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THE CLASSIFICATION OF FACTORS IS NOT SMOOTH

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1. Introduction. There is a natural Borel structure on the set F of all factors on a separable Hilbert space [**3**]. Let $\hat{F}$ denote the algebraic isomorphism classes in F together with the quotient Borel structure. Now that various non-denumerable families of mutually non-isomorphic factors are known to exist [**1**; **6**; **8**; **10**; **11**; **12**; **13**], the most obvious question to be resolved is whether or not $\hat{F}$ is smooth (i.e. is there a countable family of Borel sets which separate points). We answer this question negatively by an explicit construction. To each infinite sequence $\{a_k\}$ of zeroes and ones we associate a factor $M\{a_k\}$ which is given as an infinite tensor product of type I_2 factors. Using techniques given by Araki and Woods [**1**], we prove that $M\{a_k\}$ and $M\{b_k\}$ are isomorphic if and only if $a_k = b_k$ except for at most a finite number of indices k. It then follows from a straightforward Borel argument that $\hat{F}$ is not smooth.

Section 2 contains some definitions and known properties of ITPFI factors (factors constructible as infinite tensor products of type I factors). In Section 3 we prove our main result. Section 4 contains some concluding remarks.

We shall use the following notation. If H is a Hilbert space then $B(H)$ denotes the set of all bounded linear operators on H. The statement "$a_k = b_k$ (a.a.)" means that the equality holds except for at most a finite number of indices k. If the von Neumann algebras M and N are algebraically isomorphic we write $M \sim N$. We assume that the reader is familiar with the standard notation and terminology for von Neumann algebras.

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2. ITPFI factors. For the sake of completeness we recall some definitions and results pertaining to ITPFI factors (see [**1**] for a more complete discussion). Let $H = \otimes_{n=1}^{\infty} (H_n, \Omega_n)$ be the infinite tensor product of the Hilbert spaces H_n which contains the product vector $\otimes \Omega_n$, $\Omega_n \in H_n$, $0 < \Pi\|\Omega_n\| < \infty$. Let π_n be the canonical mapping from $B(H_n)$ to $B(H)$ defined by $\pi_n S = (\otimes_{m \neq n} 1_m) \otimes S$ where $S \in B(H_n)$ and 1_m is the identity operator on H_m. Given $\otimes(H_n, \Omega_n)$ and type I factors $M_n \subset B(H_n)$ we define the factor

$$\otimes(M_n, \Omega_n) = \{\pi_n M_n; n = 1, 2, \ldots\}''$$

Any factor constructible in this manner is called an ITPFI factor. By the

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eigenvalue list of a vector Ω relative to a type I factor M we mean the list $(\lambda_1, \lambda_2, \ldots)$ of eigenvalues of the nonnegative trace class operator ρ in M defined by

$$\text{Trace}\, \rho A = (A\Omega, \Omega), A \in M$$

ordered such that $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq 0$. We denote it by $\text{Sp}(\Omega/M)$. $\text{Sp}(\Omega/M)$ gives a complete set of unitary invariants for the pair (M, Ω).

In the remainder of this paper $\dim H_n = 4$ and M_n is a type I_2 factor. Let $0 \leqq x \leqq 1$, $\lambda = (1 + x)^{-1}$. We define factors $R_x = \otimes(M_n, \Omega_n)$ where $\text{Sp}(\Omega_n/M_n) = (\lambda, 1 - \lambda)$ independent of n. For any factor M we define the algebraic invariant $\rho(M)$ as the set of all $0 \leqq x \leqq 1$ such that $R_x \sim R_x \otimes M$. For the examples we shall consider in Section 3 the following notation is convenient.

Definition 2.1. Given $0 \leqq l_1 < l_2 < \ldots, l_j \rightarrow \infty$, and nonnegative integers $N_1, N_2, \ldots$, let

$$\lambda_n = (1 + e^{-l_j})^{-1}, N_1 + \ldots + N_{j-1} < n \leqq N_1 + \ldots + N_j$$

We denote the factor $\otimes(M_n, \Omega_n)$ where $\text{Sp}(\Omega_n/M_n) = (\lambda_n, 1 - \lambda_n)$ by $M[l_j, N_j]$.

The proof of Theorem 3.3 is based on the following result [**1**, Lemma 11.7].

