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REGULAR REES MATRIX SEMIGROUPS AND REGULAR DUBREIL–JACOTIN SEMIGROUPS

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Abstract

A partially ordered semigroup S is said to be a Dubreil–Jacotin semigroup if there is an isotone homomorphism θ of S onto a partially ordered group such that $\{x \in S: x^2\theta \leq x\theta\}$ has a greatest member. In this paper we investigate the structure of regular Dubreil–Jacotin semigroups in which the imposed partial order extends the natural partial order on the idempotents. The main tool used is a local structure theorem which is introduced in Section 2. This local structure theorem applies to many other contexts as well.

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1. Partially ordered regular semigroups

It is well known that the idempotents of a regular semigroup S can be partially ordered by setting

$$e\omega f \quad \text{if and only if } e = ef = fe.$$

This partial order is called the natural partial order on the idempotents of S. We shall say that a partially ordered semigroup S is *naturally partially ordered* if the imposed partial order $\leq$ extends the natural partial order on the idempotents; that is, if

$$e = ef = fe \quad \text{implies } e \leq f.$$

Note there is no assumption in the definition that $e \leq f$ implies $e = ef = fe$. As we shall see, the existence of a partial order, extending the natural partial

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order on the idempotents, imposes significant structural limitations on regular semigroups.

K. S. S. Nambooripad [10] has shown that the relation $\leqslant$ defined by:

$$a \leqslant b \quad \text{if and only if } a = aa'b \text{ and } a \in bS,$$

for some inverse a' of a in S, is a partial order on any regular semigroup S. This partial order extends the natural partial order on the idempotents of S but, in general, it is not compatible with multiplication. Indeed, Nambooripad shows that $\leqslant$ turns S into a partially ordered semigroup if and only if S obeys one, and thus all, of the conditions described in Lemma 1.1.

LEMMA 1.1. *Let S be a regular semigroup. Then the following conditions on S are equivalent*:

i) $|S(e, f)| = 1$ *for each* $e^2 = e, f^2 = f \in S$;

ii) *eSe is inverse for each idempotent $e \in S$*;

iii) *S does not contain a copy of B^1 for any non trivial rectangular band B.*

The notation $S(e, f)$ in Lemma 1.1 denotes the sandwich set of e and f; thus $S(e, f) = \{g^2 = g \in S\colon\ egf = ef \text{ and } ge = g = fg\}$. The importance of the sandwich set stems from the following useful observation, due to Nambooripad [9]. In this lemma, $V(a)$ denotes the set of inverses of a.

LEMMA 1.2. *Let S be a regular semigroup and let $a, b \in S$; let $a' \in V(a)$, $b' \in V(b)$. Then, for each $g \in S(a'a, bb')$, $b'ga' \in V(ab)$.*

COROLLARY 1.3. *Let S be a regular semigroup and let $x_i \in S$, $1 \leqslant i \leqslant n$; let $x_1' \in V(x_1)$, $x_n' \in V(x_n)$. Then $V(x_1x_2 \cdots x_n) \cap x_n'Sx_1' \neq \varnothing$.*

Because of condition (ii), in Lemma 1.1, we shall say that a regular semigroup is *locally inverse* if it obeys any one of the conditions of Lemma 1.1. The subsemigroups eSe, $e^2 = e$, are called the *local submonoids* of S. Nambooripad [10] used the term pseudo inverse to describe locally inverse semigroups.

PROPOSITION 1.4. *Let S be a regular semigroup. Then S can be turned into a partially ordered semigroup, in which the imposed partial order extends the natural partial order on the idempotents, if and only if S is locally inverse.*

PROOF. Nambooripad's result [10], quoted above, shows that the relation $a \leqslant b$ if and only if $a = aa'b$ and $a \in bS$, for some $a' \in V(a)$, turns S into a naturally partially ordered semigroup provided S is locally inverse.

Conversely, suppose that S is a naturally partially ordered semigroup under $\leqslant$ and let $u, v \in eSe$ where u, v, e are idempotents and, say, $u\mathcal{R}v$. Then $u, v\omega e$ imply $u, v \leqslant e$. But then $u\mathcal{R}v$ implies $u = vu \leqslant ve = v$ and $v = uv \leqslant ue = u$ so that $u = v$. Hence each $\mathcal{R}$-class of eSe contains exactly one idempotent. Similarly, each $\mathcal{L}$-class contains exactly one idempotent. Hence eSe is inverse.

