THE THREE KERNELS OF A COMPACT SEMIRING [1]

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A *topological semiring* is a system $(S, +, \cdot)$ where S is a Hausdorff space, $(S, +)$ and $(S, \cdot)$ are topological semigroups (i.e., $+$ and $\cdot$ are continuous associative binary operations on S) and the distributive laws

$$x \cdot (y+z) = (x \cdot y)+(x \cdot z),$$
$$(x+y) \cdot z = (x \cdot z)+(y \cdot z),$$

hold for all x, y, z in S. The operations $+$ and $\cdot$ are called *addition* and *multiplication* respectively.

If $(S, +, \cdot)$ is a compact semiring, the semigroup $(S, +)$ has a kernel $K[+]$ (i.e., an ideal which is contained in every other ideal) whose topological and algebraic structure has been completely determined (see Wallace [13]). We shall call $K[+]$ the *additive kernel* of the semiring. It is natural to ask what information can be given about the multiplication of members of $K[+]$ and, in particular, to wonder whether $K[+]$ is a subsemiring of S. Also $(S, \cdot)$ has a kernel $K[\cdot]$, the *multiplicative kernel* of the semiring, and one can ask similar questions about the addition of members of $K[\cdot]$. The main aim of this paper is to examine these problems.

In Theorem 15 of [11], Selden has shown that when $(S, +, \cdot)$ is a compact semiring there is a set K which is minimal with respect to being an ideal of both $(S, +)$ and $(S, \cdot)$. This set K can perhaps justifiably be called the *kernel* of the semiring. It is shown here in Theorem 9 that $K = S+K[\cdot]+S$.

Throughout this paper, $E[+]$ and $E[\cdot]$ will denote the sets of additive and multiplicative idempotents of a semiring $(S, +, \cdot)$; when S is compact, each is non-empty (Theorem 1.1.10 of [9] or Lemma 4 of [8]). Notice also that $E[+]$ is a multiplicative ideal in any semiring, for if $x \in E[+]$ and $y \in S$,

$$xy+xy = (x+x)y = xy$$

so that $xy \in E[+]$, and similarly $yx \in E[+]$.

We shall often make use of the fact that a compact semigroup which

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is algebraically a group is a topological group ([9], Theorem 1.1.8 or [8], Theorem 1).

1. The multiplicative kernel $K[\cdot]$

Suppose that $(S, +, \cdot)$ is a compact semiring with multiplicative kernel $K[\cdot]$. If $E' = E[\cdot] \cap K[\cdot]$, it is well known (see, for example, [8] or [9]) that $(eSe, \cdot)$ is a compact group if $e \in E'$, that $eSe \cap fSf$ is empty if $e, f \in E'$ and $e \neq f$, and that

$$K[\cdot] = \bigcup_{e \in E'} eSe.$$

If $e \in E'$, we see that for all x, y in S,

$$exe+eye = (ex+ey)e = e(x+y)e \in eSe,$$

so that eSe is a compact subsemiring which is multiplicatively a group. Hence its structure has been completely determined in Theorem 1 of [10]. Further, if $e, f \in E'$, then, because $(K[\cdot], \cdot)$ is completely simple (Theorem 2 of [8]), there exist a in eSf and b in fSe with $ab = e$ and $ba = f$ such that the function $\varphi : eSe \to fSf$ given by $\varphi(x) = bxa$ is a homeomorphism and multiplicative isomorphism onto fSf (see, for example, [2], Lemma 8.2). But if $x, y \in eSe$,

$$\varphi(x+y) = b(x+y)a = (bx+by)a = bxa+bya = \varphi(x)+\varphi(y),$$

and so φ is also an additive isomorphism. Thus the two semirings eSe and fSf are topologically isomorphic.

If $\mathscr{L}$ is the space of minimal left ideals of $(S, \cdot)$, it is known (see [8] or [9]) that if $L \in \mathscr{L}$ then $L = Se$ for some $e \in E'$, that if $L_1, L_2 \in \mathscr{L}$ either $L_1 = L_2$ or $L_1 \cap L_2$ is empty, and that $K[\cdot]$ is the union of all L in $\mathscr{L}$. Because Se is clearly a compact subsemiring, it follows that any $L \in \mathscr{L}$ is a compact subsemiring which is multiplicatively left simple. Further, if $L_1, L_2 \in \mathscr{L}$, let $L_1 = Se$ and $L_2 = Sf$ for $e, f \in E'$, and let $a \in eSf$ and $b \in fSe$ be such that $ab = e$ and $ba = f$. If $\psi(x) = xa$ for all x in Se, it is easily seen that ψ maps Se into Sf. But if $y \in Sf$, then $yb \in Se$ and

$$\psi(yb) = (yb)a = y(ba) = yf = y$$

since f is a right identity for Sf ([12], Theorem 1). Hence ψ maps Se onto Sf and has an inverse $\psi^{-1}(y) = yb$. Also, for all x, y in Se,

$$\psi(x+y) = (x+y)a = xa+ya = \psi(x)+\psi(y).$$

Thus L_1 and L_2 are homeomorphic subsemirings which are additively isomorphic. The minimal right ideals have similar properties.

As $E[+]$ is a non-empty multiplicative ideal, it must contain the minimal such ideal $K[\cdot]$.

The following theorem summarizes the above discussion.

THEOREM 1. *Let* $(S, +, \cdot)$ *be a compact semiring with multiplicative kernel* $K[\cdot]$.

(i) *If* e, f *are distinct multiplicative idempotents in* $K[\cdot]$ *then* eSe *and* fSf *are topologically isomorphic disjoint compact subsemirings which are multiplicatively groups;* $K[\cdot]$ *is the union of all such subsemirings.*

(ii) *If* I_1, I_2 *are two distinct minimal left (right) multiplicative ideals of* S *then* I_1, I_2 *are disjoint, homeomorphic, additively isomorphic subsemirings which are multiplicatively left (right) simple;* $K[\cdot]$ *is the union of all such minimal left (right) ideals.*

(iii) $K[\cdot] \subset E[+]$.

(iv) $K[\cdot] \subset K[\cdot]+K[\cdot]$.

Although $K[\cdot]$ is the union of several subsemirings, it need not itself be a subsemiring of S, as can be seen from the following example.

EXAMPLE 1. Let $S = \{a, b, c, d, e\}$ with the discrete topology and define addition and multiplication on S by means of the following tables.

$+$	a	b	c	d	e
a	a	b	a	e	e
b	b	b	b	b	b
c	a	b	c	d	e
d	e	b	d	d	e
e	e	b	e	e	e

$\cdot$	a	b	c	d	e
a	a	a	c	c	a
b	b	b	d	d	b
c	a	a	c	c	a
d	b	b	d	d	b
e	b	b	d	d	b

It can be readily checked that $(S, +, \cdot)$ is a compact semiring in which $K[\cdot] = \{a, b, c, d\}$ while $K[\cdot]+K[\cdot] = \{a, b, c, d, e\}$.

In view of the occurrence above of compact semirings which are multiplicatively left simple, it is natural to have a closer look at such semirings. The following theorem, however, appears only to scratch the surface.

THEOREM 2. *Let* $(S, +, \cdot)$ *be a compact semiring in which* $(S, \cdot)$ *is left simple, and let* e' *be any multiplicative idempotent. Then each* x *in* S *can be written uniquely in the form* $e\alpha$ *where* $e \in E[\cdot]$ *and* α *belongs to the multiplicative group* $G = e'S$.