LEMMA 2.2. *Let* $0 < \theta < \infty$, $M = M[l_j, N_j]$. *For each* j *choose an integer* p_j *such that* $|\delta_j|$ *is a minimum where*

$$\delta_j = p_j\theta - l_j.$$

Then $e^{-\theta} \in \rho(M)$ *if and only if*

$$\sum_{j=1}^{\infty} N_j e^{-l_j} \delta_j^2 < \infty.$$

3. A family of factors. Let G denote the Borel space of all sequences $a = \{a_k\}$, $a_k = 0, 1$ with the product Borel structure, Δ the Borel subset of sequences a such that $a_k = 0$(a.a.). Using the binary decimal expansion we can identify G with the unit interval on the real line with the usual Borel structure, and Δ with the binary rationals. G is a compact group under addition mod 1. We define an equivalence relation on G by $a \sim b$ if and only if $a - b \in \Delta$ (i.e., $a_k = b_k$(a.a.)). We give $\hat{G} = G/\Delta$ the quotient Borel structure. By Theorem 7.2 of [**7**], $\hat{G}$ is not countably separated. We will construct a Borel map M from G into F such that $M(a) \sim M(b)$ if and only if $a \sim b$, and which is a Borel isomorphism of G onto MG. It will then follow that there is a one-to-one Borel map $\hat{M}$ from $\hat{G}$ into $\hat{F}$, which implies that $\hat{F}$ is not countably separated.

Definition 3.1. For each $a \in G$ we define a factor $M(a)$ as follows. We define a sequence of integers m_k, N_k. Let $m_1 = 3$. Given m_k, choose N_k, m_{k+1} such that

$$N_k \geqq (m_k + 1)^2 e^{m_k!} > N_k - 1 \tag{3.1}$$

$$(m_{k+1} + 1)! > [(m_k + 1)!]^3 \tag{3.2}$$

and m_{k+1} is odd. Let $H = \otimes(H_n, \Omega_n)$ where $\dim H_n = 4$. We define λ_n, $n = 1, 2, \ldots$ as follows:
Let

$$\sum_{j=1}^{k-1} N_j < n \leqslant \sum_{j=1}^{k} N_j \tag{3.3}$$

and let

$$\lambda_n = \begin{cases} (1 + e^{-m_k!})^{-1} & \text{if } a_k = 1 \\ 1 & \text{if } a_k = 0. \end{cases} \tag{3.4}$$

Choose a type I_2 factor $M_n(a)$ on each H_n such that $\mathrm{Sp}(\Omega_n/M_n(a)) = (\lambda_n, 1 - \lambda_n)$. We now define

$$M(a) = \otimes(M_n(a), \Omega_n). \tag{3.5}$$

We remark that $M(a)$ is type I_∞ if $a \in \Delta$, otherwise $M(a)$ is type III (see [**1**, Lemma 2.14]).

LEMMA 3.2. *The map M is Borel.*

Proof. By the Corollary to Theorem 2 of [**3**] it is sufficient to show that there is a sequence of operators $T_k(a) \in M(a)$ such that

$$\{T_k(a); k = 1, 2, \ldots\}'' = M(a)$$

for each a, and the maps $a \to (x, T_k(a)y)$ are Borel for all $k = 1, 2, \ldots$ and all $x, y \in H$. Note that any type I_2 factor is generated by 4 partial isometries, and that each $M_n(a)$ depends on only one coordinate a_k where k is determined by (3.3). Thus each $M_n(a)$ is generated by 4 operators $T_{nm}(a_k)$, $m = 1, 2, 3, 4$. Clearly the maps

$$a \to a_k \to (x, T_{nm}(a_k)y), \; m = 1, 2, 3, 4$$

are Borel for all $x, y \in H$. Since $T_{nm}(a)$ for all n, m generate $M(a)$, the map M is Borel.

THEOREM 3.3. *$M(a) \sim M(b)$ if and only if $a \sim b$.*

Proof. If $a_k = b_k$(a.a.) then $\mathrm{Sp}(\Omega_n/M_n(a)) = \mathrm{Sp}(\Omega_n/M_n(b))$(a.a.) and $M(a) \sim M(b)$ (use Lemma 2.13 of [**1**]).

