (9)} \\ &= p_r E^r u - \sum_{j=n}^{r-1} p_r E^q v_j E^r w_j, \quad \text{as } Lw_j = 0, \\ &= p_r \left(\sum_{j=n}^{r-1} \Delta(E^q v_j) E^r w_j \right), \quad \text{by (9) with } k = r-1, \\ &= v, \quad \text{by (8)}. \end{aligned}$$

As a result of Lemma 5.1, we see that to solve $Lu = v$ in the case where L has order at least 2, it is sufficient to solve the equations (7), (8) for the v_j, $n \leqq j \leqq r - 1$, and put these v_j in (6). To this end we define, if $r - n \geqq 2$, $C_j(w_n, \ldots, w_{r-1})(x)$ for $x \in A$ and $n \leqq j \leqq r - 1$, to be the determinant obtained from the Casorati $C(w_n, \ldots, w_{r-1})(x)$ by deleting row $r - n$ (the last row) and column $j - n + 1$ (the column containing w_j). If $r - n = 1$ then $j = n$ and we take $C_n(w_n)$ to be identically 1.

We now let

$$(10) \qquad w_j^* = (-1)^{r-j+1} \frac{EC_j(w_n, \ldots, w_{r-1})}{p_r EC(w_n, \ldots, w_{r-1})}, \quad n \leqq j \leqq r - 1,$$

$$(11) \qquad H(x, y) = \sum_{j=n}^{r-1} w_j(x) w_j^*(y), \quad x, y \in A.$$

Using the definition of w_j^*, we see that in the case $r - n \geqq 2$,

$$H(x, y) = \frac{(-1)^{r-n+1} \begin{vmatrix} w_n(x) & \ldots\ldots & w_{r-1}(x) \\ w_n(g^{n+1}y) & \ldots\ldots & w_{r-1}(g^{n+1}y) \\ \vdots & & \vdots \\ w_n(g^{r-1}y) & \ldots\ldots & w_{r-1}(g^{r-1}y) \end{vmatrix}}{\begin{vmatrix} w_n(g^{n+1}y) & \ldots\ldots & w_{r-1}(g^{n+1}y) \\ \vdots & & \vdots \\ w_n(g^r y) & \ldots\ldots & w_{r-1}(g^r y) \end{vmatrix}} \quad x, y \in A.$$

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In the case $r - n = 1$, we have

$$H(x, y) = w_n(x)/p_{n+1}(y)w_n(g^{n+1}y), \quad x, y \in A.$$

The function H is known as the (one sided) *Green's function* of L, a terminology which will be justified in the sequel. In general, the value of $H(x, y)$ will depend on the choice of the fundamental set of solutions of $Lu = 0$ used to define H in (11). However, the significant aspect of this is that if $y = g^k x$ for some $k \in \mathbf{Z}$, then the value of $H(x, y)$ is independent of the choice of the fundamental set w_j, $n \leqq j \leqq r - 1$. This is proved later. We shall continue to refer to the Green's function H of L and discuss first some useful properties which follow easily from the above expressions for $H(x, y)$.

THEOREM 5.2. *Let H be the Green's function of the difference operator L, where L is given by* (4). *Then*

(i) $L(x \to H(x, y)) = 0$, *for each* $y \in A$;

(ii) *If* $r - n \geqq 2$, $x \in A$ *and* $n + 1 \leqq k \leqq r - 1$, *then*

$$H(g^k x, x) = H(x, g^{-k}x) = 0;$$

(iii) $H(g^n x, x) = -1/p_n(x)$, $x \in A$;

(iv) $H(g^r x, x) = 1/p_r(x)$, $x \in A$.

LEMMA 5.3. *Let $n \leqq q \leqq r - 1$ and let α, I be as in Theorem* 3.1, *H as in Theorem* 5.2. *Then for*

$$x \in \bigcup_{j=n}^{r-1} g^{-q+j}(C)$$

and $k \in I(x)$, $H(x, g^{-q+k}x) = 0$.

Proof. Let $x \in g^{-q+j}(C)$, for some $n \leqq j \leqq r - 1$. Then $\alpha(x) = q - j$. We may take $I(x) \neq \emptyset$, that is $j \neq q$.

If $g - j \geqq 1$, then $\alpha(x) = \{0, 1, \ldots, q-j-1\}$, so if $k \in I(x)$ and $s = q - k$, we have $n + 1 \leqq s \leqq r - 1$. If $q-j \leqq -1$, then $I(x) = \{q-j, \ldots, -1\}$ and if $k \in I(x)$, $s = q - k$, we have again $n + 1 \leqq s \leqq r - 1$. Hence, in either case, if $k \in I(x)$,

$$H(x, g^{-q+k}x) = H(g^s g^{-q+k}x, g^{-q+k}x) = 0,$$

by Theorem 5.2 (ii), since $n+1 \leqq s \leqq r - 1$.