$(S, +)$ *and* $(E[\cdot], +)$ *are idempotent semigroups,* $e'S$ *is a subsemiring and, if* $e, f \in E[\cdot]$ *and* $\alpha, \beta \in G$, *there exists* g *in* $E[\cdot]$ *so that*

$$e\alpha+f\beta = g(\alpha+\beta).$$

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Moreover, if $E[\cdot] \sim \Sigma\{E_\gamma; \gamma \in \Gamma\}$ *is the structure decomposition of* $(E[\cdot], +)$ *in the sense of page* 262 *of* [6], *then* $S \sim \Sigma\{E_\gamma G; \gamma \in \Gamma\}$ *is the structure decomposition of* $(S, +)$. *Also, if* $e, f \in E_\gamma$ *for some* $\gamma \in \Gamma$ *and* $\alpha, \beta \in G$, *then*

$$e\alpha + f\beta = (e+f)(\alpha+\beta).$$

PROOF. If $x \in S$, then, because Sx is a left ideal, it follows that $Sx = S$. Hence the first paragraph is a special case of Theorem 1 of [12]. That $(S, +)$ is an idempotent semigroup and $e'S$ is a subsemiring follows from Theorem 1 as here $K[\cdot] = S$.

Let $e, f \in E[\cdot]$ and $\alpha, \beta \in G$, and let $e\alpha + f\beta = g\delta$ for some $g \in E[\cdot]$ and $\delta \in G$. Then

$$(e'g)\delta = e'(g\delta) = e'(e\alpha+f\beta) = (e'e)\alpha + (e'f)\beta.$$

But as each multiplicative idempotent is a right identity for S ([12], Theorem 1) and e' is the identity of G,

$$\delta = \alpha + \beta$$

as required. In particular,

$$e+f = ee'+fe' = g(e'+e') = ge' = g$$

for some $g \in E[\cdot]$, and it follows that $(E[\cdot], +)$ is a semigroup.

If $(T, +)$ is any idempotent semigroup and we define a relation P by xPy if and only if $x+y+x = x$ and $y+x+y = y$, then P is an equivalence relation on T. If $T_\gamma (\gamma \in \Gamma)$ are the equivalence classes modulo P, then each of the sets T_γ is an additive semigroup and we say that the structure decomposition of $(T, +)$ is $T \sim \Sigma\{T_\gamma; \gamma \in \Gamma\}$ (see [6], page 262). It follows from Theorem 1 of [10] that $\delta+\rho+\delta = \delta$ for all $\delta, \rho \in G$. Hence if $e \in E[\cdot]$ and $\alpha \in G$, we see that

$$e = ee' = e(e'+\alpha+e') = ee'+e\alpha+ee' = e+e\alpha+e,$$
$$e\alpha = e(\alpha+e'+\alpha) = e\alpha+ee'+e\alpha = e\alpha+e+e\alpha$$

and so $eP(e\alpha)$. Thus if $e, f \in E[\cdot]$ and $\alpha, \beta \in G$, it follows from the transitivity of P that $(e\alpha)P(f\beta)$ if and only if ePf, and therefore the structure decomposition of S is as stated. If $e, f \in E_\gamma$ and $\alpha, \beta \in G$, then it follows from the distributive laws that

$$(e+f)(\alpha+\beta) = e\alpha + (e\beta+f\alpha) + f\beta.$$

But as each of $e\alpha$, $e\beta+f\alpha$, $f\beta$ is in $E_\gamma G$, it follows from Lemma 2 of [6] that

$$(e+f)(\alpha+\beta) = e\alpha + f\beta.$$

If $(S, +, \cdot)$ is a compact semiring, $H_e = eSe$ is one of the maximal multiplicative subgroups in $K[\cdot]$ and $\mathscr{L}$ and $\mathscr{R}$ denote the spaces of minimal left and right multiplicative ideals, then $K[\cdot]$ is homeomorphic with

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$H_e \times \mathscr{L} \times \mathscr{R}$ (see [13]). We have seen in Theorem 1 that $K[\cdot]$ is a subsemiring if either $\mathscr{L}$ or $\mathscr{R}$ has only one member. If H_e is degenerate, then $K[\cdot]$ need not be a subsemiring (see Example 1), but we have the following result.

THEOREM 3. *Let $(S, +, \cdot)$ be a compact semiring in which the maximal multiplicative subgroups of $K[\cdot]$ are degenerate, and let $\mathscr{L}$ and $\mathscr{R}$ denote the spaces of minimal left and right multiplicative ideals. It follows from [13] that for each $L \in \mathscr{L}$ and $R \in \mathscr{R}$, $L \cap R$ is a single element of $K[\cdot]$ and conversely that to each $x \in K[\cdot]$ there is a unique $L \in \mathscr{L}$ and a unique $R \in \mathscr{R}$ with $\{x\} = L \cap R$.*

There exist binary operations $\otimes$, $\circ$ on $\mathscr{L}$, $\mathscr{R}$ respectively such that

(i) *$(\mathscr{L}, \otimes)$ and $(\mathscr{R}, \circ)$ are idempotent semigroups;*

(ii) *for all $L_1, L_2 \in \mathscr{L}$ and $R_1, R_2 \in \mathscr{R}$,*

$$(L_1 \cap R_1)+(L_2 \cap R_1) = (L_1 \otimes L_2) \cap R_1$$

and

$$(L_1 \cap R_1)+(L_1 \cap R_2) = L_1 \cap (R_1 \circ R_2);$$

(iii) *if a binary operation $\oplus$ is defined on $K[\cdot]$ by*

$$(L_1 \cap R_1) \oplus (L_2 \cap R_2) = (L_1 \otimes L_2) \cap (R_1 \circ R_2)$$

for all $L_1, L_2 \in \mathscr{L}$ and $R_1, R_2 \in \mathscr{R}$, then $(K[\cdot], \oplus, \cdot)$ is a topological semiring and

$$(L_1 \cap R_1)+(L_2 \cap R_2) = (L_1 \cap R_1) \oplus (L_2 \cap R_2)$$

for all $L_1, L_2 \in \mathscr{L}$ and $R_1, R_2 \in \mathscr{R}$ for which $(L_1 \cap R_1)+(L_2 \cap R_2) \subset K[\cdot]$.

If $nK[\cdot]$ represents the set of all members of S which are the sum of n members of $K[\cdot]$ and $K' = \bigcup_{n=1}^{\infty} nK[\cdot]$, then $(K', +, \cdot)$ is a subsemiring of S, $K' \cdot K' = K[\cdot]$ and multiplication in K' is given by

$$\begin{aligned}[(L_1 \cap R_1)+(L_2 \cap R_2)+ \cdots +(L_m \cap R_m)]\\ \cdot\ [(L'_1 \cap R'_1)+(L'_2 \cap R'_2)+ \cdots +(L'_n \cap R'_n)]\\ = (L'_1 \otimes L'_2 \otimes \cdots \otimes L'_n) \cap (R_1 \circ R_2 \circ \cdots \circ R_m)\end{aligned}$$

where $L_i, L'_j \in \mathscr{L}$ and $R_i, R'_j \in \mathscr{R}$ for $1 \leqq i \leqq m$ and $1 \leqq j \leqq n$.

PROOF. If $L_1, L_2 \in \mathscr{L}$ and $R_1, R_2 \in \mathscr{R}$ then

$$(L_1 \cap R_1) \cdot (L_2 \cap R_2) \subset S \cdot L_2 \subset L_2$$

and

$$(L_1 \cap R_1) \cdot (L_2 \cap R_2) \subset R_1 \cdot S \subset R_1$$

so that

$$(L_1 \cap R_1) \cdot (L_2 \cap R_2) = L_2 \cap R_1.$$

Let $L_1, L_2 \in \mathscr{L}$. If $R, R' \in \mathscr{R}$ then, because R and R' are subsemirings (Theorem 1), there exist $L, L' \in \mathscr{L}$ with

$$(L_1 \cap R)+(L_2 \cap R) = L \cap R, \quad (L_1 \cap R')+(L_2 \cap R') = L' \cap R'.$$

Thus

$$\begin{aligned} L' \cap R' = (L_1 \cap R')+(L_2 \cap R') &= (L_1 \cap R') \cdot (L_1 \cap R)+(L_1 \cap R') \cdot (L_2 \cap R) \\ &= (L_1 \cap R') \cdot [(L_1 \cap R)+(L_2 \cap R)] \\ &= (L_1 \cap R') \cdot (L \cap R) = L \cap R', \end{aligned}$$

so that $L = L'$. Hence the minimal left ideal containing $(L_1 \cap R)+(L_2 \cap R)$ is independent of R. We define $L_1 \otimes L_2$ to be this left ideal. It is clear that $(\mathscr{L}, \otimes)$ is a semigroup which is idempotent because $K[\cdot] \subset E[+]$ (Theorem 1). Also

$$(L_1 \cap R_1)+(L_2 \cap R_1) = (L_1 \otimes L_2) \cap R_1$$

for all $L_1, L_2 \in \mathscr{L}$ and $R_1 \in \mathscr{R}$.