Proposition 1.5. *Let S be a partially ordered regular semigroup.*

i) *If S is naturally ordered then $f \leqslant e$ implies $f = fef$.*

ii) *If S has a greatest idempotent u then S is naturally ordered if and only if $f = fuf$ for each idempotent $f \in S$.*

Proof. Suppose that u, v are idempotents of a partially ordered semigroup S and let $u \leqslant v$. Then

$$uv \cdot uv = uv \cdot u \cdot v \leqslant uv \cdot v \cdot v = uv = u \cdot u \cdot uv \leqslant u \cdot v \cdot uv = uv \cdot uv$$

so that uv is idempotent; similarly vu is idempotent. Further $u \leqslant vu$ so that $u \cdot vu = uvu$ is idempotent.

i) Suppose now that $\leqslant$ extends ω and that $f \leqslant e$. Then fef is idempotent and $f \leqslant fef$. But fef idempotent implies $fef\omega f$ and so, because $\leqslant$ extends ω, $fef \leqslant f$. Hence $f = fef$.

ii) Suppose that S has a greatest idempotent u; then $f \leqslant u$ for each idempotent f. Hence $f = fuf$ provided that S is naturally ordered. Conversely, suppose $v = vuv$ for each idempotent v and let e, f be idempotents with $e\omega f$. Then $e \leqslant u$ implies $eue \leqslant u$ so

$$e = eue = feuef \leqslant fuf = f.$$

Thus $\leqslant$ extends ω and S is naturally ordered.

Corollary 1.6. *Let S be a naturally ordered regular semigroup with a greatest idempotent u. Then $S = SuS$.*

Proof. Let $a \in S$, $a' \in V(a)$. Then $aa' = aa'uaa'$ so that $a = aa'ua \in SuS$.

Corollary 1.6 is relevant to our present investigations because any regular Dubreil–Jacotin semigroup S has a greatest idempotent. Indeed, suppose that θ is an isotone homomorphism onto a partially ordered group G and let $u = \max\{x \in S\colon x^2\theta \leqslant x\theta\}$. Then, for $u' \in V(u)$ we have $uu' \leqslant u$, since uu' is idempotent, so that $1 = (uu')\theta \leqslant u\theta \leqslant 1$, where 1 denotes the identity of G. Hence $u \in 1\theta^{-1}$ which is, consequently, a partially ordered regular semigroup with a greatest element u. Lemma 1.7 shows that u is an idempotent of S.

LEMMA 1.7 [8]. *Let S be a partially ordered regular semigroup with a greatest element u. Then $u^2 = u$ and uSu is a semilattice. Further the imposed partial order on S coincides with the natural partial order.*

In a sequence of papers [2], [3], [4], [5] T. S. Blyth has investigated the structure of various classes of regular Dubreil–Jacotin semigroups. These are perfect in the sense of [5] and Blyth shows, [5] equations (2) and (10), that $e = eue$ for each idempotent e, where u denotes the greatest idempotent. Hence we have

COROLLARY 1.8. *Let S be a perfect Dubreil–Jacotin semigroup. Then the imposed partial order extends the natural partial order on the idempotents of S. Further, if u denotes the greatest idempotent of S then $S = SuS$.*

Naturally ordered regular semigroups having a greatest idempotent have many properties analogous to those of the orthodox Dubreil–Jacotin semigroups considered in [3], [4], [8]. For example, each element in such a semigroup S has a greatest inverse. [Let u denote the greatest idempotent of S and let $x \in S$, $x' \in V(x)$. Then, since $e = eue$ for each idempotent e, $ux'u \in V(x)$. Further, for any $x'' \in V(x)$ we have $x'' = x''xx'xx'' \leqslant ux'u$.] However, [8], Theorem 5.5 shows that these semi-groups need not be orthodox. Indeed, we have the following proposition which generalizes [5], Theorem 2.

PROPOSITION 1.9. *Let S be a partially ordered regular semigroup with a greatest idempotent u. Then the following are equivalent*:

(a) *S is naturally ordered and orthodox*;

(b) *u is a middle unit, that is $xuy = xy$ for each $x, y \in S$.*

PROOF. Suppose that (a) holds. Then, by Proposition 1.5, $e = eue$ for each idempotent $e \in S$. Hence, by [8], Theorem 5.4, u is a middle unit of S.

Suppose, on the other hand, that (b) holds and let B be the subsemigroup generated by the idempotents of S. Then u is the greatest element of B and $x^2 = xux$ for each $x \in B$. Hence, by [8], Corollary 5.6, B is a band. Thus B is orthodox and, by Proposition 1.5, it is naturally ordered.