THEOREM 5.4. *Let L be given by* (4), *let H be the Green's function of L, let α, I be as described in Theorem* 3.1 *and let $q \in \mathbf{Z}$, $n \leqq q \leqq r - 1$. Let $v \in F(A)$ be given. Then if*

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$$u(x) = -(\text{sign } \alpha(x)) \sum_{k \in I(x)} v(g^{-q+k} x)H(x, g^{-q+k} x), \quad x \in A,$$

then $Lu = v$ *on* A *and* $u = 0$ *on* $\cup_{j=n}^{r-1} g^{-q+j}(C)$, *that is,* $E^k u = 0$ *on* C *for* $n - q \leqq k \leqq r - q - 1$.

Proof. The $r - n$ equations in (7), (8) may be solved for $\Delta(E^q v_j)$ using Cramer's rule. We find that

$$\Delta(E^q v_j) = v w_j^*, \ n \leqq j \leqq r - 1, \text{ or}$$

$$\Delta v_j = E^{-q} v E^{-q} w_j^*, \ n \leqq j \leqq r - 1.$$

Using Theorem 3.1 we may solve this for v_j subject to the condition that $v_j = 0$ on C to obtain

$$v_j(x) = -(\text{sign } \alpha(x)) \sum_{k \in I(x)} v(g^{-q+k} x) w_j^*(g^{-q+k} x), \quad x \in A.$$

Now we have

$$u(x) = \sum_{j=n}^{r-1} v_j(x) w_j(x)$$

$$= -(\text{sign } \alpha(x)) \sum_{k \in I(x)} v(g^{-q+k} x) \sum_{j=n}^{r-1} w_j(x) w_j^*(g^{-q+k} x),$$

$$= -(\text{sign } \alpha(x)) \sum_{k \in I(x)} v(g^{-q+k} x) H(x, g^{-q+k} x).$$

Also, Lemma 5.3 shows that $u = 0$ on $\bigcup_{j=n}^{r-1} g^{-q+j}(C)$.

6. Adjoints and the Lagrange bracket. In this section we shall introduce the concept of the adjoint of a difference operator. By a *bilinear form* B we shall mean a bilinear function $B:F(A) \times F(A) \to F(A)$ which is of the form

$$(12) \qquad B(u, v) = \sum_{-p \leqq i,j \leqq p} f_{ij} E^i u E^j v, \quad u, v \in F(A),$$

where $p \in \mathbf{Z}$ and $f_{ij} \in F(A)$. We also adopt the following notation for summations:

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$$\boldsymbol{\Sigma}_{i=0}^{p} = \sum_{i=0}^{p}, \text{ if } p \geqq 0$$

$$= 0, \text{ if } p = -1,$$

$$= -\sum_{i=p+1}^{-1}, \text{ if } p \leqq -2.$$

LEMMA 6.1. *Let* $s_i \in F(A)$, $n \leqq i \leqq r$, *and define*

$$Mu = \sum_{i=n}^{r} s_i E^i u, \quad u \in F(A).$$

(i) *If there are functions* u_j, $n \leqq j \leqq r$, *so that* $Mu_j = 0$ *for each* j *and* $C(u_n, \ldots, u_r)$ *does not vanish on* A, *then* $s_i = 0$, $n \leqq i \leqq r$.

(ii) *If* $Mu = 0$ *on* A *for all* $u \in F(A)$, *then* $s_i = 0$, $n \leqq i \leqq r$.

Proof. (i) Suppose $s_i \neq 0$ for some i. We may assume that $s_n(x) \neq 0$ for some $x \in A$ and define

$$B = \{x | s_n(x) \neq 0\}.$$

Let r_1 be the largest integer, $n \leqq r_1 \leqq r$, so that $s_{r_1}(x) \neq 0$ for some $x \in B$. Let

$$B_0 = \{x | s_n(x) s_{r_1}(x) \neq 0\}.$$

B_0 is non void and $B_0 \subseteq B$. Now let $t_j = s_j$ on B_0 and $t_j = 1$ on $A - B_0$, $n \leqq j \leqq r_1$. If $r_1 + 1 \leqq j \leqq n$, let $t_j = 0$. For $u \in F(A)$ let

$$Nu = \sum_{i=n}^{r_1} t_i E^i u.$$

Then $Nu_j = Mu_j = 0$ on B_0, $n \leqq j \leqq r$, or

$$\sum_{i=n}^{r} t_i(x) u_j(g^i x) = 0, \quad n \leqq j \leqq r, \ x \in B_0.$$

Since $C(u_n, \ldots, u_r)$ does not vanish on A, we deduce that $t_i(x) = 0$ for $x \in B_0$, $n \leqq i \leqq r$, which contradicts the assumption that

$$t_n(x) t_{r_1}(x) = s_n(x) s_{r_1}(x) \neq 0, \quad x \in B_0.$$

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Hence $s_i = 0$ for all i and so (i) is proved.

