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THE COMPLETE CONTINUITY PROPERTY AND FINITE DIMENSIONAL DECOMPOSITIONS

MARIA GIRARDI AND WILLIAM B. JOHNSON

ABSTRACT. A Banach space $\mathfrak{X}$ has the complete continuity property (CCP) if each bounded linear operator from L_1 into $\mathfrak{X}$ is completely continuous (*i.e.*, maps weakly convergent sequences to norm convergent sequences). The main theorem shows that a Banach space failing the CCP has a subspace with a finite dimensional decomposition which fails the CCP. If furthermore the space has some nice local structure (such as fails cotype or is a lattice), then the decomposition may be strengthened to a basis.

1. **Introduction.** Given a property of Banach spaces which is hereditary, it is natural to ask whether a Banach space has the property if every subspace with a basis (or with a finite dimensional decomposition) has the property. The motivation for such questions is of course that it is much easier to deal with Banach spaces which have a basis (or at least a finite dimensional decomposition) than with general spaces. In this note we consider these questions for the *complete continuity property* (CCP), which means that each bounded linear operator from L_1 into the space is completely continuous (*i.e.*, carries weakly convergent sequences into norm convergent sequences).

The CCP is closely connected with the Radon-Nikodým property (RNP). Since a representable operator is completely continuous, the RNP implies the CCP; however, the Bourgain-Rosenthal space [BR] has the CCP but not the RNP. Bourgain [B1] showed that a space failing the RNP has a subspace with a finite dimensional decomposition which fails the RNP. Wessel [W] showed that a space failing the CCP has a subspace with a basis which fails the RNP. It is open whether a space has the RNP (respectively, CCP) if every subspace with a basis has the RNP (respectively, CCP).

Our main theorem shows that if $\mathfrak{X}$ fails the CCP, then there is an operator $T: L_1 \longrightarrow \mathfrak{X}$ that behaves like the identity operator $I: L_1 \longrightarrow L_1$ on the Haar functions $\{h_j\}$. Specifically, there is a sequence $\{x_n^*\}$ in the unit ball of $\mathfrak{X}^*$ such that x_n^* keeps the image of each Haar function along the n-th-level large (*i.e.*, $x_n^*(Th_{2^n+k}) > \delta > 0$) and the natural blocking $\{\mathrm{sp}(Th_{2^n+k} : k = 1, \ldots, 2^n)\}_n$ of the images of the Haar functions is a finite dimensional decomposition for some subspace $\mathfrak{X}_0$. Note that $\mathfrak{X}_0$ fails the CCP since T is not completely continuous (T keeps the Rachemacher functions larger than δ in norm). Thus a space failing the CCP has a subspace with a finite dimensional decomposition which fails the CCP. In the language of Banach space geometry, the theorem says that in any

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Banach space which fails the CCP grows a separated δ-tree with a difference sequence naturally blocking into a finite dimensional decomposition. If furthermore the space has some nice local structure (such as fails cotype or is a lattice), then modifications produce a separated δ-tree growing inside a subspace with a basis.

Throughout this paper, $\mathfrak{X}$ denotes an arbitrary Banach space, $\mathfrak{X}^*$ the dual space of $\mathfrak{X}$, and $S(\mathfrak{X})$ the unit sphere of $\mathfrak{X}$. The triple (Ω, Σ, μ) refers to the Lebesgue measure space on $[0, 1]$, Σ^+ to the sets in Σ with positive measure, and L_1 to $L_1(\Omega, \Sigma, \mu)$. All notation and terminology, not otherwise explained, are as in [DU].