2. The local structure theorem

Let S be a regular semigroup and let I, Λ be sets; let P be a $\Lambda \times I$ matrix over S. Then the set of all triples $(i, s, \lambda) \in I \times S \times \Lambda$ is a semigroup under the multiplication

$$(i, s, \lambda)(j, t, \mu) = (i, sp_{\lambda j}t, \mu).$$

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This semigroup, which is denoted by $\mathfrak{M}(S; I, \Lambda; P)$, is not, in general, regular. However, as we shall see, the set of regular elements forms a subsemigroup; thus a regular semigroup. We shall denote it by $\mathfrak{R}\mathfrak{M}(S; I, \Lambda P)$ and call it a *regular Rees matrix semigroup* over S. If $S = S^1$ and 1 is an entry of P, then we say that $\mathfrak{R}\mathfrak{M}(S; I, \Lambda; P)$ is *unital*.

LEMMA 2.1. *Let S be a regular semigroup, I, Λ sets and let P be a $\Lambda \times I$ matrix over S. Then*

(i) *$(i, s, \lambda) \in \mathfrak{M}(S; I, \Lambda; P)$ is regular if and only if $V(s) \cap p_{\lambda j} S p_{\mu i} \neq \varnothing$ for some $j \in I$, $\mu \in \Lambda$;*

(ii) *$\mathfrak{R}\mathfrak{M}(S; I, \Lambda; P) = \{(i, s, \lambda)\colon V(s) \cap p_{\lambda j} S p_{\mu i} \neq \varnothing$ for some $j \in I$, $\mu \in \Lambda\}$ is a regular subsemigroup of $\mathfrak{M}(S; I, \Lambda; P)$.*

PROOF. Let $(j, t, \mu) \in \mathfrak{M}(S; I, \Lambda; P)$; then $(j, t, \mu) \in V((i, s, \lambda))$ implies $p_{\lambda j} t p_{\mu i} \in V(s)$. Conversely, if $p_{\lambda j} t p_{\mu i} \in V(s)$ then $(i, s, \lambda)(j, t, \mu)(i, s, \lambda) = (i, s, \lambda)$. Hence (i) holds.

(ii) Suppose (i, s, λ), (j, t, μ) are regular. Then, by (i), there exist $s' \in V(s) \cap p_{\lambda k} S p_{\nu i}$, $t' \in V(t) \cap p_{\mu h} S p_{\rho j}$ for some $k, h \in I$, $\mu, \rho \in \Lambda$. Hence, by Corollary 1.3, $V(s p_{\lambda j} t) \cap t' S s' \neq \varnothing$; thus $V(s p_{\lambda j} t) \cap p_{\mu h} S p_{\nu i} \neq \varnothing$. Since $(i, s, \lambda)(j, t, \mu) = (i, s p_{\lambda j} t, \mu)$ this implies $(i, s, \lambda)(j, t, \mu)$ is regular.

EXAMPLE 2.2. Let S be the chain $\{1 > a > b > 0\}$, $I = \Lambda\{1, 2\}$ and let P be the 2×2 matrix $\left(\begin{smallmatrix} 1 & a \\ b & 0 \end{smallmatrix}\right)$. Then $(i, s, \lambda) \in \mathfrak{R}\mathfrak{M}(S; I, \Lambda; P)$ if and only if $s = s' \in p_{\lambda j} S p_{\mu i}$ for some $j \in I$, $\mu \in \Lambda$. Since multiplication in S is min, it follows that $(i, s, \lambda) \in \mathfrak{R}\mathfrak{M}(S; I, \Lambda; P)$ if and only if $s \leqslant p_{\lambda j} p_{\mu i}$. That is, s is less than the λ, i entry of P^2 regarded as a matrix over the lattice S. Since $P^2 = \left(\begin{smallmatrix} 1 & a \\ b & b \end{smallmatrix}\right)$ we find that $\mathfrak{R}\mathfrak{M}(S; I, \Lambda; P)$ has eleven elements, whereas $\mathfrak{M}(S; I, \Lambda; P)$ has 16;

$$
\begin{aligned}
&(1, 1, 1)\\
&(1, a, 1), (2, a, 1)\\
(1, b, 1), (1, b, 2), (2, b, 1), &(2, b, 2)^*\\
(1, 0, 1), (1, 0, 2), (2, 0, 1), &(2, 0, 2).
\end{aligned}
$$

Only $(2, b, 2)$ is not idempotent.