To prove (ii), observe from Corollary 4.3 that there are u_j, $n \leqq j \leqq r$, so that $C(u_n, \ldots, u_r)$ does not vanish on A. The result follows from (i).

THEOREM 6.2. *Let L be given by* (4). *Then there is a unique difference operator L^* which has the property that for each $s \in \mathbf{Z}$, there is a unique bilinear form B_s such that*

$$vLu - E^s(uL^*v) = \Delta(B_s(u, v)), \quad u, v \in F(A).$$

We also have

$$(13) \quad L^* = \sum_{k=-r}^{-n} E^k p_{-k} E^k, \text{ and}$$

$$(14) \quad B_s(u, v) = \sum_{k=n}^{r} \sum_{j=0}^{k-s-1} E^{j+s} uE^{j-k+s}(p_k v).$$

Proof. To prove uniqueness of L^* and B_s, given existence, we show that

$$E^s(uL^*v) = -\Delta(B_s(u, v)), \quad \text{for } u, v \in F(A),$$

implies $L^* = 0$ and $B_s = 0$. Hence, with this assumption and with B_s given by (12), we may write

$$B_s(u, v) = \sum_{i=-p}^{p} b_i E^i u, \quad \text{where } b_i = \sum_{j=-p}^{p} f_{ij} E^j v.$$

We now have

$$\Delta B_s(u, v) = \sum_{i=-p+1}^{p} E^i u(Eb_{i-1} - b_i) + E^{p+1} uEb_p - E^{-p} ub_{-p}$$
$$= -E^s(uL^*v), \quad u, v \in F(A).$$

If v is fixed and we let

$$Mu = \Delta(B_s(u, v)) + E^s(uL^*v), \quad u \in F(A),$$

we deduce from Lemma 6.1 that $b_i = 0$, $-p \leqq i \leqq p$. Hence $B_s = 0$ and $L^* = 0$.

To establish existence of B_s and L^* we let B_s be given by (14), L^* be given by (13) and observe

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$$\Delta\left(\sum_{j=0}^{k-s-1} E^{j+s}uE^{j-k+s}(p_k v)\right) = E^k up_k v - E^s uE^{-k+s}(p_k v),$$
$$k \geqq s;$$
$$\Delta\left(\sum_{j=0}^{k-s-1} E^{j+s}uE^{j-k+s}(p_k v)\right) = E^k up_k v - E^s uE^{-k+s}(p_k v),$$
$$k \leqq s-1.$$

Adding these for $n \leqq k \leqq r$ and using (4) we obtain the desired result.

Note that L^* does not depend on s whereas B_s does. B_s is called the *Lagrange bracket for L of order s*. The difference operator L^* is called the *adjoint* of L and since $(E^{-r}p_r)(E^{-n}p_r)$ does not vanish on A, it is of the same order as L itself. We now list some easily established properties of the adjoint. L, M will denote difference operators of the form (4).

THEOREM 6.3.

(i) $(L+M)^* = L^* + M^*$,
(ii) $(\alpha L)^* = \alpha L^*$, $\alpha \in \mathbf{R}$,
(iii) $(LM)^* = M^*L^*$,
(iv) $(L^*)^* = L$.

Proof. The results follow either by direct calculation or by using the uniqueness result in Theorem 6.2.

