Canad. Math. Bull. Vol. **32** (1), 1989

TREE SELF-EMBEDDINGS

BY
DAVID ROSS

ABSTRACT. Elementary proofs are given of the following two statements: (1) Every infinite tree of height at most ω properly embeds into itself. (2) There is a tree of height $\omega + 1$ that does not properly embed into itself.

0. **Introduction.** Simple proofs are given of the following two statements:
(1) Every infinite tree of height at most ω properly embeds into itself
(2) There is a tree of height $\omega + 1$ that does not properly embed into itself.

While statement (1) is an immediate consequence of a difficult theorem of Nash-Williams [3, 4, 2], this special case is of independent interest, and can be proved as an elementary consequence of Kruskal's embedding theorem [5].

1. **Definitions.** A *tree* is a strict partial order $(T, <)$ such that for every $x \in T$, $\{y \in T: y \leqq x\}$ is well ordered by $<$. It will be convenient to assume that every tree has a least element r_T.

The *height of* $x \in T$ is the order type of $\{y \in T: y \leqq x\}$. The *height of* T is the supremum of the heights of elements of T. For $x \in T$, $T(x) = \{y \in T; y \geqq x\}$, $S(x) = \{y > x$: if $x < z \leqq y$ then $z = y\}$, $S_\infty(x) = \{y \in S(x)$: $T(y)$ infinite$\}$, and $S_f(x) = S(x) - S_\infty(x)$.

A *branch* of T is a maximal linearly ordered subset of T. Note that if T has height ω, every infinite branch can be written $\{x_n: n \in \mathbf{N}\}$ with $x_0 = r_T$ and $x_{n+1} \in S_\infty(x_n)$. An *essential antichain* of T is a subset G of $\{x \in T$: $T(x)$ is infinite$\}$ with the property that for every $x \neq y \in G$, neither $x \in T(y)$ nor $y \in T(x)$. Observe that $\sup\{\,\|G\|$: G an essential antichain of $T\} \geqq \sup\{\,\|S_\infty(x)\|: x \in T\}$, where $\|A\|$ is the cardinality of A.

For V, W subsets of trees, say that V *embeds* in W, $V \Rightarrow W$, if for some injection $\theta: V \to W$, and every $x, y \in V$, $x < y$ if and only if $\theta(x) < \theta(y)$. Say that V *properly embeds* in W, $V \Rightarrow_0 W$, provided $V \Rightarrow W'$ for some proper subset W' of W.

Received by the editors October 21, 1986, and, in final revised form, June 30, 1988.
The author is supported by an SERC grant.
AMS Subject Classification (1980): Primary 02K04; Secondary 02K06.

2. **Results.**

LEMMA 1. (Kruskal) *If* $\{T_n: n \in \mathbf{N}\}$ *is a sequence of finite trees then* $T_i \Rightarrow T_j$ *for some* $i < j$.

LEMMA 2. *If* $\{T_n: n \in \mathbf{N}\}$ *is a sequence of finite trees, then for some* $N \in \mathbf{N}$ *and all* $n \geqq N$, $E(n) = \{i \neq n: T_n \Rightarrow T_i\}$ *is infinite.*

PROOF. Else without loss of generality $E(n)$ is finite for all n. By Lemma 1, $\{n: E(n) = \phi\}$ is finite. Since for every n with $E(n) \neq \phi$, $T_n \Rightarrow T_i$ for some i with $E(i) = \phi$, every T_n embeds in one of a finite number of finite trees, so there are only finitely many trees in $\{T_n: n \in \mathbf{N}\}$, giving a contradiction.

LEMMA 3. *If* $\{T_n: n \in \mathbf{N}\}$ *is a sequence of finite trees, then there is a nontrivial increasing* $\psi: \mathbf{N} \rightarrow \mathbf{N}$ *such that for all* $n \in \mathbf{N}$, $T_n \Rightarrow T_{\psi(n)}$.

PROOF. Take N as in Lemma 2. For $n \leqq N$ put $\psi(n) = n$. For $n > N$, define $\psi(n)$ inductively so that $\psi(n + 1)$ is the least $i > \psi(n)$ with $T_{n+1} \Rightarrow T_i$.

