Canad. Math. Bull. Vol. **28** (1), 1985

TWO INEQUALITIES FOR PLANAR CONVEX SETS

BY

GEORGE TSINTSIFAS

ABSTRACT. B. Grünbaum, J. N. Lillington and lately R. J. Gardner, S. Kwapien and D. P. Laurie have considered inequalities defined by three concurrent straight lines in the interior of a planar compact convex set. In this note we prove two elegant conjectures by R./J. Gardner, S. Kwapien and D. P. Laurie.

1. **Introduction**. Trying to establish a conjecture of B. Grünbaum [2], J. N. Lillington [3] came up with some interesting problems concerning the division of a planar compact convex set by three concurrent lines. In [1] R. J. Gardner, S. Kwapien and D. P. Laurie solve a conjecture of J. N. Lillington [3] and propose the following two new conjectures concerning area inequalities for planar convex sets.

Let X be a planar compact convex set and L_1, L_2, L_3 three concurrent lines through the interior point 0, which divide X into six regions with areas $|X_i|$, $|Y_i|$, $i = 1, 2, 3$ (see figure 1).
(Here and throughout we denote by $|E|$ the area of the set E. Values of i lying outside the set $\{1, 2, 3\}$ are defined by $i \equiv i + 3$).

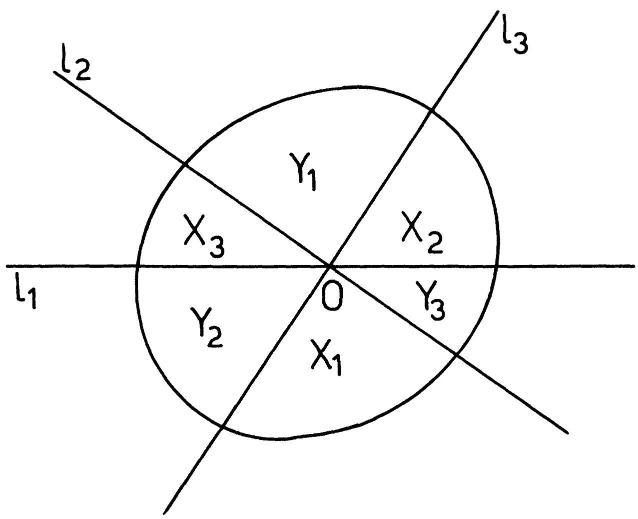

FIG. 1.

Received by the editors May 31, 1983 and, in final revised form, March 13, 1984.
AMS Subject Classification 52A10, 52A40.

We define

$$Q(X) = \frac{|X_1|}{|Y_1|} + \frac{|X_2|}{|Y_2|} + \frac{|X_3|}{|Y_3|},$$

$$P(X) = \frac{|X_1| + |X_2|}{|Y_3|} + \frac{|X_2| + |X_3|}{|Y_1|} + \frac{|X_3| + |X_1|}{|Y_2|}$$

R. J. Gardner, S. Kwapien and D. P. Laurie conjectured:

1. $Q(X) \geq 3/2$,
2. $P(X) \geq 3$.

2. **Proof of the first conjecture**. We originally obtained a trigonometrical proof for the first conjecture using an affine transformation and high school mathematics only. Below we give another proof, following R. J. Gardner's, S. Kwapien's and D. P. Laurie's formulation.

THEOREM 1. $Q(X) \geq 3/2$.

PROOF. The line L_i intersects X at the points p_i, q_i, $i = 1, 2, 3$. Let $A_1 = q_1p_2 \cap p_1q_3$, $A_2 = q_2p_3 \cap p_2q_1$, $A_3 = q_3p_1 \cap p_3q_2$.

CASE 1. *Suppose O is an interior point of the triangle $A_1A_2A_3$ (see figure 2). Obviously we have*

$$Q(X) \geq Q(A_1A_2A_3). \tag{1}$$

We use areal coordinates, setting

$$A_1 = (1, 0, 0), \qquad A_2 = (0, 1, 0), \qquad A_3 = (0, 0, 1)$$

$O = (\chi_1, \chi_2, \chi_3)$, where $\chi_1 + \chi_2 + \chi_3 = |A_1A_2A_3| = 1$, $\chi_i \geq 0$. We take L_i to be the line

$$\chi'_{i+1} - \chi_{i+1} = \lambda_i(\chi_{i-1} - \chi'_{i-1}), \qquad \lambda_i > 0$$