THEOREM 6.4. *Let L be given by* (4) *with* $r-n \geqq 1$. *Let H be the Green's function of L and* H^* *the Green's function of* L^*. *Then*

$$H(g^i x, g^j x) = -H^*(g^j x, g^i x), \quad i, j \in \mathbf{Z},\ x \in A.$$

Proof. Let $y \in A$ and let $u(x) = H(x, y)$. Let $z \in A$ and put $v(x) = H^*(x, g^r z)$. Then we have

$$B_r(u, v) = -\sum_{k=n}^{r-1} \sum_{j=k-r}^{-1} E^{j+r}uE^{j-k+r}(p_k v). \tag{15}$$

Hence, by Theorem 5.2, since $E^{j+r}u(x) = H(g^{j+r}x, y)$,

$$B_r(u, v)(y) = -u(g^n y, y)p(y)v(y)$$
$$= v(y) = H^*(y, g^r z).$$

Also,

$$B_r(u, v)(z) = -\sum_{k=n}^{r-1} E^k u(z)E^0(p_k v)(z)$$

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$$= u(g^r z)p_r(z)v(z) - \left(\sum_{k=n}^{r} p_k(z)u(g^k z)\right) v(z)$$

$$= H(g^r z, y)p_r(z)H^*(z, g^r z), \text{ as } Lu = 0,$$

$$= -H(g^r z, y), \text{ by Theorem 5.2.}$$

Now $Lu = L^*v = 0$ on A so we deduce that $\Delta(B_r(u, v)) = 0$ on A. Hence if $y = g^k z$ for some $k \in \mathbf{Z}$,

$$B_r(u, v)(y) = B_r(u, v)(z)$$

so that

$$H^*(y, g^r z) = -H(g^r z, y),$$

from which the result follows.

COROLLARY 6.5. *If $y = g^k x$ for some $k \in \mathbf{Z}$, then the value of $H(x, y)$ is independent of the fundamental set of solutions of $Lu = 0$ used to define H.*

Proof. If H^* is the Green's function of L^* calculated from a particular fundamental set of solutions of $L^*u = 0$, Theorem 6.4 shows that $H(x, y) = -H^*(y, x)$ regardless of the fundamental set of solutions used to define H.

THEOREM 6.6. *Let L be given by* (4) *where $r - n \geqq 1$ and let w_j, $n \leqq j \leqq r - 1$, be a fundamental set of solutions of $Lu = 0$. Then the functions w_j^*, $n \leqq j \leqq r - 1$, given by* (10), *form a fundamental set of solutions of $L^*u = 0$.*

Proof. Let H, H^* be the Green's functions of L, L^* respectively. If $i \in \mathbf{Z}$, $x \in A$, we have

$$0 = (L^*(H^*(\cdot, g^i x)))(x) = \sum_{k=-r}^{-n} E^k p_{-k}(x)H^*(g^k x, g^i x)$$

$$= \sum_{k=-r}^{-n} E^k p_{-k}(x)H(g^i x, g^k x), \quad \text{by Theorem 6.4,}$$

$$= -(L^*(H(g^i x, \cdot)))(x)$$

$$= -\sum_{j=n}^{r-1} w_j(g^i x)L^*w_j^*(x), \quad \text{by (11).}$$

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Since $C(w_n, \ldots, w_{r-1})$ does not vanish on A we deduce that $L^*w_j^* = 0$ on A, $n \leqq j \leqq r - 1$.

To prove that the w_j^* are fundamental, let $z \in A$ and put $u = H(\cdot, z)$. Then $Lu = 0$ and if $L^*v = 0$ we have from Theorem 6.2 that

$$\Delta(B_r(u, v)) = 0.$$

It follows from (15) and Theorem 5.2 that $B_r(u, v)(z) = v(z)$. Hence

$$\begin{aligned} v(z) &= B_r(u, v)(z) \\ &= \sum_{j=n}^{r-1} B_r(w_j, v)(z)w_j^*(z), \text{ by (11)}, \\ &= \sum_{j=n}^{r-1} f_j(z)w_j^*(z), \end{aligned}$$

where

$$f_j = B_r(w_j, v), \; n \leqq j \leqq r-1.$$

Since

$$\Delta f_j = \Delta(B_r(w_j, v)) = vLw_j - E^r(w_j L^*v) = 0,$$

we see that each f_j is g-invariant, so by Theorem 4.4, the w_j^*, $n \leqq j \leqq r - 1$, form a fundamental set of solutions of $L^*u = 0$.

7. Conjugate solutions of $Lu = 0$ **and** $L^*u = 0$. Let $u, v \in F(A)$ be such that $Lu = L^*v = 0$ on A. From Theorem 6.1 we see that if $s \in \mathbf{Z}$, then

$$\Delta(B_s(u, v)) = 0 \text{ on } A.$$

Thus $B_s(u, v)$ is a g-invariant function and it shall be shown that $B_s(u, v)$ is independent of s. Accordingly, if $Lu = L^*v = 0$ on A we say that u and v are *conjugate solutions of* $Lu = 0$ *and* $L^*v = 0$ if $B_s(u, v) = 0$ for some, and hence all, $s \in \mathbf{Z}$.