REMARK. In lemmas 1-3 the injections $T_i \Rightarrow T_j$ may be defined in such a way that the least element of T_i is taken to the least element of T_j.

LEMMA 4. *Suppose* T *is a tree of height at most* ω, $x \in T$, $\{x_n: n \in \mathbf{N}\} \subseteq S(x)$, *and* $W \Rightarrow_0 W$, *where* $W = \cup_n T(x_n)$. *Then* $T \Rightarrow_0 T$.

PROOF. Let $\theta_W: W \rightarrow W$ be a proper embedding, and let $\theta: T \rightarrow T$ be θ_W on W and the identity on $T - W$. It suffices to show that θ is order preserving. Let $y, z \in T$ with $y < z$. There are three cases.

CASE 1. $y, z \in T - W$. Then $\theta(y) = y < z = \theta(z)$.

CASE 2. $y, z \in W$. Then $\theta(y) = \theta_W(y) < \theta_W(z) = \theta(z)$.

CASE 3. $y \in T - W$, $z \in W$. Then $z \in T(x_n)$ for some n, and $\theta(y) = y \leqq x < \theta_W(x_n) \leqq \theta_W(z) = \theta(z)$.

THEOREM 1. *If* T *is an infinite tree of height at most* ω *then* $T \Rightarrow_0 T$.

PROOF. There are several cases.

CASE 1. For some $x \in T$, $S_f(x)$ is infinite. Let $\{x_n: n \in \mathbf{N}\} \subseteq S_f(x)$, and put $T_n = T(x_n)$. Take $\psi: \mathbf{N} \rightarrow \mathbf{N}$ from Lemma 3, properly embed $\cup_n T_n$ into itself by embedding T_n into $T_{\psi(n)}$, and apply Lemma 4.

CASE 2. For some branch $\{x_n: n \in \mathbf{N}\}$ of T, and some $N \in \mathbf{N}$, $S_\infty(x_i) = \{x_{i+1}\}$ whenever $i \geqq N$. Let $T_n = T(x_{N+n}) - T(x_{N+n+1})$ for $n \in \mathbf{N}$; evidently each T_n is finite. Apply Lemma 3 as in Case 1 to properly embed $\cup_n T_n$ into itself, and as in Lemma 4 extend this embedding to all of T.

CASE 3. Otherwise. Induct on $\alpha = \sup\{\, \|G\|: G$ an essential antichain of $T\}$. Clearly $\alpha > 0$. If α is finite then $\alpha = \|G\|$ for some essential antichain G of T.

https://doi.org/10.4153/CMB-1989-005-7 Published online by Cambridge University Press

Let $x_0 \in G$. Since $(G - \{x_0\}) \cup S_\infty(x_0)$ is an essential antichain, $S_\infty(x_0) = \{x_1\}$ for some x_1. Similarly, $S_\infty(x_1) = \{x_2\}$ for some x_2. Continue to obtain a sequence $\{x_n: n \in \mathbf{N}\}$ with $S_\infty(x_n) = \{x_{n+1}\}$. The result follows by Case 2.

Suppose then that α is infinite. By induction and the hypothesis that Case 1 fails, assume that for every $x \in T$ with $T(x)$ infinite, $\alpha = \sup\{\,||G||: G$ an essential antichain of $T(x)\,\}$, and $S_f(x)$ is finite. In particular, $||S(x)|| \leqq \alpha$ for every $x \in T$. Consider two cases.

CASE 3a. $||S(x)|| < \alpha$ for every $x \in T$. For every x with $T(x)$ infinite, and every $\delta < \alpha$, there is an essential antichain $G_\delta(x)$ of $T(x)$ with cardinality δ. Define $\theta: T \rightarrow T$ inductively as follows. Put $\theta(r_T) = y$ for some $y > r_T$ with $T(y)$ infinite. Once $\theta(x)$ is defined with $T(\theta(x))$ infinite, let $\delta = ||S(x)|| < \alpha$, and define θ on $S(x)$ to be an injection of $S(x)$ into $G_\delta(\theta(x))$. It is easy to verify that θ is a proper embedding of T into itself.