and

$$p_i = (p^i_1, p^i_2, p^i_3), \qquad q_i = (q^i_1, q^i_2, q^i_3).$$

An easy calculation gives:

$$p^i_{i-1} = \chi_{i-1} + \chi_{i+1}/\lambda_i, \quad p^i_i = \chi_i + \left(1 - \frac{1}{\lambda_i}\right)\chi_{i+1}, \quad p^i_{i+1} = 0,$$

$$q^i_{i-1} = 0, \quad q^i_i = \chi_i + (1 - \lambda_i)\chi_{i-1}, \quad q^i_{i+1} = \chi_{i+1} + \lambda_i\chi_{i-1},$$

and

$$\begin{aligned} |Op_{i-1}q_{i+1}| &= \chi_i^2\left(\lambda_{i+1} + \left(\frac{1}{\lambda_{i-1}}\right) - 1\right) \\ |A_ip_{i+1}Oq_{i-1}| &= (\chi_{i-1} + \chi_{i+1})^2 - \lambda_{i-1}\chi^2_{i+1} - \chi^2_{i-1}/\lambda_{i+1}, \qquad i = 1, 2, 3. \end{aligned} \tag{2}$$

These expressions were obtained in [1].

Now using the inequality

$$k_1 a_1^2 + k_2 a_2^2 \geq \frac{k_1 k_2 (a_1 + a_2)^2}{k_1 + k_2}, \tag{3}$$

where $k_1, k_2, a_1, a_2 \in R$, $k_1 + k_2 > 0$, we have

$$\lambda_{i-1}\chi_{i+1}^2 + \frac{\chi_{i-1}^2}{\lambda_{i+1}} \geq \frac{\frac{\lambda_{i-1}}{\lambda_{i+1}}(\chi_{i+1} + \chi_{i-1})^2}{\lambda_{i-1} + \frac{1}{\lambda_{i+1}}}. \tag{3a}$$

Consequently, from (2) and (3a) we obtain:

$$\frac{|Op_{i-1}q_{i+1}|}{|A_i p_{i+1} O q_{i-1}|} \geq \left[\frac{\chi_i}{\chi_{i-1} + \chi_{i+1}}\right]^2 \left[\lambda_{i+1} + \frac{1}{\lambda_{i-1}}\right],$$

or,

$$Q(A_1A_2A_3) \geq \sum_{i=1}^{3} \left[\frac{\chi_i}{\chi_{i-1} + \chi_{i+1}}\right]^2 \left[\lambda_{i+1} + \frac{1}{\lambda_{i-1}}\right]$$

or,

$$Q(A_1A_2A_3) \geq \sum_{i=1}^{3} \left(\left[\frac{\chi_{i-1}}{\chi_i + \chi_{i+1}}\right]^2 \lambda_i + \left[\frac{\chi_{i+1}}{\chi_i + \chi_{i-1}}\right]^2 \frac{1}{\lambda_i}\right),$$

We have now

$$\left[\frac{\chi_{i-1}}{\chi_i + \chi_{i+1}}\right]^2 \lambda_i + \left[\frac{\chi_{i+1}}{\chi_i + \chi_{i-1}}\right]^2 \frac{1}{\lambda_i} \geq \frac{2\chi_{i-1}\chi_{i+1}}{(\chi_i + \chi_{i+1})(\chi_{i-1} + \chi_i)} \tag{4}$$

and consequently

$$Q(A_1A_2A_3) \geq \sum_{i=1}^{3} \frac{2\chi_{i-1}\chi_{i+1}}{(\chi_i + \chi_{i+1})(\chi_{i-1} + \chi_i)}.$$

We use now the known inequality

$$\sum_{i=1}^{3} \frac{2\chi_{i-1}\chi_{i+1}}{(\chi_i + \chi_{i+1})(\chi_i + \chi_{i-1})} \geq \frac{3}{2} \tag{5}$$

and finally obtain $Q(X) \geq 3/2$.

The equality holds if and only if X is a triangle and L_1, L_2, L_3 are parallel straight lines through the centroid to the sides respectively. This can be seen by making use of (1), (3a), (4) and (5).