If $a \nsim b$ then there is a sequence $k_1 < k_2 < \ldots$ such that either $a_{k_j} = 0$, $b_{k_j} = 1$ or $a_{k_j} = 1$, $b_{k_j} = 0$ for all j. Without loss of generality we can take

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$a_{k_j} = 0, b_{k_j} = 1, j = 1, 2, \ldots$. Let

$$(3.6) \qquad \theta = n_1 \prod_{j=1}^{\infty} [1 - (n_j/n_{j+1})]^{-1}$$

where

$$(3.7) \qquad n_j = (m_{k_j} + 1)!$$

It follows from (3.2) that the infinite product in (3.6) converges. For any $j = 1, 2, \ldots$ we have

$$(3.8) \qquad \theta = n_j Q_j^{-1}(1 + \epsilon_j)$$

where

$$Q_1 = 1$$

$$(3.9) \qquad Q_j = \prod_{s=1}^{j-1} [(n_{s+1}/n_s) - 1], j = 2, 3, \ldots$$

$$(3.10) \qquad 1 + \epsilon_j = \prod_{s=j}^{\infty} [1 - (n_s/n_{s+1})]^{-1}.$$

We will use Lemma 2.2 to prove that $e^{-\theta} \in \rho(M(a))$, $e^{-\theta} \notin \rho(M(b))$. In order to do this we note that by construction we can write

$$M(a) = M[m_k!, a_k N_k] \otimes P(a),\ M(b) = M[m_k!, b_k N_k] \otimes P(b)$$

where $P(a)$, $P(b)$ are tensor products of type I_2 factors where the eigenvalue lists are all $(1, 0)$, and hence $P(a)$, $P(b)$ are type I (use Lemma 2.14 of [**1**]). It follows from Lemmas 11.4 and 11.5 of [**1**] that $\rho'(M(a)) = \rho'(M[m_k!, a_k N_k])$, $\rho'(M(b)) = \rho'(M[m_k!, b_k N_k])$ where $\rho'(M) = \rho(M) \cap [0, 1)$. Thus we need estimates on

$$(3.11) \qquad \delta_k = \inf_p |p\theta - m_k!|$$

where the infimum is taken over integers p.

Case 1. $k \notin (k_1, k_2, \ldots)$, $k > k_1$: Such a k need not exist but if it does there is an integer s such that

$$(3.12) \qquad k_s < k < k_{s+1}.$$

Let

$$(3.13) \qquad p = Q_s m_k!/(m_{k_s} + 1)!.$$

Note that p is an integer. Equations (3.7), (3.8) with $j = s$ and equations (3.11), (3.13) give

$$(3.14) \qquad \delta_k \leqq |p\theta - m_k!| = m_k! \epsilon_s.$$

We now derive an estimate on ϵ_s. It follows from the power series for $\log(1 + x)$ that if $0 < x \leqq \frac{1}{2}$ we have

$$(3.15) \qquad -\tfrac{3}{2}x < \log(1 - x) < -x$$

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and

$$\tfrac{3}{4}x < \log(1 + x) < x. \tag{3.16}$$

Equations (3.10) and (3.15) give

$$\begin{aligned} \log(1 + \epsilon_s) &= -\sum_{j=s}^{\infty} \log[1 - (n_j/n_{j+1})] \\ &< \tfrac{3}{2} \sum_{j=s}^{\infty} n_j/n_{j+1}. \end{aligned} \tag{3.17}$$

Equations (3.2), (3.7), (3.12) give

$$n_s/n_{s+1} < [(m_k + 1)!]^{-2} \tag{3.18}$$

and for $t > 0$,

$$\begin{aligned} n_{s+t}/n_{s+t+1} &< [(m_{k_{s+t}} + 1)!]^{-2} \\ &< 2^{-t}[(m_k + 1)!]^{-2}. \end{aligned} \tag{3.19}$$

From (3.17)–(3.19) we have

$$\log(1 + \epsilon_s) < \tfrac{3}{2}[(m_k + 1)!]^{-2} \sum_{t=0}^{\infty} 2^{-t} < 3[m_k!]^{-2}, \tag{3.20}$$

and from (3.16) and (3.20) it follows that

$$\epsilon_s < 4[m_k!]^{-2}. \tag{3.21}$$

By (3.14), (3.21)

$$\delta_k < 4/m_k!, \tag{3.22}$$

and from (3.1), (3.22) we obtain

$$N_k e^{-m_k!}\delta_k^2 < 16[(m_k + 1)^2 + e^{-m_k!}][m_k!]^{-2}. \tag{3.23}$$

Equations (3.2) and (3.23) yield

$$\sum_{k \notin \{k_1, k_2, \ldots\}} N_k e^{-m_k!}\delta_k^{\,2} < \infty.$$

It follows that

$$\sum a_k N_k e^{-m_k!}\delta_k^{\,2} < \infty \tag{3.24}$$

and thus $e^{-\theta} \in \rho(M[m_k!, a_k N_k])$ by Lemma 2.2.