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2. **Operator view-point.** A system $\mathcal{A} = \{A_k^n \in \Sigma : n = 0, 1, 2, \ldots \text{ and } k = 1, \ldots, 2^n\}$ is a *dyadic splitting* of $A_1^0 \in \Sigma^+$ if each A_k^n is partitioned into the two sets A_{2k-1}^{n+1} and A_{2k}^{n+1} of equal measure for each admissible n and k. Thus the collection $\pi_n = \{A_k^n : k = 1, \ldots, 2^n\}$ of sets along the n-th-level partition A_1^0 with π_{n+1} refining π_n and $\mu(A_k^n) = 2^{-n}\mu(A_1^0)$. To a dyadic splitting corresponds a (normalized) Haar system $\{h_j\}_{j\geq 1}$ where

$$h_1 = \frac{1}{\mu(A_1^0)} 1_{A_1^0} \quad \text{and} \quad h_{2^n+k} = \frac{2^n}{\mu(A_1^0)} (1_{A_{2k-1}^{n+1}} - 1_{A_{2k}^{n+1}})$$

for $n = 0, 1, 2, \ldots$ and $k = 1, \ldots, 2^n$.

A set N in the unit sphere of the dual of a Banach space $\mathfrak{X}$ is said to norm a subspace $\mathfrak{X}_0$ within $\tau > 1$ if for each $x \in \mathfrak{X}_0$ there is $x^* \in N$ such that $\|x\| \leq \tau x^*(x)$. It is well known and easy to see that a sequence $\{\mathfrak{X}_j\}$ of subspaces of $\mathfrak{X}$ forms a finite dimensional decomposition with constant at most τ provided that for each $n \in \mathbb{N}$ the space generated by $\{\mathfrak{X}_1, \ldots, \mathfrak{X}_n\}$ can be normed by a set from $S(\mathfrak{X}_{n+1}^{\perp})$ within $\tau_n > 1$ where $\Pi\tau_n \leq \tau$.

THEOREM 1. *If the bounded linear operator $T\colon L_1 \longrightarrow \mathfrak{X}$ is not completely continuous and $\{\tau_n\}_{n\geq 0}$ is a sequence of numbers larger than 1, then there exist*

(A) a dyadic splitting $\mathcal{A} = \{A_k^n\}$

(B) a sequence $\{x_{t_n}^*\}_{n\geq 0}$ in $S(\mathfrak{X}^*)$

(C) a finite set $\{y_{n,i}^*\}_{i=1}^{p_n}$ in $S(\mathfrak{X}^*)$ for each $n \geq 0$

such that for the Haar system $\{h_j\}_{j\geq 1}$ corresponding to $\mathcal{A}$, for some $\delta > 0$, and each $n \geq 0$,

(1) $x_{t_n}^*(Th_{2^n+k}) > \delta$ for $k = 1, \ldots, 2^n$

(2) $\{y_{n,i}^*\}_{i=1}^{p_n}$ norms $\mathrm{sp}(Th_j : 1 \leq j \leq 2^n)$ within τ_n

(3) $y_{n,i}^*(Th_{2^n+k}) = 0$ for $k = 1, \ldots, 2^n$ and $i = 1, \ldots, p_n$.

Note that if $\Pi\tau_n$ is finite, then conditions (2) and (3) guarantee that the natural blocking $\{\mathrm{sp}(Th_j : 2^{n-1} < j \leq 2^n)\}_{n\geq 0}$ forms a finite dimensional decomposition with constant at most $\Pi\tau_n$.

The proof uses the following standard lemma which, for completeness, we shall prove later.

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LEMMA 2. *If $A \in \Sigma^+$ and $\{g_i\}_{i=1}^n$ is a finite collection of L_1 functions, then an extreme point u of the set $C \equiv \{f \in L_1 : |f| \leq 1_A$ and $\int_A f g_i \, d\mu = 0$ for $i = 1, \ldots, n\}$ has the form $|u| = 1_A$.*

PROOF OF THEOREM 1. Let $T: L_1 \longrightarrow \mathfrak{X}$ be a norm one operator that is not completely continuous. Then there is a sequence $\{r_t\}$ in L_1 and a sequence $\{x_t^*\}$ in $S(\mathfrak{X}^*)$ satisfying:

(a) $\|r_t\|_{L_\infty} \leq 1$

(b) r_t converges to 0 weakly in L_1

(c) $4\delta \leq x_t^* T r_t$ for some $\delta > 0$.