We say that a homomorphism θ of a regular semigroup S onto a regular semigroup T is a *local isomorphism* if θ maps each eSe, $e^2 = e$, isomorphically into T; in this case, we call T a *locally isomorphic image* of S.

LEMMA 2.3. *Let θ be a homomorphism of a regular semigroup S onto a regular semigroup T.*

(i) *If θ is a local isomorphism then each idempotent $\theta \circ \theta^{-1}$ class is a rectangular band.*

(ii) *If S is locally inverse, and each idempotent $\theta \circ \theta^{-1}$ class is a rectangular band, then θ is a local isomorphism.*

PROOF. (i) Nambooripad [10] shows that every idempotent class of a congruence ρ on S, is completely simple if and only if $e\omega f$, $e\rho f$ together imply $e = f$. Suppose $e\theta = f\theta$ where e, f are idempotents and $e\omega f$ then, since θ is one to one on fSf, $e = f$. Hence each idempotent $\theta \circ \theta^{-1}$ class is completely simple. Further, since θ is one to one on each fSf, these classes have trivial subgroups. That is, they are rectangular bands.

(ii) Let $a, b \in eSe$, where $e^2 = e$, and suppose that $a\theta = b\theta$. Then, for each $a' \in V(a) \cap eSe$, we have $aa'\theta = ba'\theta$. Thus aa' and ba' belong to a rectangular sub-band of eSe. Since eSe is inverse, this implies $aa' = ba'$ so that $a = ba'a \in bS$. Dually $a = aa'b \in Sb$ and, likewise, $b \in Sa$. Hence $a\mathcal{L}b$ and so, since $a = ba'a$, $a = b$.

THEOREM 2.4. (Local Structure Theorem.) *Let S be a regular semigroup and suppose that $S = SeS$ for some idempotent e. Then S is a locally isomorphic image of a unital regular Rees matrix semigroup over eSe.*

PROOF. For each idempotent $u \in S$, there exists $r_u \in S$, $r'_u \in V(r_u)$ such that $r_u r'_u = u$ and $r'_u r_u \omega e$, since $u \in SeS$. For each pair of idempotents v, u, let $p_{v,u} = r'_v r_u$; then, since $r'_v r_v$, $r'_u r_u \omega e$, we have $p_{v,u} \in eSe$. We may further assume, without loss of generality, that $r'_e = r_e = e$.

Let I be a set of idempotents of S, such that for each idempotent $v \in S$ there exists $u \in I$ with $v = uv$. Likewise, let Λ be a set of idempotents such that for each $v^2 = v \in S$ there exists $u \in \Lambda$ with $v = vu$ and let P be the $\Lambda \times I$ matrix with entries $p_{v,u}$. Then we can form the (unital) regular Rees matrix semigroup $\mathfrak{R}\mathfrak{M}(eSe; I, \Lambda; P)$ over eSe. Its entries consist of all triples (u, s, v) in $I \times eSe \times \Lambda$ for which $V(s) \cap r'_v r_v S r'_u r_u \neq \varnothing$, since $r'_v r_v e = r'_v r_v$, $er'_u r_u = r'_u r_u$.

Define θ: $\mathfrak{R}\mathfrak{M}(eSe; I, \Lambda; P) \to S$ by $(u, s, v)\theta = r_u s r'_v$. Then

$$\begin{aligned}((u, s, v)(x, t, y))\theta &= (u, sp_{v,x}t, y)\theta = r_u s p_{v,x} t r'_y \\ &= r_u s r'_v r_x t r'_y = (u, s, v)\theta(x, t, y)\theta\end{aligned}$$

so that θ is a homomorphism of $T = \mathfrak{R}\mathfrak{M}(eSe; I, \Lambda; P)$ into S.

Now, let $s \in S$. Then, because of the choice of I, Λ, $s \in uSv$ for some $u \in I$, $v \in \Lambda$ so that $s = r_u r'_u s r_v r'_v = r_u t r'_v$ where $t = r'_u s r_v$. By Corollary 1.3, $t \in eSe$

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has an inverse in $r'_v S r_u = r'_v r_v S r'_u r_u = r'_v r_w S r'_z r_u$ for some $w \in I$, $z \in \Lambda$. Hence $(u, t, v) \in \mathfrak{R}\mathfrak{M}(eSe; I, \Lambda; P)$. But $(u, t, v)\theta = r_u t r_v = s$. Hence θ is onto.