Similarly if $R_1, R_2 \in \mathscr{R}$ and $L \in \mathscr{L}$, the minimal right ideal containing $(L \cap R_1)+(L \cap R_2)$ is independent of L. We define $R_1 \circ R_2$ to be this right ideal so that $(\mathscr{R}, \circ)$ is an idempotent semigroup and

$$(L_1 \cap R_1)+(L_1 \cap R_2) = L_1 \cap (R_1 \circ R_2)$$

for all $L_1 \in \mathscr{L}$ and $R_1, R_2 \in \mathscr{R}$.

If $L_i, L'_j \in \mathscr{L}$ and $R_i, R'_j \in \mathscr{R}$ for $1 \leqq i \leqq m$ and $1 \leqq j \leqq n$ then

$$\begin{aligned} &[(L_1 \cap R_1)+\cdots+(L_m \cap R_m)] \cdot [(L'_1 \cap R'_1)+\cdots+(L'_n \cap R'_n)] \\ &\quad = (L_1 \cap R_1) \cdot [(L'_1 \cap R'_1)+\cdots+(L'_n \cap R'_n)] \\ &\qquad +\cdots+(L_m \cap R_m) \cdot [(L'_1 \cap R'_1)+\cdots+(L'_n \cap R'_n)] \\ &\quad = [(L'_1 \cap R_1)+\cdots+(L'_n \cap R_1)] \\ &\qquad +\cdots+[(L'_1 \cap R_m)+\cdots+(L'_n \cap R_m)] \\ &\quad = [(L'_1 \otimes L'_2 \otimes \cdots \otimes L'_n) \cap R_1]+\cdots+[(L'_1 \otimes L'_2 \otimes \cdots \otimes L'_n) \cap R_m] \\ &\quad = (L'_1 \otimes L'_2 \otimes \cdots \otimes L'_n) \cap (R_1 \circ R_2 \circ \cdots \circ R_m). \end{aligned}$$

This shows that $K' \cdot K' \subset K[\cdot]$. It is now a simple matter to check that K' is a subsemiring of S. That $K' \cdot K' = K[\cdot]$ follows because $K[\cdot] \subset K'$ and $K[\cdot] \cdot K[\cdot] = K[\cdot]$.

It can be easily checked that $(K[\cdot], \oplus, \cdot)$ is a topological semiring. If $L_1, L_2 \in \mathscr{L}$ and $R_1, R_2 \in \mathscr{R}$ are such that $(L_1 \cap R_1)+(L_2 \cap R_2) \subset K[\cdot]$ then, since each element of $K[\cdot]$ is a multiplicative idempotent,

$$\begin{aligned} (L_1 \cap R_1)+(L_2 \cap R_2) &= [(L_1 \cap R_1)+(L_2 \cap R_2)] \cdot [(L_1 \cap R_1)+(L_2 \cap R_2)] \\ &= (L_1 \otimes L_2) \cap (R_1 \circ R_2) = (L_1 \cap R_1) \oplus (L_2 \cap R_2). \end{aligned}$$

We conclude this section by considering in detail the multiplicative kernel of any compact connected semiring which is a subset of the plane.

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Hunter has identified all compact connected simple semigroups which are subsets of the plane ([4], Theorem 5); we paraphrase his result. (A multiplication is left trivial (right trivial) if $xy = x$ ($xy = y$) for all x, y.)

THEOREM 4. *Let T be a compact connected simple semigroup which is a subset of the plane. Then*

(a) *multiplication in T is left trivial or right trivial; or*

(b) *T is the cartesian product of two arcs, multiplication in the first being left trivial and in the second right trivial; or*

(c) *T is the cartesian product of a simple closed curve with left trivial (right trivial) multiplication and an arc with right trivial (left trivial) multiplication; or*

(d) *T is the circle group; or*

(e) *T is the cartesian product of the circle group C and an arc A with trivial multiplication.*

Suppose firstly that $(S, +, .)$ is a compact connected semiring which is a subset of the plane and we wish to know whether $K[\cdot]$ is a subsemiring. Because $K[\cdot]$ is connected ([9], Lemma 2.4.1), we see that $(K[\cdot], .)$ must be topologically isomorphic with one of the semigroups (a)—(e) of Theorem 4. If $(K[\cdot], .)$ is given by (a), (d) or (e) of Theorem 4 then at least one of $\mathscr{L}$ and $\mathscr{R}$ is degenerate and so $K[\cdot]$ must be a subsemiring of S (Theorem 1), while if $(K[\cdot], \cdot)$ is given by (b) or (c) of Theorem 4 we are unable to say whether or not $K[\cdot]$ must be a subsemiring of S. Notice, however, that in the special case where $S \subset R_1$, $(K[\cdot], \cdot)$ must be as in (a) and so $K[\cdot]$ must be a subsemiring of S.

On the other hand, if the multiplicative kernel of a compact connected semiring in the plane is a subsemiring, it is a compact connected semiring which is multiplicatively simple. We identify here all compact connected, multiplicatively simple semirings which are subsets of the plane. It is clear that the multiplicative semigroup of such a semiring must be one of the semigroups in Theorem 4. We examine in turn the possible additions on each one.

If $(T, \cdot)$ is given by case (a) of Theorem 4, it is clear that $(T, +, \cdot)$ is a topological semiring if and only if $(T, +)$ is an idempotent topological semigroup. In the special case where $T \subset R_1$, T must be a closed interval. Now Paalman-de Miranda has listed all semigroups $(T, +)$ on a closed interval T of R_1 for which $T+T = T$ (see § 2.6 of [9]). Any idempotent semigroup has this latter property and all idempotent semigroups on a closed interval of R_1 can be identified from his results.

If $(T, \cdot)$ is given by (b) or (c) of Theorem 4 and $\mathscr{L}$ and $\mathscr{R}$ are the spaces of minimal left and right ideals of $(T, \cdot)$ then it follows from Theorem

3 that the only additions of semirings $(T, +, \cdot)$ are those of the form

$$(L_1 \cap R_1)+(L_2 \cap R_2) = (L_1 \otimes L_2) \cap (R_1 \circ R_2)$$

where $(\mathscr{L}, \otimes)$ and $(\mathscr{R}, \circ)$ are idempotent semigroups. In (b) both $\mathscr{L}$ and $\mathscr{R}$ are arcs, while in (c) one is an arc and the other a circle. The only idempotent semigroups on the circle are the trivial ones ([7], page 280) and all idempotent semigroups on an arc can be identified from § 2.6 of [9].

If $(T, \cdot)$ is the circle group then, because a circle is not the topological product of two non-degenerate continua, we see from [10], Theorem 1 that the only possible additions of a semiring $(T, +, \cdot)$ are the trivial ones.

If $(T, \cdot)$ is given by (e) of Theorem 4 we have the following result.

LEMMA 1. *Let $(T, \cdot)$ be the cartesian product of the circle group C and an arc A with trivial multiplication. Then $(T, +, \cdot)$ is a topological semiring if and only if there exist additions $\otimes$ and $\circ$ of semirings on C and A respectively such that, for all α, β in C and x, y in A,*

$$(\alpha, x)+(\beta, y) = (\alpha \otimes \beta, x \circ y).$$

Note that $\circ$ is an addition of a semiring on A if and only if $(A, \circ)$ is an idempotent semigroup and so all possibilities for $\circ$ can be identified from § 2.6 of [9]; $\otimes$ must of course be trivial.

PROOF. The sufficiency of the condition given is clear.