Throughout this section, w_j, $n \leqq j \leqq r - 1$, will denote a fundamental set of solutions of $Lu = 0$ on A, where L is given by (4) and $r - n \geqq 1$. Let w_j^*, $n \leqq j \leqq r - 1$, be given by (10). By Theorem 6.5 the w_j^* form a fundamental set of solutions of $L^*u = 0$. Our main purpose is to show that for $i \neq j$, w_i and w_j^* are conjugate solutions of $Lu = 0$ and $L^*v = 0$. A corresponding result for differential equations was proved in [**10**].

LEMMA 7.1 *For* $u, v \in F(A)$ *and* $s \in \mathbf{Z}$ *we have*

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(i) $B_s(u, v) - B_{s+1}(u, v) = E^s(uL^*v)$,
(ii) $EB_s(u, v) - B_{s+1}(u, v) = vLu$.

Proof. We have

$$\Delta(B_s(u, v) - E^s(uL^*v)) = vLu - E^s(uL^*v) - E^{s+1}(uL^*v) + E^s(uL^*v)$$
$$= vLu - E^{s+1}(uL^*v)$$
$$= \Delta(B_{s+1}(u, v)).$$

Now $B_s(u, v) - E^s(uL^*v)$ is a bilinear form in u, v so we deduce from the uniqueness of B_{s+1} in Theorem 6.1 that

$$B_{s+1}(u, v) = B_s(u, v) - E^s(uL^*v),$$

so that (i) holds. (ii) may be proved in a like manner. Both (i) and (ii) may also be proved by direct calculation.

COROLLARY 7.2. *If $Lu = L^*v = 0$ on A then $B_s(u, v)$ is a g-invariant function on A which is independent of s.*

If $r - n \geqq 2$ we define $R_{i,j}(x)$, $x \in A$, $n \leqq i, j \leqq r - 1$, to be the minor of the entry $w_i(g^j x)$ in the Casorati determinant

$$\det (w_i(g^j x))_{n \leqq i,j \leqq r-1}.$$

If $r - n = 1$, we let $R_{n,n}(x) = 1$ for $x \in A$.

LEMMA 7.3 *Let $n \leqq i, j \leqq r - 1$. Then*

(i) $R_{i,n} = ER_{i,r-1}$, $r - n \geqq 1$

(ii) $R_{i,j} = (-1)^{r-n-1} \dfrac{p_r}{p_n} ER_{i,j-1} + (-1)^{j-n} \dfrac{p_j}{p_n} ER_{i,r-1}$,

$n + 1 \leqq j \leqq r - 1$, $r - n \geqq 2$.

Proof. (i) is clear from the definitions. To establish (ii), observe that since $Lw_k = 0$ we may replace the entry $w_k(g^n x)$ in $R_{i,j}(x)$ by

$$-\frac{p_r(x)}{p_n(x)} w_k(g^r x) - \frac{p_j(x)}{p_n(x)} w_k(g^j x),$$

for $n \leqq k \leqq r - 1$, $k \neq i$. If the resulting determinant is then expanded as the sum of two determinants, the result follows easily.

LEMMA 7.4. *For $n \leqq i, j \leqq r - 1$ we have*

$$(16) \qquad \sum_{k=j+1}^{r} E^{j-k}(p_k w_i^*) = (-1)^{i+j} R_{i,j} \Big/ C(w_n, \ldots, w_{r-1}).$$

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Proof. For brevity, C will be used in place of $C(w_n, \ldots, w_{r-1})$. From (10) we have

$$w_i^* = (-1)^{i+r-1}R_{i,n}/p_rEC, \quad n \leqq i \leqq r-1.$$

When $j = r - 1$, the left hand side of (16) is

$$E^{-1}(p_rw_i^*) = (-1)^{i+r-1}E^{-1}R_{i,n}/C,$$
$$= (-1)^{i+r-1}R_{i,r-1}/C, \quad \text{by Lemma 7.3 (i)},$$

and the result holds if $j = r - 1$. Now suppose that (16) holds for some j, $n + 1 \leqq j \leqq r - 1$. Then

$$\sum_{k=j}^{r} E^{j-1-k}(p_kw_i^*) = E^{-1}(p_jw_i^*) + E^{-1}\left(\sum_{k=j+1}^{r} E^{j-k}(p_kw_i^*)\right)$$

$$= E^{-1}(p_jw_i^*) + (-1)^{i+j}E^{-1}R_{i,j}/E^{-1}C$$

$$= (-1)^{i+r-1}\frac{E^{-1}p_j}{E^{-1}p_r}\frac{E^{-1}R_{i,n}}{C}$$

$$+ (-1)^{i+j+r-n-1}\frac{E^{-1}p_r}{E^{-1}p_n}\frac{R_{i,j-1}}{E^{-1}C}$$

$$+ (-1)^{i-n}\frac{E^{-1}p_j}{E^{-1}p_n}\frac{R_{i,r-1}}{E^{-1}C},$$

by Lemma 7.3 (ii)

$$= (-1)^{i+j-1}\frac{R_{i,j-1}}{C},$$

by Theorem 4.2 and Lemma 7.3 (i). Hence (16) holds for $j - 1$, and the proof of Lemma 7.4 follows by induction.