Consider $T^* x_t^* \in L_\infty$. Since $\|r_t(T^* x_t^*)\|_{L_\infty}$ is at most 1, by passing to a subsequence we may assume that $\{r_t(T^* x_t^*)\}$ converges to some function h in the weak-star topology on L_∞. Since $\int h \, d\mu \geq 4\delta$ the set $A_1^0 \equiv [h \geq 4\delta]$ is in Σ^+. (Compare this with [B2, Proposition 5]).

We shall construct, by induction on the level n, a dyadic splitting of A_1^0 along with the desired functional. Fix $n \geq 0$.

Suppose we are given a finite dyadic splitting $\{A_k^m : m = 0, \ldots, n$ and $k = 1, \ldots, 2^m\}$ of A_1^0 up to n-th-level. This gives the corresponding Haar functions $\{h_j : 1 \leq j \leq 2^n\}$. For each $1 \leq k \leq 2^n$, we shall partition A_k^n into 2 sets A_{2k-1}^{n+1} and A_{2k}^{n+1} of equal measure (thus finding h_{2^n+k}) and find $x_{t_n}^* \in S(\mathfrak{X}^*)$ and a sequence $\{y_{n,i}^*\}_{i=1}^{p_n}$ in $S(\mathfrak{X}^*)$ such that conditions (1), (2), and (3) hold.

Find a finite set $\{y_{n,i}^*\}_{i=1}^{p_n}$ in $S(\mathfrak{X}^*)$ that norms $\mathrm{sp}(Th_j : 1 \leq j \leq 2^n)$ within τ_n. Let

$$C_k^n \equiv \left\{ f \in L_1 : |f| \leq 1_{A_k^n}, \int_{A_k^n} f \, d\mu = 0 \text{ and } \int_{A_k^n} (T^* y_{n,i}^*) f \, d\mu = 0 \text{ for } 1 \leq i \leq p_n \right\}.$$

Note that each C_k^n is a convex weakly compact subset of L_1.

Since $\{r_t\}$ tends weakly to 0, for large t there is a small perturbation $\tilde{r}_t$ of r_t so that $\tilde{r}_t 1_{A_k^n}$ is in C_k^n for each k. To see this, put

$$F = \mathrm{sp}\Big(\{1_{A_k^n}\} \cup \{(T^* y_{n,i}^*) 1_{A_k^n} : k = 1, \ldots, 2^n \text{ and } i = 1, \ldots, p_n\} \Big) \subset L_1.$$

Now pick $t_n \equiv t$ so large that for $k = 1, \ldots, 2^n$ and $i = 1, \ldots, p_n$

(d) $\int_{A_k^n} r_t(T^* x_t^*) \, d\mu \geq 2\delta\alpha_n$

(e) $|\int_\Omega r_t f \, d\mu| \leq \frac{\delta}{3}\alpha_n \|f\|$ for all f in F

where $\alpha_n = 2^{-n}\, \mu(A_1^0) \equiv \mu(A_k^n)$. Condition (d) follows from the definition of A_1^0 and the weak-star convergence of $\{r_t(T^* x_t^*)\}$ to h while condition (e) follows from (b) and the fact that F is finite dimensional.

Thus the L_∞-distance from r_t to $F^\perp \equiv \{g \in L_\infty : \int_\Omega f g \, d\mu = 0$ for each $f \in F\}$ is at most $\frac{\delta\, \alpha_n}{3}$. So there is $\tilde{r}_t \in F^\perp$ such that $\|\tilde{r}_t - r_t\|_{L_\infty}$ is less than $\delta\alpha_n$ and, as with r_t, is of L_∞-norm at most 1. Clearly $\tilde{r}_t 1_{A_k^n} \in C_k^n$ for each $k = 1, \ldots, 2^n$.

The functional $T^* x_t^* \in L_1^*$ attains its maximum on C_k^n at an extreme point u_k^n of C_k^n. By the lemma, $u_k^n = 1_{A_{2k-1}^{n+1}} - 1_{A_{2k}^{n+1}}$ for 2 disjoint sets A_{2k-1}^{n+1} and A_{2k}^{n+1} whose union is A_k^n. Furthermore, A_{2k-1}^{n+1} and A_{2k}^{n+1} are of equal measure since $\int_{A_k^n} u_k^n \, d\mu = 0$.