CASE 2. *Suppose that O is not an interior point of the triangle $A_1A_2A_3$ and that O lies in the angle A_1 of the triangle $A_1A_2A_3$. We can prove*

$$\frac{|X_2|}{|Y_2|} + \frac{|X_3|}{|Y_2|} > \frac{3}{2}.$$

PROOF. The straight lines through p_1 parallel to the line q_1p_2 intersects L_2, L_3 and A_2A_3 at the points N, M, A_3' respectively. The proof of theorem 1 in case 1 is independent of the angle A_1 of the triangle $A_1A_2A_3$. Consequently we can consider as a triangle the figure S with sides A_2A_3', the semiline A_2p_2 and the semiline $A_3'p_1$. Then, from case 1, it follows that $Q(S) > 3/2$ (this can be also proved directly).

Also, it is very easy to see that:

$$\frac{|X_1|}{|Y_1|} + \frac{|X_2|}{|Y_2|} + \frac{|X_3|}{|Y_3|} = Q(X) > Q(S).$$

So, for the second case, we have $Q(X) > 3/2$.

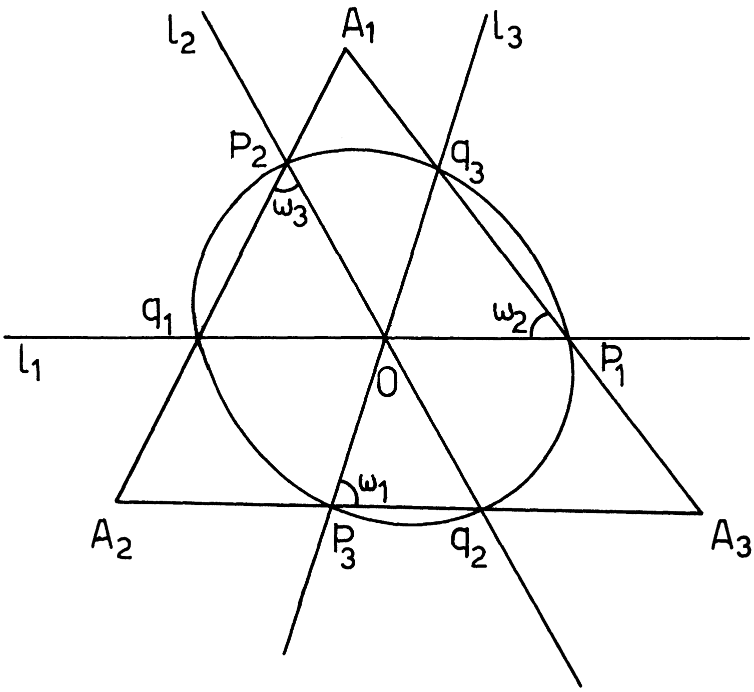

FIG. 2.

3. **Proof of the second conjecture**.

THEOREM 2. $P(X) \geq 3$.

We similarly have to investigate two cases.

CASE 1. Suppose that the point O is an interior point of the triangle $A_1A_2A_3$ (see

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figure 2). From figure 2, we can see that

$$P(X) \geq P(A_1A_2A_3).$$

We need the following lemma.

LEMMA. *Let ABC be a triangle and O, N, M points on the sides BC, CA, AB, respectively. We will show that:*

$$P_A = \frac{|BOM| + |OCN|}{|AMON|} \geq \frac{\sin(B+\omega)\sin\phi}{\sin(\omega+\phi)\sin B} + \frac{\sin(C+\phi)\sin\omega}{\sin(\omega+\phi)\sin C} - 1$$

where A, B, C are the angles of the triangle ABC and $\sphericalangle BOM = \omega$ $\sphericalangle CON = \phi$.

The equality holds if and only if MN is parallel to BC.

PROOF OF THE LEMMA. It is elementary to see that:

$$\frac{|BOM|}{|ABC|} = \frac{\overline{BM}\cdot\overline{BO}}{c\cdot a} = \frac{d_1\cdot\overline{BO}^2}{c\cdot a},$$

$$\frac{|OCN|}{|ABC|} = \frac{\overline{CN}\cdot\overline{CO}}{b\cdot a} = \frac{d_2\cdot\overline{CO}^2}{b\cdot a},$$

where

$$d_1 = \frac{\sin\omega}{\sin(B+\omega)}, \qquad d_2 = \frac{\sin\phi}{\sin(C+\phi)}$$

and a, b, c are the sides of the triangle ABC. Using the key inequality (3) we obtain

$$\frac{|BOM| + |OCN|}{|ABC|} \geq \frac{a\cdot d_1\cdot d_2}{bd_1 + cd_2}.$$

Consequently

$$(6) \qquad P_A = \frac{1}{\dfrac{|ABC|}{|BOM| + |OCN|} - 1} \geq \frac{1}{\dfrac{bd_1 + cd_2}{ad_1d_2} - 1}.$$