THEOREM 7.5. *Let w_i, $n \leqq i \leqq r - 1$, be a fundamental set of solutions of $Lu = 0$ on A and w_i^*, $n \leqq i \leqq r - 1$, be the fundamental set of solutions of $L^*v = 0$ given by* (10). *Then if $s \in \mathbf{Z}$*

$$B_s(w_i, w_j^*) = 0, \text{ if } i \neq j, \text{ and}$$

$$B_s(w_i, w_i^*) = 1.$$

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Proof. We shall prove that the result is true if $s = n$, we have from Theorem 6.2 that

$$B_n(u, v) = \sum_{k=n+1}^{r} \sum_{j=0}^{k-n-1} E^{j+n}uE^{j-k+n}(p_k v), \quad u, v \in F(A).$$

Hence

$$B_n(w_i, w_p^*) = \sum_{k=n+1}^{r} \sum_{j=n}^{k-1} E^j w_i E^{j-k}(p_k w_p^*)$$

$$= \sum_{j=n}^{r-1} \sum_{k=j+1}^{r} E^j w_i E^{j-k}(p_k w_p^*)$$

$$= \sum_{j=n}^{r-1} E^j w_i(-1)^{p+j} R_{p,j}/C, \quad \text{by Lemma 7.4,}$$

$$= 1, \text{ if } p = i,$$

$$= 0, \text{ if } p \neq i.$$

Since $Lw_i = L^*w_p^* = 0$, it follows from Corollary 7.2 that the theorem holds for all $s \in \mathbf{Z}$.

8. Factorization of difference operators. Let L be a difference operator of the form (4). Let q_i, $n_1 \leqq i \leqq r_1$, and s_i, $n_2 \leqq i \leqq r_2$, be functions in $F(A)$ and let

(17) $$Q = \sum_{i=n_1}^{r_1} q_i E^i,$$

(18) $$S = \sum_{i=n_2}^{r_2} s_i E^i.$$

Now we have

(19) $$SQ = \sum_{i=n_2}^{r_2} \sum_{j=n_1}^{r_1} s_i E^i q_j E^{i+j} = \sum_{k=n_1+n_2}^{r_1+r_2} \left(\sum_{i+j=k} s_i E^i q_j \right) E^k.$$

We write $L = SQ$ if $ly = S(Qy)$ for all $y \in F(A)$. In this case we have $n = n_1 + n_2$, $r = r_1 + r_2$, $s_{n_2} s_{r_2} q_{n_1} q_{r_1}$ does not vanish on A and the order of L is the sum of the orders of Q and S. We say that L can be *factorized as SQ*

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if $L = SQ$ and S, Q both have order at least one. This is to avoid trivial factorizations of the form

$$L = E(E^{-1}L) = (LE^{-1})E.$$

It should be noted that if L can be factorized as SQ, where S, Q are given in (17), (18), then there is no loss of generality in assuming that $n \leqq n_1 \leqq r_1 \leqq r$. This is because

$$L = SQ = (SE^k)(E^{-k}Q) \quad \text{for } k \in \mathbf{Z}$$

and we may replace Q by $E^{-k}Q$ for a suitably chosen k.

The sequel corresponds to results obtained in [**2**, **9**] for differential operators.

THEOREM 8.1. *The following are equivalent conditions on* L.

(i) *L can be factorized.*

(ii) *There are solutions* $y_{n_1}, \dots, y_{r_1-1}$, *of* $Ly = 0$ *where* $n \leqq n_1 < r_1 \leqq r$ *and* $r_1 - n_1 < r - n$ *so that* $C(y_{n_1}, \dots, y_{r_1-1})$ *does not vanish on* A.

When (ii) *holds,* L *can be factorized as* SQ *where* Q *has order* $r_1 - n_1$.