CASE 3b. Otherwise. Without loss of generality (by Lemma 4), for every x with $T(x)$ infinite there is a $y \in T(x)$ with $||S_\infty(y)|| = ||S(y)|| = \alpha$. In particular, for every x with $T(x)$ infinite there is an essential antichain $G_\alpha(x)$ of $T(x)$ with cardinality α. Proceed as in Case 3a, substituting $G_\alpha(\theta(x))$ for $G_\delta(\theta(x))$ in the construction of θ. The theorem is proved.

The next result shows that Theorem 1 is in some sense the best possible.

THEOREM 2. *There is a tree T of height $\omega + 1$ such that $T \not\Rightarrow_0 T$.*

PROOF. There is a unique (up to isomorphism) tree V of height ω such that $||S(x)|| = 2$ for every $x \in V$. Let B be the set of branches of V. Extend the order $<$ on V to $V \cup B$ by putting $x < b$ whenever $x \in b \in B$. If $W \subseteq B$ is nonempty then $V \cup W$ is a tree of height $\omega + 1$.

Let Φ be the set of nontrivial embeddings of V into itself. (Note that proper embeddings are nontrivial, but not necessarily vice versa). Every $\psi \in \Phi$ extends uniquely to an embedding $\bar{\psi}$ of $V \cup B$, where $\bar{\psi}(b) = \{y \in V: y < \psi(x)$ for some $x \in b\}$. Since $||V|| = \aleph_0$, $||\Phi|| \leqq c$ (the cardinality of the continuum), so Φ can be enumerated (with repetition if necessary) by $\Phi = \{\psi_\alpha: \alpha < c\}$. Observe that $B_\alpha = \{b \in B: \bar{\psi}_\alpha(b) \neq b\}$ has cardinality c.

Cardinality considerations make it possible to inductively define disjoint subsets $\{p_\alpha: \alpha < c\}$ and $\{q_\alpha: \alpha < c\}$ of B such that $q_\alpha = \bar{\psi}_\alpha(p_\alpha)$; indeed, take $p_\alpha \in B_\alpha - (\{q_i: i < \alpha\} \cup \{\bar{\psi}_\alpha^{-1}(p_i): i < \alpha\})$ and put $q_\alpha = \bar{\psi}_\alpha(p_\alpha)$. Put $T = V \cup \{p_\alpha: \alpha < c\}$.

Suppose (for a contradiction) that $\bar{\psi}$ properly embeds T into itself. It is easy to verify that $\bar{\psi}$ must take V into V and $T - V$ into $T - V$. Moreover, if $\bar{\psi}$ is the identity on V then it must be the identity on $T - V$. Since $\bar{\psi}$ is proper, it is not the identity on V, so $\bar{\psi} = \bar{\psi}_\alpha$ for some α. But then $\bar{\psi}(p_\alpha) = \bar{\psi}_\alpha(p_\alpha) = q_\alpha \notin T$, a contradiction.

https://doi.org/10.4153/CMB-1989-005-7 Published online by Cambridge University Press

REMARK. The tree T in Theorem 2 is *rigid* in the sense of ([1], 4.22).

ACKNOWLEDGEMENTS. The author is grateful to the referee for pointing out an error in the original statement and proof of Lemma 4.

REFERENCES

1. T. Jech, *Set theory*, Academic Press (New York) 1978.

2. R. Laver, *Better-quasi-orderings and a class of trees*, Studies in Foundations and Combinatorics, Advances in Mathematics Supplementary Studies, Vol. **1**, Academic Press (New York) (1978), pp. 31-48.

3. C. St. J. A. Nash-Williams, *On well-quasi-ordering infinite trees*, Proc. Camb. Phil. Soc. **61** (1965), pp. 697-720.

4. ———, *On better-quasi-ordering transfinite sequences*, Proc. Camb. Phil. Soc. **64** (1968), pp. 273-290.

5. J. B. Kruskal, *Well quasi-ordering, the tree theorem, and Vazsony's conjecture*, Trans. Amer. Math. Soc. **95** (1960), pp. 210-225.

DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF HULL, HULL, HU6 7RX, ENGLAND