Conversely, suppose that $(T, +, \cdot)$ is a topological semiring. We give the proof in the case where A has left trivial multiplication; the other case is similar. That is, we suppose that multiplication in T is given by

$$(\alpha, x)(\beta, y) = (\alpha\beta, x)$$

for all α, β in C and x, y in A; note that T is multiplicatively left simple. We shall denote the identity of C by δ.

Clearly $E[\cdot] = \{(\delta, x)|x \in A\}$ is topologically isomorphic to A. By Theorem 2, $(E[\cdot], +)$ is a semigroup and thus $E[\cdot]$ is a subsemiring. Hence there is an addition $\circ$ of a semiring on A such that

$$(\delta, x)+(\delta, y) = (\delta, x \circ y).$$

Thus if $\alpha \in C$,

$$\begin{aligned}(\alpha, x \circ y) = (\delta, x \circ y)(\alpha, x) &= [(\delta, x)+(\delta, y)](\alpha, x)\\ &= (\alpha, x)+(\alpha, y).\end{aligned}$$

Also, if we pick any $x \in A$, $(\delta, x) \cdot T = C \times \{x\}$ is topologically isomorphic with C and is a subsemiring. Hence there is an addition $\otimes$ of a semiring on C such that

$$(\alpha, x)+(\beta, x) = (\alpha \otimes \beta, x).$$

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Thus if $y \in A$,

$$(\alpha \otimes \beta, y) = (\delta, y)[(\alpha, x)+(\beta, x)] = (\alpha, y)+(\beta, y).$$

As $\otimes$ must be trivial, we shall assume (without loss of generality) that it is left trivial.

Let $x, y \in A$. If $(\delta, x)+(\alpha, y) = (\delta, x)+(\beta, y)$, then

$$\begin{aligned}(\delta, x)+(\beta\alpha^{-1}, y) &= [(\delta, x)+(\alpha^{-1}, x)]+(\beta\alpha^{-1}, y)\\ &= (\delta, x)+[(\alpha^{-1}, x)+(\beta\alpha^{-1}, y)]\\ &= (\delta, x)+[(\delta, x)+(\beta, y)](\alpha^{-1}, y)\\ &= (\delta, x)+[(\delta, x)+(\alpha, y)](\alpha^{-1}, y)\\ &= (\delta, x)+(\alpha^{-1}, x)+(\delta, y)\\ &= (\delta, x)+(\delta, y).\end{aligned}$$

Thus $\qquad H = \{\alpha | \alpha \in C \quad \text{and} \quad (\delta, x)+(\alpha, y) = (\delta, x)+(\delta, y)\}$

is a closed subgroup of C. Now if $\alpha \in C$, it follows from Theorem 2 that there exists a multiplicative idempotent g such that

$$(\delta, x)(\delta, y)+(\delta, y)(\alpha, y) = g[(\delta, y)+(\alpha, y)];$$

i.e., there exists $u \in A$ such that

$$(\delta, x)+(\alpha, y) = (\delta, u)[(\delta, y)+(\alpha, y)] = (\delta, u)+(\alpha, u) = (\delta, u).$$

As u depends on α, we put $u = \varphi(\alpha)$. Then φ is a continuous mapping of C into A and so we can choose α, β arbitrarily close with $\alpha \neq \beta$ and $\varphi(\alpha) = \varphi(\beta)$. But then $\beta\alpha^{-1} \in H$ and $\beta\alpha^{-1}$ is arbitrarily close, but not equal, to δ. Hence δ is an accumulation point of H and it follows easily from the fact that H is closed that $H = C$. Hence

$$(\delta, x)+(\alpha, y) = (\delta, x)+(\delta, y)$$

for all α in C and x, y in A. Thus if $\alpha, \beta \in C$ and $x, y \in A$,

$$\begin{aligned}(\alpha, x)+(\beta, y) &= [(\delta, x)+(\beta\alpha^{-1}, y)](\alpha, y)\\ &= [(\delta, x)+(\delta, y)](\alpha, y)\\ &= (\delta, x \circ y)(\alpha, y) = (\alpha, x \circ y) = (\alpha \otimes \beta, x \circ y).\end{aligned}$$

2. The additive kernel $K[+]$

In Example 2 a compact semiring in which $K[+]$ is not a subsemiring is given, while Theorem 5 gives a necessary and sufficient condition for $K[+]$ to be a subsemiring of S.

EXAMPLE 2. $([0, 1], +, \cdot)$, where $x+y = \max(x, y)$ and $x \cdot y = 0$, has $K[+] = \{1\}$ and $(K[+])^2 = \{0\}$.

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THEOREM 5. *If* $(S, +, \cdot)$ *is a compact semiring with additive kernel* $K[+]$, *then either* $K[+]$ *is a subsemiring of* S *or* $S^2 \cap K[+]$ *is empty.*

PROOF. Suppose $S^2 \cap K[+]$ is not empty. Then there exist x, y in S with $xy \in K[+]$. But then $z = x+xy$ and $w = y+xy$ are both in $K[+]$ and

$$zw = (x+xy)(y+xy) = xy+xy^2+x^2y+xyxy \in xy+S \subset K[+].$$

Now because $z \in K[+]$, $S+z+S$ is an additive ideal contained in $K[+]$ so that $S+z+S = K[+]$, and similarly $S+w+S = K[+]$. Thus if $k_1, k_2 \in K[+]$, there exist s_1, s_2, s_3, s_4 in S such that

$$k_1 = s_1+z+s_2, \qquad k_2 = s_3+w+s_4.$$

Hence
$$\begin{aligned} k_1k_2 &= (s_1+z+s_2)(s_3+w+s_4) \\ &= (s_1s_3+s_1w+s_1s_4+zs_3)+zw+(zs_4+s_2s_3+s_2w+s_2s_4) \\ &\in S+zw+S = K[+], \end{aligned}$$

and $K[+]$ is a subsemiring.

We now give a characterization of any additive kernel $K[+]$ which is a subsemiring of a compact semiring S. Since $(K[+], +)$ is then a compact simple semigroup, this amounts to characterizing all compact, additively simple semirings, which we do in the following theorem.

THEOREM 6. *Let* $(T, +)$ *be a compact simple semigroup in which* $\mathscr{L}$ *and* $\mathscr{R}$ *denote the spaces of minimal left and right ideals, and, for each* $L \in \mathscr{L}$ *and* $R \in \mathscr{R}$, $\tau(L, R)$ *denotes the identity of the maximal group* $L \cap R$ *(see* [13]).

Then $(T, +, \cdot)$ *is a topological semiring if and only if* $\cdot$ *is a binary operation on* T *such that there exist* $L' \in \mathscr{L}$, $R' \in \mathscr{R}$ *and binary operations* $\otimes$ *and* $\circ$ *on* $\mathscr{L}$ *and* $\mathscr{R}$ *respectively for which*

(i) $(L' \cap R', +, \cdot)$ *is a topological semiring,*

(ii) $(\mathscr{L}, \otimes)$ *and* $(\mathscr{R}, \circ)$ *are topological semigroups,*

(iii) *the relations*

$$(1) \quad \varphi(L \otimes M, R \circ V) = \varphi(L \otimes M, R \circ T)-\varphi(L \otimes N, R \circ T)+\varphi(L \otimes N, R \circ V),$$

$$(2) \quad \varphi(L \otimes N, T \circ V) = \varphi(L \otimes N, R \circ V)-\varphi(M \otimes N, R \circ V)+\varphi(M \otimes N, T \circ V)$$

hold for all $L, M, N \in \mathscr{L}$ *and* $R, T, V \in \mathscr{R}$,

(iv) $$\alpha \cdot \beta = \beta \cdot \alpha = \tau(L', R')$$

for all $\alpha \in L' \cap R'$ *and all* $\beta \in \pm\, \varphi(\mathscr{L} \otimes \mathscr{L}, \mathscr{R} \circ \mathscr{R})$,

(v) $$\gamma+\alpha \cdot \beta = \alpha \cdot \beta+\gamma$$

for all $\alpha, \beta \in L' \cap R'$ *and all* $\gamma \in \pm\, \varphi(\mathscr{L} \otimes \mathscr{L}, \mathscr{R} \circ \mathscr{R})$, *and*

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(vi) $\psi(\alpha, L, R)\cdot\psi(\beta, M, T) = \psi(\alpha\cdot\beta - \varphi(L\otimes M, R\circ T), L\otimes M, R\circ T)$ *for all* $\alpha, \beta \in L'\cap R'$, $L, M \in \mathscr{L}$ *and* $R, T \in \mathscr{R}$, *where, for all* $\alpha \in L'\cap R'$, $L\in\mathscr{L}$ *and* $R\in\mathscr{R}$,

(a) $\varphi(L, R)$ *denotes the member* $\tau(L, R')+\tau(L', R)$ *of* $L'\cap R'$, *and*

(b) $\psi(\alpha, L, R)$ *denotes the member* $\tau(L', R)+\alpha+\tau(L, R')$ *of* T.