Condition (3) holds since for $i = 1, \ldots, p_n$ and $k = 1, \ldots, 2^n$

$$y^*_{n,i}(Th_{2^n+k}) = \alpha_n^{-1} \int_{A^n_k} (T^* y^*_{n,i}) u^n_k \, d\mu = 0.$$

Condition (1) follows from the observations that

$$x^*_{t_n}(Th_{2^n+k}) = \alpha_n^{-1}\,(T^* x^*_{t_n}) u^n_k \geq \alpha_n^{-1}\,(T^* x^*_{t_n})(\tilde{r}_t 1_{A^n_k})$$

and

$$|(T^* x^*_{t_n})(\tilde{r}_t 1_{A^n_k}) - (T^* x^*_{t_n})(r_t 1_{A^n_k})| \leq \|\tilde{r}_t - r_t\|_{L_1} < \delta\, \alpha_n$$

and

$$(T^* x^*_{t_n})(r_t 1_{A^n_k}) \geq 2\delta \alpha_n. \qquad \blacksquare$$

PROOF OF LEMMA 2. Fix a function f of C such that $|f| \neq 1_A$. Find a positive α and a subset B of A with positive measure such that $|f 1_B| < 1 - \alpha$.

Let $\tilde{\Sigma} = B \cap \Sigma$. Consider the measures $\lambda_i \colon \tilde{\Sigma} \longrightarrow \mathbb{R}$ given by $\lambda_i(E) \equiv \int_E g_i \, d\mu$. Define the measure $\lambda \colon \tilde{\Sigma} \longrightarrow \mathbb{R}^{n+1}$ by

$$\lambda(E) = \big(\lambda_1(E), \ldots, \lambda_n(E), \mu(E)\big).$$

Liapounoff's Convexity Theorem gives a subset B_1 of B satisfying $\lambda(B_1) = \frac{1}{2}\lambda(B) + \frac{1}{2}\lambda(\emptyset)$. Set $B_2 = B \setminus B_1$. Note that

$$\lambda_i(B_1) = \frac{1}{2}\lambda_i(B) = \lambda_i(B_2) \quad \text{and} \quad \mu(B_1) = \frac{1}{2}\mu(B) = \mu(B_2)$$

for $i = 1, \ldots, n$. Set

$$f_1 = f + \alpha(1_{B_1} - 1_{B_2}) \quad \text{and} \quad f_2 = f + \alpha(1_{B_2} - 1_{B_1}).$$

Clearly f_1 and f_2 are in C and $f = \frac{1}{2} f_1 + \frac{1}{2} f_2$. Thus f is not an extreme point of C. $\blacksquare$

3. **Geometric view-point.** Consider a non-completely-continuous operator $T \colon L_1 \longrightarrow \mathfrak{X}$ along with the corresponding Haar system $\{h_j\}$ from Theorem 1. Let $\{I^n_k = [\frac{k-1}{2^n}, \frac{k}{2^n})\}_{n,k}$ be the usual dyadic splitting of $[0, 1]$ with corresponding Haar functions $\{\tilde{h}_j\}_{j \geq 1}$. Consider the map $\tilde{T} \equiv T \circ S$ where $S \colon L_1 \longrightarrow L_1$ is the isometry that takes $\tilde{h}_j$ to h_j. Theorem 1 gives that there is a sequence $\{x^*_n\}_{n \geq 0}$ in $S(\mathfrak{X}^*)$ and a subspace $\mathfrak{X}_0$ of $\mathfrak{X}$ such that

(1) $x^*_n(\tilde{T}\tilde{h}_{2^n+k}) > \delta$ for some $\delta > 0$

(2) $\{\mathrm{sp}(\tilde{T}\tilde{h}_j : 2^{n-1} < j \leq 2^n)\}_{n \geq 0}$ is a finite dimensional decomposition of $\mathfrak{X}_0$ with constant at most $1 + \tau$.

The next corollary follows from the observation that $\tilde{T}$ is not completely continuous and $\tilde{T} L_1 \subset \mathfrak{X}_0$.