Case 2. $k = k_j$ for some j: Let

$$r = Q_j/(m_k + 1). \tag{3.25}$$

By construction $m_k + 1$ is even. It follows from (3.2), (3.7) and (3.9) that n_{s+1}/n_s is always even and thus Q_j is always odd. Hence r is not an integer, and the integer p giving the infimum for δ_k satisfies

$$|p - r| \geqq (m_k + 1)^{-1} \tag{3.26}$$

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Equations (3.7), (3.8), (3.25) give

$$(3.27) \qquad |r\theta - m_k!| = m_k!\epsilon_j.$$

The same argument used to derive (3.21) yields that

$$(3.28) \qquad \epsilon_j < 4[m_k!]^{-2}.$$

Equations (3.11), (3.26–28) give

$$(3.29) \qquad \begin{aligned} \delta_k = |p\theta - m_k!| &\geqq |(p - r)\theta| - |r\theta - m_k!| \\ &> \theta(m_k + 1)^{-1} - 4/m_k!, \end{aligned}$$

and from (3.1) and (3.29) we obtain

$$(3.30) \qquad N_k e^{-m_k!}\delta_k^2 > \theta^2 - 8\theta(m_k + 1)/m_k! + 16(m_k + 1)^2(m_k!)^{-2}.$$

Since $m_k \to \infty$ (see (3.2)) it follows that

$$(3.31) \qquad \sum b_k N_k e^{-m_k!}\delta_k^2 \geqq \sum_{j=1}^{\infty} N_{k_j} e^{-m_k!}\delta_{k_j}^2 = \infty$$

and thus $e^{-\theta} \notin \rho(M[m_k!, b_k N_k])$ by Lemma 2.2. Since ρ is an algebraic invariant we have $M(a) \nsim M(b)$.

THEOREM 3.4. *$\hat{F}$ is not countably separated.*

Proof. Let Π_G, Π_F be the quotient maps from $G \to \hat{G}$, $F \to \hat{F}$. Since M is a one-to-one Borel function from the standard Borel space G into the standard Borel space F, its range MG is a Borel subset of F and M is a Borel isomorphism of G onto MG [**7**, Theorem 3.2]. Since M respects the equivalence relations (Theorem 3.3), it defines a map $\hat{M}$ from $\hat{G}$ into $\hat{F}$ such that $\hat{M}\Pi_G = \Pi_F M$. We now prove that $\hat{M}$ is a Borel map from $\hat{G}$ onto $\hat{M}\hat{G}$ with its relative Borel structure in $\hat{F}$. A Borel set in $\hat{M}\hat{G}$ is of the form $X \cap \hat{M}\hat{G}$ where X is Borel in $\hat{F}$. Then $\Pi_F^{-1}(X) \cap MG$ is Borel in MG, and $M^{-1}(\Pi_F^{-1}(X) \cap MG)$ is Borel in G. But $M^{-1}(X \cap \hat{M}\hat{G}) = \Pi_G(M^{-1}(\Pi_F^{-1}(X) \cap MG))$ which is Borel in $\hat{G}$. Thus $\hat{M}$ is Borel. Now $\hat{F}$ countably separated would imply that $\hat{M}\hat{G}$ is countably separated which would imply that $\hat{G}$ is countably separated (since $\hat{M}$ is Borel). But since $\hat{G}$ is not countably separated [**7**, Theorem 7.2], the theorem follows.

4. Concluding remarks. Our result is analogous to the fact, first proved by Glimm [**5**], that a separable locally compact group is type I if and only if it has a smooth dual. Actually Glimm proved the stronger result that the dual is not metrically smooth (i.e. not metrically countably separated) if the group is not type I. (A Borel space X is called metrically countably separated if, given any finite Borel measure μ, there is a μ-null Borel set N such that $X - N$ is countably separated.) Since our method of proof involves an explicit construction quite similar to that used by Glimm, one might expect that it could be used to show that $\hat{F}$ is not metrically countably separated. In fact, Nielsen [**9**] has extended the argument of Theorem 3.4 to yield the existence of a von Neumann algebra which is not "centrally smooth" (see [**4**]). This implies that $\hat{F}$ is not metrically countably separated.

Of course we have only shown that the classification of ITPFI factors is not smooth. It remains open whether the classification of type II factors, non-hyperfinite type III factors etc. is smooth or not. While present techniques seem inadequate to decide this, it seems likely that the answer is no.

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