PROOF. The structure of $(T, +)$ has been given by Wallace in [13]. In particular,

(a) the functions $\tau: \mathscr{L}\times\mathscr{R}\to E[+]$ and $\varphi: \mathscr{L}\times\mathscr{R}\to L'\cap R'$ are continuous;

(b) the function $\psi: (L'\cap R')\times\mathscr{L}\times\mathscr{R}\to T$ is a homeomorphism and

(3) $$\psi(\alpha, L, R)+\psi(\beta, M, T) = \psi(\alpha+\varphi(L, T)+\beta, M, R)$$

for all $\alpha, \beta \in L'\cap R'$, $L, M\in\mathscr{L}$ and $R, T\in\mathscr{R}$;

(c) if $x\in T$, the minimal left (right) additive ideal containing x is $\lambda(x) = S+x$ $(\rho(x) = x+S)$ and the function

$$\lambda: T\to\mathscr{L} \qquad (\rho: T\to\mathscr{R})$$

is continuous;

(d) if $\psi(\alpha, L, R) = x$ then $\lambda(x) = L$ and $\rho(x) = R$.

Sufficiency. Suppose that $\cdot$ is a binary operation of the kind described in the theorem. Because ψ is a homeomorphism it follows that the values of $\cdot$ on the whole of T are completely specified by (vi) in terms of $\otimes$, $\circ$ and the values of $\cdot$ on $L'\cap R'$. It is an easy matter to check that $(T, +, \cdot)$ is a topological semiring.

Necessity. We suppose now that $(T, +, \cdot)$ is a topological semiring. As usual we shall omit the multiplication symbol $\cdot$ where it causes no confusion to do so.

Because $(T, \cdot)$ is a compact semigroup, it contains at least one idempotent f ([9], Theorem 1.1.10 or [8], Lemma 3). There exist $L'\in\mathscr{L}$ and $R'\in\mathscr{R}$ such that f belongs to the additive group $L'\cap R'$ ([9], Theorem 1.2.6 or [8]). We will put $H = L'\cap R'$ and let e denote the identity, $\tau(L', R')$, of the group $(L'\cap R', +)$. Then $H = e+S+e$ ([9], Theorem 1.2.6 or [8]), and clearly $H = f+S+f$. Now for all $s_1, s_2\in S$,

$$\begin{aligned}(f+s_1+f)(f+s_2+f)&\\ &= f^2+(fs_2+f^2+s_1f+s_1s_2+s_1f+f^2+fs_2)+f^2\\ &= f+s+f\end{aligned}$$

for some $s\in S$, and we see that $(H, +, \cdot)$ is a subsemiring. But $E[+]\cap H$, which is a multiplicative ideal of H, is just $\{e\}$. Hence $ex = xe = e$ for all x in H. This means that if $\alpha, \beta\in H$ then

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$$\alpha\beta+(-\alpha)\beta = [\alpha+(-\alpha)]\beta = e\beta = e$$

and so $-(\alpha\beta) = (-\alpha)\beta$, and similarly $-(\alpha\beta) = \alpha(-\beta)$.

If $L, M \in \mathscr{L}$, let $x \in L$, $y \in M$ and define $L \otimes M$ to be $\lambda(xy) = S+xy$. To see that $\otimes$ is well defined, let $x' \in L$, $y' \in M$ and we show that $\lambda(x'y') = \lambda(xy)$. For as $L = S+x$, $M = S+y$, there are s_1, s_2 in S with

$$x' = s_1+x, \qquad y' = s_2+y.$$

Thus

$$x'y' = (s_1+x)(s_2+y) = (s_1s_2+xs_2+s_1y)+xy \in S+xy$$

as required. It should be noted that $L' \otimes L' = L'$ since $e = ee \in L' \otimes L'$. The associativity of $\otimes$ is clear because if $L, M, N \in \mathscr{L}$ and $x \in L$, $y \in M$, $z \in N$, then

$$xyz \in [(L \otimes M) \otimes N] \cap [L \otimes (M \otimes N)].$$

The continuity of $\otimes$ follows because, in particular,

$$L \otimes M = \lambda(\tau(L, R') \cdot \tau(M, R')).$$

Hence $(\mathscr{L}, \otimes)$ is a topological semigroup. Similarly one can define a topological semigroup $(\mathscr{R}, \circ)$ with analagous properties. Because

$$\psi(\alpha, L, R) \in L \cap R \quad \text{and} \quad \psi(\beta, M, T) \in M \cap T$$

it follows that

$$\psi(\alpha, L, R) \cdot \psi(\beta, M, T) \in (L \otimes M) \cap (R \circ T)$$

and so

$$\psi(\alpha, L, R) \cdot \psi(\beta, M, T) = \psi(\gamma, L \otimes M, R \circ T)$$

for some $\gamma \in H$. In what follows we shall omit $\otimes, \circ$ where it causes no confusion to do so.

In all that follows we let $\alpha, \beta \in L' \cap R'$, $L, M, N \in \mathscr{L}$, $R, T, V \in \mathscr{R}$.

Because $\tau(L, R)$ is a member of the multiplicative ideal $E[+]$, it follows that $\tau(L, R) \cdot \psi(\alpha, M, T) \in E[+]$. On the other hand,

$$\tau(L, R) \cdot \psi(\alpha, M, T) \in LM \cap RT$$

and so we see that

$$\tau(L, R) \cdot \psi(\alpha, M, T) = \tau(LM, RT). \tag{4}$$

Similarly,

$$\psi(\alpha, L, R) \cdot \tau(M, T) = \tau(LM, RT). \tag{4'}$$

It is clear from (3) that $\psi(\alpha, L, R) \in E[+]$ if and only if $\alpha = -\varphi(L, R)$.

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Thus

$$\tau(L, R) = \psi(-\varphi(L, R), L, R) \tag{5}$$

and so, from (4), (4′),

$$\psi(-\varphi(L, R), L, R) \cdot \psi(\alpha, M, T) = \psi(-\varphi(LM, RT), LM, RT), \tag{6}$$

$$\psi(\alpha, L, R) \cdot \psi(-\varphi(M, T), M, T) = \psi(-\varphi(LM, RT), LM, RT). \tag{6′}$$

Hence

$$\begin{aligned}\psi(-\varphi(L, R), L, R) &\cdot \{\psi(\alpha, M, T)+\psi(\beta, N, V)\}\\ &= \psi(-\varphi(L, R), L, R) \cdot \psi(\alpha+\varphi(M, V)+\beta, N, T)\\ &= \psi(-\varphi(LN, RT), LN, RT)\end{aligned}$$

while

$$\begin{aligned}\{\psi(-\varphi(L, R), L, R) &\cdot \psi(\alpha, M, T)\}+\{\psi(-\varphi(L, R), L, R) \cdot \psi(\beta, N, V)\}\\ &= \psi(-\varphi(LM, RT), LM, RT)+\psi(-\varphi(LN, RV), LN, RV)\\ &= \psi(-\varphi(LM, RT)+\varphi(LM, RV)-\varphi(LN, RV), LN, RT).\end{aligned}$$