COROLLARY 3. *A Banach space failing the* CCP *has a subspace with a finite dimensional decomposition (with constant arbitrarily close to 1) that fails the* CCP.

A *tree* in $\mathfrak{X}$ is a system of the form $\{x_k^n : n = 0, 1, \ldots; k = 1, \ldots, 2^n\}$ satisfying

$$x_k^n = \frac{x_{2k-1}^{n+1} + x_{2k}^{n+1}}{2}.$$

Associated to a tree is its difference system $\{d_j\}_{j\geq 1}$ where $d_1 = x_1^0$ and

$$d_{2^n+k} = \frac{x_{2k-1}^{n+1} - x_{2k}^{n+1}}{2}.$$

There is a one-to-one correspondence between the bounded linear operators T from L_1 into $\mathfrak{X}$ and bounded trees $\{x_k^n\}$ growing in $\mathfrak{X}$. This correspondence is realized by $T(\tilde{h}_j) = d_j$.

A tree is a *δ-Rademacher tree* if $\| \sum_{k=1}^{2^n} d_{2^n+k}\| \geq 2^n\delta$. A tree is a *separated δ-tree* if there exists a sequence $\{x_n^*\}_{n\geq 0}$ in $S(\mathfrak{X}^*)$ such that $x_n^*(d_{2^n+k}) > \delta$. Clearly, a separated δ-tree is also a δ-Rademacher tree. The operator corresponding to a δ-Rademacher tree is not completely continuous since the image of the Rademacher functions stay large in norm. Thus if a bounded δ-Rademacher tree (or separated δ-tree) grows in $\mathfrak{X}$, then $\mathfrak{X}$ fails the CCP.

In any Banach space failing the CCP, a bounded δ-Rademacher tree grows (see [G1] for a direct proof); in fact, even a bounded separated δ-tree grows (see [G2] for an indirect proof). The proof of Theorem 1 is a *direct* proof that *if $\mathfrak{X}$ fails* CCP *then a bounded separated δ-tree, with a difference sequence naturally blocking into a finite dimensional decomposition, grows in $\mathfrak{X}$.*

4. **From decompositions to bases.** As previously mentioned, we do not know whether a space failing the CCP necessarily contains a subspace with a basis which fails the CCP. However, if the space has some nice local properties, then the proof of Theorem 1 can be modified to show this is so.

We now introduce some local properties. A Banach space is said to have the (K,n)-*local basis property* if each of its n-dimensional subspaces has a finite dimensional superspace which has a basis with basis constant at most K. A Banach space is said to have the $(*$-$K)$-*property* provided that each of its finite codimensional subspaces contains a finite codimensional subspace which has, for each n, the (K,n)-local basis property. A Banach space is said to have the $(**$-$K)$-*property* provided that, for each n, each of its finite codimensional subspaces contains a finite codimensional subspace (depending on n) with the (K,n)-local basis property. Clearly, the $(*$-$K)$-property implies the $(**$-$K)$-property, but Szarek's spaces [S] show that the properties are not equivalent. We do not know any example of a space which fails the $(**$-$K)$-property for all K.

Spaces failing cotype (*i.e.*, containing ℓ_∞^n uniformly for all n), have the $(*$-$K)$-property. In fact, if $\mathfrak{X}$ fails cotype and Z is a finite codimensional subspace of $\mathfrak{X}$, then for any finite dimensional subspace W of $\mathfrak{X}$ there is a finite dimensional subspace Y of

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Z such that $W + Y$ has a basis with basis constant less than, say, 10. To see this, use the fact ([P], [JRZ]) that W is $(1+\epsilon)$-complemented in a finite dimensional space which has a basis with basis constant less than $1+\epsilon$ and embed the complement to W in that space into $Z \cap {}^{\perp}F$, where F is a finite subset of $\mathfrak{X}^*$ which $(1+\epsilon)$-norms W. This is possible because finite codimensional subspaces of $\mathfrak{X}$ must contain ℓ_∞^n uniformly for all n and hence [J] contain even $(1+\epsilon)$-isomorphs of ℓ_∞^n for all n.