Proof. Suppose (i) holds, that $L = SQ$ where S, Q are given by (17), (18) and that $n \leqq n_1 < r_1 \leqq r$ with $r_1 - n_1 < r - n$. Let y_i, $n_1 \leqq i \leqq r_1 - 1$, be a fundamental set of solutions of $Qy = 0$. then $C(y_{n_1}, \dots, y_{r_1-1})$ does not vanish on A (ii) holds.

Now suppose (ii) holds and put

$$Qy = \frac{C(y_n, \dots, y_{r_1-1}, y)}{C(y_{n_1}, \dots, y_{r_1-1})}, \quad y \in F(A).$$

Then Q is of the form (17) for suitable q_i. Also $Qy_i = 0$, $n_1 \leqq i \leqq r_1 - 1$. We wish to select functions s_j, $n - n_1 \leqq j \leqq r - r_1$, so that if S is given by (18), $L - SQ$ has order $r_1 - n_1 - 1$ at most. Now

$$L - SQ = \sum_{k=n}^{r} \left(p_k - \sum_{i+j=k} s_i E^i q_j \right) E^k.$$

Consider, then the equations

$$(20) \qquad p_k = \sum_{i+j=k} s_i E^i q_j, \quad r - r_2 + n_2 \leqq k \leqq r,$$

where $r_2 = r - r_1$, $n_2 = n - n_1$. Since $q_{r_1} = 1$, if we take $k = r$ in (20), the resulting equation may be solved for s_{r_2}. Using $k = r - 1, \dots, r - r_2 +$

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n_2 in that order, we may solve (20) to obtain uniquely determined functions s_{r_2-r+k}, $r - r_2 + n_2 \leqq k \leqq r$. Then we have

$$L - SQ = \sum_{k=n}^{r-r_2+n_2-1} \left(p_k - \sum_{i+j=k} s_i E^i q_j\right) E^k,$$

so that $L - SQ$ has order $r_1 - n_1 - 1$ at most. Since $(L - SQ)v_i = 0$, $n_1 \leqq i \leqq r_1 - 1$, we deduce from Lemma 6.1 (i) that $L - SQ = 0$, that is $L = SQ$.

LEMMA 8.2. *Let Q, S be given by* (17), (18) *and suppose that s_{n_2} does not vanish on A. Then if $SQ = 0$ on $F(A)$, $Q = 0$ on $F(A)$.*

Proof. By (19) and Lemma 6.1 (ii), we deduce that

$$\sum_{i+j=k} s_i E^i q_j = 0, \quad n_1 + n_2 \leqq k \leqq r_1 + r_2,$$

where $n_1 \leqq j \leqq r_1$, $n_2 \leqq i \leqq r_2$. Now let

$$t = \min (r_1 + n_2, r_2 + n_1).$$

Then we have

$$\sum_{i=n_2}^{k-n_1} s_i E^i q_{k-i} = 0, \quad n_1 + n_2 \leqq k \leqq t.$$

Taking $k = n_1 + n_2$ we see that $q_{n_1} = 0$. Consecutively using the other values of k we find that $q_i = 0$, $n_2 \leqq i \leqq t - n_2$.

If $t = r_1 + n_2$ we have $t - n_2 = r$, so that $Q = 0$, while if $t < r_1 + n_2$ we also have the equations

$$\sum_{i=n_2}^{r_2} s_i E^i q_{k-i} = 0, \quad t + 1 \leqq k \leqq r_1 + n_2,$$

and we deduce that $q_i = 0$, $t - n_2 + 1 \leqq i \leqq r_1$. Again this gives $Q = 0$.

THEOREM 8.3. *Suppose that L can be factorized as $L = SQ$ where L, S, Q are given respectively by* (4), (17), (18). *Let v_i, $n_1 \leqq i \leqq r_1 - 1$, be a fundamental set of solutions of $Qv = 0$ and let $n_1 + 1 \leqq -p \leqq r_1$. Let B_p be the bilinear form described in Theorem 6.2. Then*

$$S^* v = \sum_{j=n_1}^{r_1-1} v_j^* B_p(v_j, v), \quad v \in F(A).$$

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where v_j^* is obtained from (10) *with n_1, r_1, q_{r_1} and v_j in lieu of n, r, p_r and w_j respectively.*