A comparison of these two expressions yields (1). (2) follows from a similar consideration of

$$\{\psi(\alpha, L, R)+\psi(\beta, M, T)\} \cdot \psi(-\varphi(N, V), N, V).$$

It follows from (5) and the definition of φ that

$$\begin{aligned}\varphi(L, R') &= \tau(L, R')+\tau(L', R')\\ &= \psi(-\varphi(L, R'), L, R')+\psi(-\varphi(L', R'), L', R')\\ &= \psi(-\varphi(L, R')+\varphi(L, R')-\varphi(L', R'), L', R')\\ &= \psi(-\varphi(L', R'), L', R') = \tau(L', R').\end{aligned}$$

Similarly $\varphi(L', R) = \tau(L', R')$ and so

$$\varphi(L, R') = \varphi(L', R) = \tau(L', R') = e. \tag{7}$$

Now

$$\begin{aligned}\{\tau(L', R)+\alpha\} &\cdot \{\tau(L', T)+\beta\}\\ &= \tau(L', R) \cdot \tau(L', T)+\alpha \cdot \tau(L', T)+\tau(L', R) \cdot \beta+\alpha\beta\\ &= \tau(L'L', RT)+\tau(L'L', R'T)+\tau(L'L', RR')+\alpha\beta \quad \text{[by (4), (4′)]}\\ &= \tau(L', RT)+\tau(L', R'T)+\tau(L', RR')+\alpha\beta\\ &= \psi(e, L', RT)+\psi(e, L', R'T)+\psi(e, L', RR')+\psi(\alpha\beta, L', R')\\ &\qquad \text{[by (5), (7)]}\\ &= \psi(e+\varphi(L', R'T)+e+\varphi(L', RR')+e+\varphi(L', R')+\alpha\beta, L', RT)\\ &\qquad \text{[by (3)]}\\ &= \psi(\alpha\beta, L', RT) \qquad \text{[by (7)].}\end{aligned}$$

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Also

$$\tau(L', R)+\alpha \in (L' \cap R)+(L' \cap R') \subset (L' \cap R)$$

and so, by (4′) and (5),

$$\{\tau(L', R)+\alpha\} \cdot \tau(M, R') = \tau(L'M, RR') = \psi(-\varphi(L'M, RR'), L'M, RR').$$

Similarly

$$\tau(L, R') \cdot \{\tau(L', T)+\beta\} = \psi(-\varphi(LL', R'T), LL', R'T).$$

Thus

$$\begin{aligned}\psi(\alpha, L, R) \cdot \psi(\beta, M, T)& \\ &= \{(\tau(L', R)+\alpha)+\tau(L, R')\} \cdot \{(\tau(L', T)+\beta)+\tau(M, R')\} \\ &= \{\tau(L', R)+\alpha\} \cdot \{\tau(L', T)+\beta\}+\{\tau(L', R)+\alpha\} \cdot \tau(M, R') \\ &\quad +\tau(L, R') \cdot \{\tau(L', T)+\beta\}+\tau(L, R') \cdot \tau(M, R') \\ &= \psi(\alpha\beta, L', RT)+\psi(-\varphi(L'M, RR'), L'M, RR') \\ &\quad +\psi(-\varphi(LL', R'T), LL', R'T)+\psi(-\varphi(LM, R'), LM, R') \\ &= \psi(\alpha\beta-\varphi(L'M, RR')+\varphi(L'M, R'T)-\varphi(LL', R'T), LM, RT) \\ &\hspace{18em} \text{[by (3), (7)].}\end{aligned}$$

But if we put L, L', M, R', R, T for L, M, N, R, T, V respectively in (2) we see that

$$\varphi(LM, RT) = \varphi(LM, R'T)-\varphi(L'M, R'T)+\varphi(L'M, RT).$$

If we put L, M, L', R', R', T for L, M, N, R, T, V respectively in (1) we see that

$$\begin{aligned}\varphi(LM, R'T) &= \varphi(LM, R'R')-\varphi(LL', R'R')+\varphi(LL', R'T) \\ &= \varphi(LM, R')-\varphi(LL', R')+\varphi(LL', R'T) \\ &= \varphi(LL', R'T), \qquad \text{[by (7)]}\end{aligned}$$

and similarly, by putting L', M, L', R, R', T for L, M, N, R, T, V respectively in (1) we see that

$$\varphi(L'M, RT) = \varphi(L'M, RR').$$

Hence

$$\varphi(LM, RT) = \varphi(LL', R'T)-\varphi(L'M, R'T)+\varphi(L'M, RR')$$

and so

$$\psi(\alpha, L, R) \cdot \psi(\beta, M, T) = \psi(\alpha\beta-\varphi(LM, RT), LM, RT).$$

This proves (vi).

If we proceed in a similar manner to calculate $\psi(\alpha, L, R) \cdot \psi(\beta, M, T)$ as

$$\{\tau(L', R)+(\alpha+\tau(L, R'))\} \cdot \{\tau(L', T)+(\beta+\tau(M, R'))\}$$

we find that

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$$\psi(\alpha, L, R) \cdot \psi(\beta, M, T) = \psi(-\varphi(LM, RT)+\alpha\beta, LM, RT).$$

A comparison of the two expressions for $\psi(\alpha, L, R) \cdot \psi(\beta, M, T)$ yields

(8) $$-\varphi(LM, RT)+\alpha\beta = \alpha\beta-\varphi(LM, RT).$$

If we take the additive inverse of each side we have

$$-(\alpha\beta)+\varphi(LM, RT) = \varphi(LM, RT)-(\alpha\beta),$$

or equivalently,

(9) $$(-\alpha)\beta+\varphi(LM, RT) = \varphi(LM, RT)+(-\alpha)\beta$$

(v) follows from (8) and (9).

From (vi),

$$\psi(\alpha, L', R') \cdot \psi(-\varphi(L, R), L, R)$$
$$= \psi(\alpha \cdot (-\varphi(L, R))-\varphi(L'L, R'R), L'L, R'R),$$

while it follows from (6′) that

$$\psi(\alpha, L', R') \cdot \psi(-\varphi(L, R), L, R) = \psi(-\varphi(L'L, R'R), L'L, R'R).$$

Thus $\alpha \cdot (-\varphi(L, R)) = e$ and so

$$\alpha \cdot \varphi(L, R) = -\{\alpha \cdot (-\varphi(L, R))\} = -e = e.$$

Similarly

$$(-\varphi(L, R)) \cdot \alpha = e = (\varphi(L, R)) \cdot \alpha$$

and we have (iv). The proof is now complete.

The conditions (1), (2), (iv) and (v) are rather cumbersome, especially the former two whose effect is not clear. However it can be shown that they are independent of one another since it is possible to construct examples where any desired one of (1), (2), (iv), (v) is false and all the others of (i)—(vi) are true. We also remark that any one or more of (iii), (iv), (v) may be replaced by the conditions

(iii′) For all $L, M, N \in \mathscr{L}$ and $R, T, V \in \mathscr{R}$,

$$\varphi(L \otimes M, R \circ T) = \varphi(L' \otimes M, R \circ R')+\varphi(L \otimes L', R' \circ T)$$
$$= \varphi(L \otimes L', R' \circ R)+\varphi(L' \otimes M, R \circ R').$$

(iv′) $\alpha \cdot \beta = \beta \cdot \alpha = \tau(L', R')$ for all $\alpha \in L' \cap R'$ and all

$$\beta \in [\varphi(\mathscr{L} \otimes L', R' \circ \mathscr{R}) \cup \varphi(L' \otimes \mathscr{L}, \mathscr{R} \circ R')].$$

(v′) $\gamma+\alpha \cdot \beta = \alpha \cdot \beta+\gamma$ for all $\alpha, \beta \in L' \cap R'$ and all

$$\gamma \in [\varphi(\mathscr{L} \otimes L', R' \circ \mathscr{R}) \cup \varphi(L' \otimes \mathscr{L}, \mathscr{R} \circ R')]$$

respectively without affecting the validity of the theorem. Note also that putting $L \otimes M = L'$ and $R \circ T = R'$ for all $L, M \in \mathscr{L}$ and $R, T \in \mathscr{R}$ is always one way of satisfying conditions (ii)—(v).