Using simple trigonometrical formulas in the triangle ABC we obtain:

$$(7) \qquad \frac{bd_1 + cd_2}{ad_1d_2} = \frac{\sin B\sin(C+\phi)}{\sin A\sin\phi} + \frac{\sin C\sin(B+\omega)}{\sin A\sin\omega}$$

$$= \frac{\sin B\sin C}{\sin A}[\cot\phi + \cot\omega] + 1.$$

From (6) and (7) it follows that

$$(8) \qquad P_A \geq \frac{\sin A}{\sin B\sin C[\cot\phi + \cot\omega]}.$$

It is very easy to see that the following identity holds

(9) $$\frac{\sin A}{\sin B \sin C[\cot \phi + \cot \omega]} = \frac{\sin (B + \omega) \sin \phi}{\sin B \sin (\phi + \omega)} + \frac{\sin (C + \phi) \sin \omega}{\sin C \sin (\phi + \omega)} - 1 .$$

Formulas (8) and (9) prove our lemma.

The equality in (3) holds when $k_1\alpha_1 = k_2\alpha_2$ or,

$$\frac{d_1 \cdot \overline{BO}}{c} = \frac{d_2 \cdot \overline{CO}}{b},$$

or,

$$\frac{\overline{BM}}{c} = \frac{\overline{CN}}{b},$$

that is, MN is parallel to BC.

We are now ready to prove the second conjecture. In figure 2 we define:

$$\measuredangle Op_3q_2 = \omega_1, \qquad \measuredangle Op_2q_1 = \omega_3, \qquad \measuredangle Op_1q_3 = \omega_2 .$$

Applying the lemma to the triangles $q_1A_1p_1$, $q_2A_2p_2$, $q_3A_3p_3$, we take,

$$\begin{aligned} P_{A_i} &= \frac{|q_iOp_{i+1}| + |Op_iq_{i-1}|}{|A_ip_{i+1}Oq_{i-1}|} \\ &\geq \frac{\sin \omega_{i-1} \sin (\pi - A_{i-1} - \omega_i + \omega_{i+1})}{\sin (\pi - A_i - \omega_{i+1}) \sin (\pi - A_{i+1} - \omega_{i-1} + \omega_i)} \\ &\quad + \frac{\sin (\pi - A_1 - \omega_{i+1} + \omega_{i-1}) \sin (\pi - A_{i-1} - \omega_i)}{\sin \omega_{i+1} \sin (\pi - A_{i+1} - \omega_{i-1} + \omega_i)} - 1 \end{aligned}$$

Now

$$P(A_1A_2A_3) = \sum_{i=1}^{3} P_{A_i},$$

or

(10) $$P(A_1A_2A_3) \geq \sum_{i=1}^{3} \left(\frac{m_{i-1}}{n_{i+1}} + \frac{n_{i+1}}{m_{i-1}}\right) - 3,$$

where,

$$m_{i-1} = \sin \omega_{i-1} \sin (\pi - A_{i-1} - \omega_i + \omega_{i+1})$$

$$n_{i+1} = \sin (\pi - A_i - \omega_{i+1}) \sin (\pi - A_{i+1} - \omega_{i-1} + \omega_i) .$$

Obviously it follows

$$P(A_1A_2A_3) \geq 2 + 2 + 2 - 3 = 3 .$$

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The equality holds, according to our lemma, if p_2q_3 is parallel to L_1, p_3q_1 is parallel to L_2 and p_1q_2 is parallel to L_3. Also taking (10) and (11) into account we find that L_1, L_2, L_3 must be parallel to the sides of $A_1A_2A_3$. Therefore we conclude that the equality holds if and only if X is a triangle, O is its centroid and L_1, L_2, L_3 are parallel to the sides, respectively.

CASE 2. *A similar argument holds as in Theorem 1, case 2.*

4. **Comments**. The theorems 1 and 2 are remarkable tools in proving inequalities on convex sets. R. J. Gardner, S. Kwapien and D. P. Laurie noticed (see [1] page 309) that their theorems 3.1 and 4.1 follow immediately from theorems 1 and 2 respectively. Also it is worthwhile to notice here that Grünbaum's inequality $f(X) \geq 1/2$ (see [2]) follows easily from theorem 1.

The author is grateful to the referee for several improvements in the paper.

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