Proof. Let

$$V(x, y) = \sum_{j=n_1}^{r_1-1} v_j(x)\, v_j^*(y), \quad x, y \in A,$$

be the Green's function of Q. Given $y \in A$, $QV(\cdot, y) = 0$ on A so that $LV(\cdot, y) = 0$ on A as $L = SQ$. Let $u(x) = V(x, y)$ for $x \in A$ and let $s \in \mathbf{Z}$ be given. Then from Theorem 6.2,

$$-uL^*v = \Delta(E^{-s} B_s(u, v)), \quad v \in F(A).$$

Hence

$$-\sum_{k \in I(x)} u(g^{p+k}x)L^*v(g^{p+k}x)$$

$$= -(\text{sign } \alpha(x))\, [E^{-s}(B_s(u, v))(g^p x)$$

$$= -E^{-s}(B_s(u, v))(g^{p+\alpha(x)}x)]$$

so that

$$(21) \quad -(\text{sign } \alpha(x)) \sum_{k \in I(x)} -V(g^{p+k}x, y)\, L^*v(g^{p+k}x)$$

$$= \sum_{j=n_1}^{r_1} v_j^*(y)E^{p-s}B_s(v_j, v)(x) - \sum_{j=n_1}^{r_1} v_j^*(y)E^{p-s}(B_s(v_j, v)(g^{\alpha(x)}x).$$

Now if $f \in F(A)$, it is immediate from the fact that $\alpha(gx) = \alpha(x) - 1$, that the function $f(g^{\alpha(x)}x)$ is g-invariant. Hence the function

$$E^{p-s}(B_s(v_j, v))(g^{\alpha(x)}x)$$

in the right hand side of (21) is g-invariant and so we have

$$Q^*\left(\sum_{j=n_1}^{r_1} v_j^*(x)E^{p-s}(B_s(v_j, v))(g^{\alpha(x)}x)\right)$$

$$= \sum_{j=n_1}^{r_1} E^{p-s}(B_s(v_j, v))(g^{\alpha(x)}x)(Q^*v_j^*)(x)$$

$$= 0, \quad \text{since } Q^*v_j^* = 0.$$

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Now we let $y = x$ in (21) and then apply Q^* to both sides of (21) using Theorems 5.4 and 6.4 with Q^* in place of L. We also use Theorem 6.3 on L to obtain

$$Q^*S^*v(x) = L^*v(x)$$
$$= Q^*\left(\sum_{j=n_1}^{r_1} v_j^*(x)E^{p-s} B_s(v_j, v)\right)(x)$$
$$= Q^*\left(\sum_{j=n_1}^{r_1} v_j^*(x) B_p(v_j, v)\right)(x),$$

by Lemma 7.1 (ii).

Thus

$$Q^*(S^*v - \sum_{j=n_1}^{r_1} v_j{}^*B_p(v_j, v)) = 0, \quad v \in F(A),$$

and the result follows from Lemma 8.2.

THEOREM 8.4. *The following conditions on L are equivalent.*

(i) *L can be factorized in the form* R^*VQ.

(ii) *There are solutions* u_i, $n_1 \leqq i \leqq r_1 - 1$ *of* $Lu = 0$ *and solutions* v_j, $n_2 \leqq j \leqq r_2 - 1$, *of* $L^*v = 0$ *so that* u_i *is conjugate to* v_j *for all* i, j *and both* $C(u_{n_1}, \dots, u_{r_1})$, $C(v_{n_2}, \dots, v_{r_2})$ *do not vanish on* A.

Proof. Suppose (i) holds and let u_i, $n_1 \leqq i \leqq r_1 - 1$ be a fundamental set of solutions of $Qu = 0$, v_j, $n_2 \leqq j \leqq r_2 - 1$, a fundamental set of solutions of $Rv = 0$. By Theorems 6.3 (iii) and 8.3 we have

$$V^*Rv = \sum_{j=n_1}^{r_1-1} u_j{}^*B_p(u_j, v),$$

while $Rv_j = 0$ implies $B_p(u_j, v_i) = 0$ by Theorems 4.4, 6.6 and Corollary 7.2 applied to Q^*. The remainder of (ii) follows from the definition of fundamental system.

Conversely, let (ii) hold. Then by Theorem 8.1 we may write $L = WQ$ where u_i, $n_1 \leqq i \leqq r_1 - 1$ form a fundamental set of solutions of $Qu = 0$. From Theorem 8.3 we deduce that

$$W^*v = \sum_{j=n_1}^{r_1-1} u_j{}^*B_p(u_j, v), \quad v \in F(A),$$

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so that $W^*v_i = 0$ for $n_2 \leqq i \leqq r_2 - 1$. Again by Theorem 8.1 we may write $W^* = V^*R$, where v_i, $n_2 \leqq i \leqq r_2 - 1$ form a fundamental set of solutions of $Rv = 0$. We now have $L = WQ$ and $W^* = V^*R$ so that $L = R^*VQ$ by Theorem 6.3.

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