In view of condition (i), Theorem 6 is to some extent dependent upon a characterization of all (compact) semirings which are additively topological groups. The following theorem gives some information about such semirings. (We use $^-$ to denote closure.)

THEOREM 7. *Let $(S, +, \cdot)$ be a topological semiring in which $(S, +)$ is a topological group with identity 0. Then there is a closed subring T of S such that $S^2 \subset T$. If $(H, +)$ is the commutator subgroup of $(S, +)$, then $(\bar{H}, +)$ is a normal subgroup of $(S, +)$, $x \cdot h = h \cdot x = 0$ for all $x \in S$, $h \in \bar{H}$, and there is a binary operation $\circ$ on $S/\bar{H}$ such that $(S/\bar{H}, +, \circ)$ is a ring and*

$$x \cdot y \in (x+\bar{H}) \circ (y+\bar{H})$$

for all x, y in S. Further, if $x_1 \in x+\bar{H}$ and $y_1 \in y+\bar{H}$, then $x_1 \cdot y_1 = x \cdot y$.

If S is also compact and connected, then $S^2 \subset \bar{H}$ and $S^3 = \{0\}$.

PROOF. Because $E[+] = \{0\}$ and $E[+]$ is a multiplicative ideal, it follows that $0x = x0 = 0$ for all x in S. Thus if $x, y \in S$,

$$xy+x(-y) = x[y+(-y)] = x0 = 0$$

so that $-(xy) = x(-y)$; similarly, $-(xy) = (-x)y$.

If $x, y, z, w \in S$ then

$$(z+x)(y+w) = (z+x)y+(z+x)w = zy+xy+zw+xw$$

and $$(z+x)(y+w) = z(y+w)+x(y+w) = zy+zw+xy+xw$$

so that $$zy+xy+zw+xw = zy+zw+xy+xw.$$

Therefore, since $(S, +)$ is a group, $xy+zw = zw+xy$. Thus if T' is the additive group generated by S^2, i.e.,

$$T' = \bigcup_{n=1}^{\infty} \{t_1+ \cdots +t_n | t_i \in S^2 \text{ for } 1 \leqq i \leqq n\},$$

then T' is an abelian subgroup of $(S, +)$. If T is the topological closure of T', then $(T, +)$ is also an abelian subgroup of $(S, +)$ ([3], Corollary 5.3). Also

$$T^2 \subset S^2 \subset T' \subset T$$

so that $(T, +, \cdot)$ is a ring.

A commutator in $(S, +)$ is an element $(x, y) = x+y-x-y$. If H_1 is the set of all commutators, then the commutator subgroup $(H, +)$ is the additive group generated by H_1, so that

$$H = \bigcup_{n=1}^{\infty} \{h_1+ \cdots +h_n | h_i \in H_1 \text{ for } 1 \leqq i \leqq n\}.$$

If $x \in S$ and $h = (y, z) \in H_1$ then

$$xh = x(y+z-y-z) = xy+xz+x(-y)+x(-z)$$
$$= xy+xz-(xy)-(xz) = 0$$

since elements of S^2 commute, and similarly $hx = 0$. Thus if $x \in S$ and $h \in H$, there are $h_1, h_2, \cdots, h_n$ in H_1 with

$$h = h_1+h_2+\cdots+h_n$$

so that

$$xh = x(h_1+h_2+\cdots+h_n) = xh_1+xh_2+\cdots+xh_n = 0$$

and also $hx = 0$. Now $(H, +)$ is a normal subgroup of $(S, +)$ and $(S/H, +)$ is abelian and thus $(\bar{H}, +)$ is a normal subgroup of $(S, +)$ and $(S/\bar{H}, +)$ is abelian (see, for example, [3], Theorems 23.8 and 5.3). It follows from the continuity of multiplication that $xh = hx = 0$ for all $x \in S$ and $h \in \bar{H}$.

Suppose $x_1 \in x+\bar{H}$ and $y_1 \in y+\bar{H}$; then there exist $h_1, h_2 \in \bar{H}$ with $x_1 = x+h_1$, $y_1 = y+h_2$. Hence

$$x_1y_1 = (x+h_1)(y+h_2) = xy+xh_2+h_1y+h_1h_2$$
$$= xy+0+0+0 = xy.$$

We define $\circ$ on $S/\bar{H}$ by

$$(x+\bar{H}) \circ (y+\bar{H}) = xy+\bar{H}.$$

The above shows that this is independent of the representatives of the cosets. Also $xy \in (x+\bar{H}) \circ (y+\bar{H})$ for all $x, y \in S$. It can be easily verified that $(S/\bar{H}, +, \circ)$ is a topological ring.

If S is compact then $\bar{H}$ and $S/\bar{H}$ are also compact. If S is connected then the set H_1 of commutators is connected so that

$$\{h_1+\cdots+h_n | h_i \in H_1 \text{ for } 1 \leqq i \leqq n\}$$

is connected for each $n \geqq 1$. Thus H and $\bar{H}$ are connected ([5], page 54) and so $S/\bar{H}$ is connected. Then because $(S/\bar{H}, +, \circ)$ is a compact connected ring, it follows from Theorem 1 of [1] that $\circ$ is trivial; i.e.,

$$(x+\bar{H}) \circ (y+\bar{H}) = \bar{H}$$

for all x, y in S. Thus $xy \in \bar{H}$ for all x, y in S and $S^3 = S^2S \subset \bar{H}S = \{0\}$.

We turn our attention again to semirings which are subsets of the plane. If the additive kernel of a compact connected semiring in the plane is a subsemiring it is a compact connected semiring which is additively simple. We characterize here all compact connected, additively simple semirings which are subsets of the plane. It is clear that the additive semigroup of such a semiring must be one of the semigroups in Theorem 4. We look in turn at the possible multiplications on each one.

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If $(T, +)$ has trivial addition, it is clear that $(T, +, \cdot)$ is a topological semiring if and only if $(T, \cdot)$ is a topological semigroup.

If $(T, +)$ is given by (b) or (c) of Theorem 4 then all its maximal subgroups are degenerate. It follows from Theorem 6 that if $\mathscr{L}$ and $\mathscr{R}$ are the spaces of minimal left and right ideals of $(T, +)$ then all the multiplications of semirings $(T, +, \cdot)$ are given as cartesian products of any semigroups on $\mathscr{L}$ and $\mathscr{R}$. (In case (b), $\mathscr{L}$ and $\mathscr{R}$ are arcs, while in case (c), one is an arc and the other a circle.)

If $(T, +)$ is the circle group and 0 denotes its identity, it follows from Theorem 1 of [1] that the only multiplication of a semiring $(T, +, \cdot)$ is given by $x \cdot y = 0$.

If $(T, +)$ is given by (e) of Theorem 4, we have the following result.

LEMMA 2. *Let $(T, +)$ be the cartesian product of the circle group $(C, \oplus)$, with identity denoted by 0, and an arc A with trivial addition. Then $(T, +, \cdot)$ is a topological semiring if and only if there exists a binary operation Δ on A such that (A, Δ) is a topological semigroup and*

$$(\alpha, x) \cdot (\beta, y) = (0, x\Delta y)$$

for all $\alpha, \beta \in C$ and $x, y \in A$.

PROOF. The sufficiency of the conditions given may be easily checked.