Banach lattices also enjoy the $(*\text{-}K)$-property. By the above observation, we need only consider lattices with cotype. Such a lattice X must be order continuous since it contains no copy of c_0. By a perturbation argument, it is enough to show that if F is a finite set of disjoint linear functionals, then $F^{\perp}$ has the local basis property with uniform constant. To see this, consider $F = \{f_1, \ldots, f_n\}$. Let X_j be the "support" of f_j; that is, let X_j be the complementary band to the band $\{x \in X : |f_j|\,|x| = 0\}$. Notice that the X_j's are disjoint since the f_j's are disjoint. Thus, $F^{\perp}$ is the disjoint sum of $Y, Y_1, \ldots, Y_n$, where each Y_j is a one codimensional subspace of the band X_j and Y is the intersection of the bands $\{x \in X : |f_j|\,|x| = 0\}$.

COROLLARY 4. *If a Banach space $\mathfrak{X}$ fails the* CCP *and enjoys the* $(**\text{-}K)$*-property, then $\mathfrak{X}$ has a subspace with a basis that fails the* CCP.

To see this, it is enough by the argument for Corollary 3 to observe that when $\mathfrak{X}$ has the $(**\text{-}K)$-property Theorem 1 can be modified by adding:

(D) finite dimensional subspaces $\{G_n\}_{n=0}^{\infty}$ of $\mathfrak{X}$

changing (2) and (3) to:

(2′) $\{y_{n,i}^*\}_{i=1}^{p_n}$ norms $\operatorname{sp}(\bigcup_{k=0}^{n} G_k)$ within τ_n

(3′) $G_{n+1} \subset {}^{\perp}\{y_{n,i}^*\}_{i=1}^{p_n}$

and adding:

(4) $\{Th_j : 2^{n-1} < j \leq 2^n\} \subset G_n$

(5) G_n has a basis with basis constant at most K.

To achieve these modifications, at the first stage in the proof of Theorem 1, let $G_0 = \operatorname{sp}(Th_1)$. Then in the inductive step in the proof, choose $\{y_{n,i}^*\}_{i=1}^{q_n}$ so that (2′) holds and, by appealing to the $(**\text{-}K)$-property, enlarge the set to $\{y_{n,i}^*\}_{i=1}^{p_n}$ where $p_n \geq q_n$ so that ${}^{\perp}\{y_{n,i}^*\}_{i=1}^{p_n}$ has the $(K, 2^n)$-local basis property. Proceed as before and then, after selecting A_k^{n+1} (thereby defining h_j for $j = 2^n + 1, \ldots, 2^{n+1}$), choose a finite dimensional space G_{n+1} such that $\{Th_j : 2^n < j \leq 2^{n+1}\} \subset G_{n+1} \subset {}^{\perp}\{y_{n,i}^*\}_{i=1}^{p_n}$ and G_{n+1} has a basis with basis constant at most K. ∎

In the last years, geometric properties such as the CCP have allowed a deeper understanding of the RNP. Two such properties are the Point of Continuity property (PCP) and the Convex Point of Continuity property (CPCP). We refer the reader to [GGMS] for the definitions and a survey of these properties; here we merely recall that the RNP implies the PCP, which implies the CPCP, which in turn implies the CCP.

Relevant for this paper is Bourgain's result [B3, Proposition 5.4] that a space failing the PCP has a subspace with a finite dimensional decomposition which fails the PCP. Similar to the situation with the CCP, additional local structure on the space can help to sharpen the decomposition to a basis.

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PROPOSITION 5. *If a Banach space $\mathfrak{X}$ fails the* PCP *and enjoys the* $(*\text{-}K)$*-property, then* $\mathfrak{X}$ *has a subspace with a basis which fails the* PCP.

