# A MAXIMAL RIEMANN SURFACE

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We let the notations be as in [3]. Then, in the category $\mathfrak{C}$ of all bordered Riemann surfaces, the following inclusion diagram holds [3, Theorem 9]:

$$M_0 \subset O_{HB} \begin{matrix} \subset O_{HD} \subset \\ \subset O_{KB} \subset \end{matrix} O_{KD} \subset M_2.$$

Further, from a theorem of Kuramochi [4] (see also Constantinescu and Cornea [2]), it easily follows that the class $O_{AD}$ is not contained in $M_2$. On the other hand, it is well known that $M_2$(which equals $O_{SB}$ for ordinary planar surfaces) is not contained in $O_{AD}$ (see Ahlfors and Beurling [1]).

Now let $\mathfrak{C}_0$ be the subcategory of bordered Riemann surfaces without planar ideal boundary. Then $\mathfrak{C}_0 \cap M_2 = M =$ the class of all maximal bordered Riemann surfaces. Hence the question whether $M$ is or not contained in $O_{AD}$ naturally arises; it was first considered by Sario [5]. This note contains the negative answer to Sario's question.

Let $X = R \cup B$ and $X_0 = R_0 \cup B_0$ be two bordered Riemann surfaces. We recall that a continuous map $f: X \to X_0$ is said to be *distinguished* if $f(B) \subset B_0$, and *proper* if, for any compact $K_0 \subset X_0$, $f^{-1}(K_0)$ is compact. Let $M_1$ be the class of all bordered Riemann surfaces with absolutely disconnected ideal boundary.

THEOREM 1. *Suppose there exists a distinguished proper conformal map* $f: X \to X_0$. *Then* $X \in M_1$ *if and only if* $X_0 \in M_1$.

*Proof.* Let $\beta$ and $\beta_0$ be the nowhere disconnecting and 0-dimensional ideal boundaries of $X$ and $X_0$. Then the spaces $X^* = X \cup \beta$ and $X_0^* = X_0 \cup \beta_0$ are compact and locally connected, and the sets $\beta$ and $\beta_0$ are nowhere disconnecting and 0-dimensional. By Lemma 2 in [3], the proper map $f: X \to X_0$ can be extended to a continuous map $f^*: X^* \to X_0^*$ satisfying $f^*(\beta) = \beta_0$ and $f^{*-1}(\beta_0) = \beta$.

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For any $x \in X$, let $o(x)$ denote the multiplicity of $f$ at $x$. As $f$ is conformal, proper and distinguished, there exists a natural number $s$ such that

$$\sum_{x \in f^{-1}(x_0)} o(x) = s,$$

for any $x_0 \in X_0$ [6, p. 126]. Let $E$ be the set of all points $x \in X$ for which $o(x) > 1$. Then $E$ is discrete in $X$, and $E_0 = f(E)$ is discrete in $X_0$.

Choose, in a parametric neighborhood on $X_0$, a disc $\overline{U}_0$ which does not meet $E_0$, and let $\overline{U} = f^{-1}(\overline{U}_0)$. Then $Y_0 = X_0 - \overline{U}_0$ is a normal neighborhood [3, Definition 8] in $X_0$ of $\beta_0$, and $Y = X - \overline{U}$ is a normal neighborhood in $X$ of $\beta$. Let $(Y_{0,n})_{n \in N}$ be a relative exhaustion [3, Definition 7] of $Y_0$ such that $\beta_{0,n}$ does not meet $E_0$ for any $n \in N$, where $\beta_{0,n} = \partial Y_{0,n} - \partial Y_0$ and where $\partial$ stands for the relative boundary. Then $(Y_n)_{n \in N}$ is a relative exhaustion of $Y$, where $Y_n = f^{-1}(Y_{0,n})$. Let $\beta_n = f^{-1}(\beta_{0,n}) = \partial Y_n - \partial Y$. Let $\alpha_0$ be a subset of $\beta_0$, $\alpha = f^{*-1}(\alpha_0)$, $\mu_{\alpha_0}$ the modulus of $Y_0$ for $\partial Y_0$ and $\alpha_0$ and $\mu_\alpha$ the modulus of $Y$ for $\partial Y$ and $\alpha$ [3, Definition 13]. It will be proved that

$$\mu_\alpha = \frac{1}{s} \mu_{\alpha_0}.$$

Let $\alpha_{0,n}$ be the minimal subcycle of $\beta_{0,n}$ which separates $\alpha_0$ from $\partial Y$. Then, since $f^*$ is continuous, $\alpha_n = f^{-1}(\alpha_{0,n})$ is the minimal subcycle of $\beta_n$ which separates $\alpha$ from $\partial Y$. Let $u_{0,n}$ and $\mu_{0,n}$ be the extremal function and the modulus of $Y_{0,n}$ for $\partial Y_0$ and $\alpha_{0,n}$, and let $u_n$ and $\mu_n$ be the extremal function and the modulus of $Y_n$ for $\partial Y$ and $\alpha_n$. By Lemma 8 in [3], we have

$$u_n = \frac{1}{s} u_{0,n} \circ f.$$

Hence $\mu_n = \frac{1}{s} \mu_{0,n}$ and so, as $n \to \infty$,

$$\mu_\alpha = \frac{1}{s} \mu_{\alpha_0},$$

as asserted. From this equality it follows that $\alpha_0$ is parabolic [3, Definition 14] if and only if $\alpha$ is parabolic. In particular, $\gamma_0 \in \beta_0$ is parabolic if and only if $f^{*-1}(\gamma_0)$ is parabolic. But it is easily seen that the set $f^{*-1}(\gamma_0)$ is finite. Thus $\gamma_0$ is parabolic if and only if all $\gamma \in f^{*-1}(\gamma_0)$ are parabolic [3, Corollary 4]. The theorem now follows.

*Remark.* An immediate corollary of Theorem 1 is the following statement: If $X_0$ is relatively planar and $X_0 \in O_{SB}$ and if there exists a distinguished proper conformal map $f: X \to X_0$, then $X$ is essentially maximal.

A direct proof of this statement, in the ordinary case, was given by Tamura [7].

THEOREM 2. *There exists a maximal ordinary Riemann surface $X \notin O_{AD}$.*

*Proof.* According to Ahlfors and Beurling [1, Theorem 16], there exists a planar ordinary Riemann surface $X_0 \in O_{SB} - O_{AD}$. As $M_1 = O_{SB}$ for planar ordinary surfaces, this $X_0$ belongs to $M_1 - O_{AD}$.

Let $E_0$ be a discrete subset of $X_0$ having the property that the closure in $X_0^*$ of $E_0$ is $E_0 \cup \beta_0$. Then there exists an ordinary Riemann surface $X$ and a proper conformal map $f: X \to X_0$ such that $f^{-1}(x_0)$ contains a single point for any $x_0 \in E_0$, and such that $o(x) = 2$ if $x \in f^{-1}(E_0)$ and $o(x) = 1$ if $x \in X - f^{-1}(E_0)$.

It is clear that $X$ has no boundary components of planar type. As $X_0 \in M_1$, $X \in M_1$ by Theorem 1, and consequently $X$ is essentially maximal. As $X_0 \notin O_{AD}$, it is easily seen that $X \notin O_{AD}$. Thus the proof is complete.

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