Conversely, suppose that $(T, +, \cdot)$ is a topological semiring. We give the proof in the case where addition on A is right trivial; the other case is similar. Thus addition on T is given by

$$(\alpha, x) + (\beta, y) = (\alpha \oplus \beta, y)$$

for all $\alpha, \beta \in C$ and $x, y \in A$. In what follows we shall use the terminology of Theorem 6. It is easily seen that, for each $x \in A$, $\{(\alpha, x) | \alpha \in C\}$ is a member of $\mathscr{L}$, and that the function $\pi : A \to \mathscr{L}$, defined by

$$\pi(x) = \{(\alpha, x) | \alpha \in C\}$$

is a homeomorphism. Also, $\mathscr{R}$ has only one member, $R_1 = T$. It follows from Theorem 6 that there exist $L' \in \mathscr{L}$, $R' \in \mathscr{R}$ and binary operations $\otimes, \circ$ on $\mathscr{L}, \mathscr{R}$ respectively such that (i)—(vi) of Theorem 6 are satisfied. Clearly $R' = R_1$, the only member of $\mathscr{R}$, and $R' \circ R' = R'$. We put

$$x\Delta y = \pi^{-1}(\pi(x) \otimes \pi(y))$$

for $x, y \in A$ and then (A, Δ) is a topological semigroup since $(\mathscr{L}, \otimes)$ is a topological semigroup and π is a homeomorphism between A and $\mathscr{L}$. For any $L \in \mathscr{L}$,

$$L \cap R' = L \cap T = L = \{(\alpha, \pi^{-1}(L)) | \alpha \in C\},$$

$(L, +)$ is a subgroup topologically isomorphic with $(C, \oplus)$ under the correspondence $(\alpha, \pi^{-1}(L)) \leftrightarrow \alpha$ and the additive unit of $L \cap R'$ is

$$\tau(L, R') = (0, \pi^{-1}(L)).$$

Thus for all $\alpha \in C$, $L \in \mathscr{L}$, it follows from the definitions of ψ and φ that

$$\psi((\alpha, \pi^{-1}(L')), L, R') = (\alpha, \pi^{-1}(L)),$$
$$\varphi(L, R') = (0, \pi^{-1}(L')).$$

We know from Theorem 6 that $L' \cap R' = L'$ is a subsemiring of T; because $(L', +)$ is topologically isomorphic with $(C, \oplus)$, multiplication on L' must be given by

$$(\alpha, \pi^{-1}(L')) \cdot (\beta, \pi^{-1}(L')) = (0, \pi^{-1}(L'))$$

for all $\alpha, \beta \in C$. Thus, by (vi) of Theorem 6,

$$\begin{aligned}
&(\alpha, x) \cdot (\beta, y) \\
&\quad = \psi((\alpha, \pi^{-1}(L')), \pi(x), R') \cdot \psi((\beta, \pi^{-1}(L')), \pi(y), R') \\
&\quad = \psi((\alpha, \pi^{-1}(L')) \cdot (\beta, \pi^{-1}(L')) \\
&\qquad -\varphi(\pi(x) \otimes \pi(y), R' \circ R'), \pi(x) \otimes \pi(y), R' \circ R') \\
&\quad = \psi((0, \pi^{-1}(L'))-(0, \pi^{-1}(L')), \pi(x) \otimes \pi(y), R') \\
&\quad = \psi((0, \pi^{-1}(L')), \pi(x) \otimes \pi(y), R') \\
&\quad = (0, \pi^{-1}(\pi(x) \otimes \pi(y))) = (0, x\Delta y)
\end{aligned}$$

as required.

3. $K[+] \cap K[\cdot]$ and $K[+] \cup K[\cdot]$

That $K[+]$ and $K[\cdot]$ need not meet is shown by Example 2. However, if $K[+]$ does meet $K[\cdot]$, we can make certain assertions.

THEOREM 8. *If* $(S, +, \cdot)$ *is a compact semiring in which*

$$L = K[+] \cap K[\cdot]$$

is not empty then

(i) *if* $e \in K[\cdot] \cap E[\cdot]$, *either* $eSe \subset L$ *or* $eSe \cap L$ *is empty;*

(ii) $L^2 \subset L$;

(iii) $K[+]$ *is a subsemiring;*

(iv) $M^2 \subset M$, *where* $M = K[+] \cup K[\cdot]$.

PROOF. Suppose that $e \in K[\cdot] \cap E[\cdot]$; then eSe is a multiplicative group and a subsemiring. Suppose that $x \in L \cap eSe$. If $y \in eSe$, it follows from [10], Theorem 1 that $y+x+y = y$. On the other hand x in L means that $y = y+x+y \in K[+]$ also. Thus $eSe \subset K[+] \cap K[\cdot]$.

Let x, y be any two members of L. As both y and y^2 are in some group eSe, it follows from (i) that $y^2 \in L$. Because x and y are in $K[+]$ there exist z, w in S with $x = z+y+w$. Hence

$$xy = zy + y^2 + wy.$$

But $y^2 \in L \subset K[+]$ and so $xy \in K[+]$. As $xy \in K[\cdot]$, we conclude that $xy \in L$.

That $K[+]$ is a subsemiring follows from Theorem 5 because

$$L^2 = L^2 \cap L \subset S^2 \cap K[+].$$

Finally,

$$\begin{aligned} M^2 &= (K[+] \cup K[\cdot])^2 \\ &= (K[+])^2 \cup (K[+]K[\cdot]) \cup (K[\cdot]K[+]) \cup (K[\cdot])^2. \end{aligned}$$

But $K[\cdot]$ is a multiplicative ideal and $K[+]$ is a subsemiring. Hence

$$M^2 \subset K[+] \cup K[\cdot] \cup K[\cdot] \cup K[\cdot] = M.$$

It can be seen by considering the following examples that all the different inclusion relations between L, $K[+]$ and $K[\cdot]$ can occur. (We use $A \subset\subset B$ to mean $A \subset B$ and $A \neq B$.)

Example 3. $([0, 1], +, \cdot)$, where $x+y = x \cdot y = x$, has

$$K[+] = K[\cdot] = [0, 1].$$

Example 4. $([0, 1], +, \cdot)$, where $x+y = \min(x, y)$, $x \cdot y = x$, has

$$\{0\} = K[+] \subset\subset K[\cdot] = [0, 1].$$

Example 5. $([0, 1], +, \cdot)$, where $x+y = x$, $x \cdot y = 0$, has

$$\{0\} = K[\cdot] \subset\subset K[+] = [0, 1].$$

Example 6. $([0, 1], +, \cdot)$, where $x \cdot y = \max(\frac{1}{2}, x)$ and

$$x+y = \begin{cases} \min(\frac{1}{2}, x) & \text{if} \quad x \leqq \frac{1}{2} \quad \text{or} \quad y \leqq \frac{1}{2}, \\ \min(x, y) & \text{if} \quad x > \frac{1}{2} \quad \text{or} \quad y > \frac{1}{2}. \end{cases}$$

It can be shown that this is a semiring with $K[+] = [0, \frac{1}{2}]$ and $K[\cdot] = [\frac{1}{2}, 1]$ so that $L \subset\subset K[+]$ and $L \subset\subset K[\cdot]$.

4. The kernel

The existence of a set which is minimal with respect to being both an additive and a multiplicative ideal of any compact semiring $(S, +, \cdot)$ was shown by Selden ([11], Theorem 15). We call this set the *kernel* K and show that $K = S + K[\cdot] + S$.

Theorem 9. *If* $(S, +, \cdot)$ *is a compact semiring then* $S + K[\cdot] + S$ *is its kernel.*

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PROOF. It is clear that $S+K[\cdot]+S$ is an additive ideal. If $s \in S$ and $t \in S+K[\cdot]+S$, there exist $s_1, s_2 \in S$ and $k \in K[\cdot]$ with $t = s_1+k+s_2$. Hence

$$st = s(s_1+k+s_2) = ss_1+sk+ss_2 \in S+K[\cdot]+S$$

since $K[\cdot]$ is a multiplicative ideal. Similarly $ts \in S+K[\cdot]+S$ and so $S+K[\cdot]+S$ is a multiplicative ideal.

Let M be any non-empty subset of S which is both a multiplicative and an additive ideal. We must show that $S+K[\cdot]+S \subset M$. Because M is a multiplicative ideal and $K[\cdot]$ is the minimal such ideal, $K[\cdot] \subset M$. Also, because M is an additive ideal, $S+M+S \subset M$. Hence

$$S+K[\cdot]+S \subset S+M+S \subset M$$

as required.

In fact if T is any set which is both an additive and multiplicative ideal, it is clear that $T+K[\cdot]+T$ is also an ideal of both types. But as

$$T+K[\cdot]+T \subset S+K[\cdot]+S = K$$

we conclude that $T+K[\cdot]+T = K$. In particular, we have the following corollary.

COROLLARY. $K = K+K[\cdot]+K$.

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