To see this, it is convenient for us to use Rosenthal's exposition of Bourgain's result [R, Remark, p. 315]. In a space $\mathfrak{X}$ failing the PCP, Rosenthal finds a "bad" bounded subset U of $\mathfrak{X}$ and $\delta > 0$ and then constructs by induction on n, for a given sequence $\{\tau_n\}$ of numbers larger than one with finite product

(A) finite subsets $\{D_n\}_{n=1}^{\infty}$ of U

(B) finite dimensional subspaces $\{F_n\}_{n=1}^{\infty}$ of $\mathfrak{X}$

(C) a finite set $\{x_{n,i}^*\}_{i=1}^{p_n}$ in $S(\mathfrak{X}^*)$ for each $n \geq 1$

such that, for $H_n \equiv \mathrm{sp}(x_{n,i}^*)_{i=1}^{p_n}$,

(1) $D_n \subset D_{n+1}$

(2) $D_n \subset F_1 + \cdots + F_n$

(3) $\{x_{n,i}^*\}_{i=1}^{p_n}$ norms $\mathrm{sp}(\bigcup_{j=1}^n F_j)$ within τ_n

(4) $F_{n+1} \subset {}^{\perp}H_n$

(5) for every $d \in D_n$ and n—$\frac{1}{n}$ neighborhood V of d, there is a $d' \in D_{n+1} \cap V$ such that $\|d - d'\| > \delta$.

He then considers the set $D \equiv \bigcup_{n=1}^{\infty} D_n$. By construction, each relatively weakly open neighborhood of $\bar{D}$ has diameter at least δ and $\{F_n\}_{n=1}^{\infty}$ forms a finite dimensional decomposition (with constant at most $\Pi\tau_n$) of a subspace which contains $\bar{D}$.

If $\mathfrak{X}$ also enjoys the $(*\text{-}K)$-property, then Rosenthal's construction can be modified by adding:

(D) finite dimensional subspaces $\{G_n\}_{n=1}^{\infty}$ of $\mathfrak{X}$

and changing (3) and (4) to:

(3′) $\{x_{n,i}^*\}_{i=1}^{p_n}$ norms $\mathrm{sp}(\bigcup_{j=1}^n G_j)$ within τ_n

(4′) $G_{n+1} \subset {}^{\perp}H_n$

and adding:

(6) $F_n \subset G_n$

(7) G_n has a basis with constant at most K.

To accomplish this, at the first stage of his construction, let $G_1 = F_1$. Then, in the inductive step, when given D_n, $\{F_j\}_{j=1}^n$, $\{G_j\}_{j=1}^n$, and $\{x_{n,i}^*\}_{i=1}^{q_n}$ satisfying (2), (3′), and (6), appeal to the $(*\text{-}K)$-property to find $\{x_{n,i}^*\}_{i=1}^{p_n}$ with $p_n \geq q_n$ such that ${}^{\perp}\{x_{n,i}^*\}_{i=1}^{p_n}$ has the (K, m)-local basis property for all m. Put $H_n = \mathrm{sp}\{x_{n,i}^*\}_{i=1}^{p_n}$. Proceed as in Rosenthal's argument to find the finite dimensional subspace F_{n+1} of ${}^{\perp}H_n$. The $(*\text{-}K)$-property then provides the desired G_{n+1}. Clearly this is sufficient. ∎

Bourgain [B3, Theorem 5.7; B1, Theorem 1] also showed that a space failing the RNP has a subspace with a finite dimensional decomposition which fails the RNP. The argument is split into two cases. In the first case, Bourgain shows that a space failing not only the RNP but also the CPCP has a subspace with a *finite dimensional decomposition* which fails the RNP. It immediately follows from the last proposition that if such a space also enjoys the $(*\text{-}K)$-property, then it has a subspace with a *basis* which fails the RNP. In the second case, Bourgain shows that a space which fails the RNP but has the CPCP contains a subspace with a finite dimensional decomposition which fails the RNP. His argument is rather delicate; the above technique for passing from a finite dimensional decomposition to a basis seems not to work.

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University of South Carolina
Department of Mathematics
Columbia, South Carolina 29208
U.S.A.
e-mail: girardi@math.scarolina.edu

Texas A&M University
Department of Mathematics
College Station, Texas 77843
U.S.A.
e-mail: WBJ7835@venus.tamu.edu

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