Finally, suppose that (u, s, v) is idempotent. Then $(x, t, y) \in (u, s, v)T(u, s, v)$ implies $x = u$, $y = v$ and $t = tp_{v,u}s = sp_{v,u}t$; further $(x, t, y)\theta = r_u t r'_v$. Suppose (u, t, v), $(u, z, v) \in (u, s, v)T(u, s, v)$ and $(u, t, v)\theta = (u, z, v)\theta$; then $r_u t r'_v = r_u z r'_v$ which, in turn, implies

$$\begin{aligned} t &= sp_{v,u}tp_{v,u}s = sr'_v r_u t r'_v r_u s \\ &= sr'_v r_u z r'_r r_u s = z. \end{aligned}$$

Thus θ is one to one on each local submonoid. That is, θ is a local isomorphism.

COROLLARY 2.5. *Let S be a regular semigroup and suppose that $S = SeS$ for some idempotent e. Then S is locally inverse if and only if it is a locally isomorphic image of a unital regular Rees matrix semigroup over an inverse monoid.*

In this case eSe is inverse and S is a locally isomorphic image of a unital regular Rees matrix semigroup over eSe.

COROLLARY 2.6. *Let S be a regular semigroup and let e be an idempotent of S. Then S is simple (bisimple) if and only if it is a locally isomorphic image of a unital regular Rees matrix semigroup over a regular simple (bisimple) monoid.*

In this case eSe is simple (bisimple) and S is a locally isomorphic image of a unital regular Rees matrix semigroup over eSe.

Corollaries 2.5 and 2.6 depend on a knowledge of Green's relations in regular Rees matrix semigroups. These are easily calculated, as the following lemma shows.

LEMMA 2.7. *Let $\mathfrak{R}\mathfrak{M}(S; I, \Lambda; P)$ be a regular Rees matrix semigroup over the regular semigroup S.*

i) *(i, s, λ) is idempotent if and only if $s = sp_{\lambda i}s$;*

ii) *let (i, s, λ), (j, t, μ) be idempotents, then*

(a) *$(i, s, \lambda)\mathfrak{R}(j, t, \mu)$ if and only if $i = j$, $s\mathfrak{R}t$,*

(b) *$(i, s, \lambda)\mathfrak{L}(j, t, \mu)$ if and only if $\lambda = \mu$, $s\mathfrak{L}t$,*

(c) *$(i, s, \lambda)\omega(j, t, \mu)$ if and only if $i = j$, $\lambda = \mu$ and $sp_{\lambda i}\omega tp_{\lambda i}$,*

(d) *$(i, s, \lambda)\mathfrak{D}(j, t, \mu)$ if and only if $s\mathfrak{D}t$,*

(e) *$(i, s, \lambda)\mathfrak{J}(j, t, \mu)$ if and only if $s\mathfrak{J}t$.*

Corollary 2.5, coupled with the main theorem of [7], can be used to give a proof of a theorem of Pastijn [11] for principally generated locally inverse semigroups. The proof, in the general case, is much more difficult.

The homomorphism θ, constructed in Theorem 2.4, is usually far from one to one. The next result specifies exactly when θ is an isomorphism. We do not need this result so the proof is omitted.

THEOREM 2.8. *Let $S = SeS$ be a regular semigroup, where $e^2 = e$ and let I, Λ, P be as in Theorem 2.4. Then θ: $\mathfrak{RM}(eSe; I, \Lambda; P) \to S$ is an isomorphism if and only if*

(i) *uS is a maximal principal right ideal of S, for each $u \in I$, and dually;*

(ii) *distinct maximal principal right (left) ideals are disjoint;*

(iii) *$t \in r'_u r_u S r'_v r_v$ implies $V(t) \cap eSe \subseteq r'_v r_v S r'_u r_u$.*

In this case $\mathfrak{RM}(eSe; SI, \Lambda; P) = \{(u, s, v): s \in r'_u r_u S r'_v r_v\}$.

Condition (iii) in Theorem 2.8 is unsatisfactory since it depends on the choice of elements $r_u, r'_u \in S$. It becomes redundant however in two situations:

(a) S is bisimple; for we may choose $r'_u r_u = e$ in this case;

(b) S is locally inverse; for then eSe is inverse and (iii) is automatic.

Theorem 2.4 leans heavily on the ideas of D. Allen [1]. Our contribution consists of defining $\mathfrak{RM}(S; I, \Lambda; P)$ and removing some of Allen's unnecessary hypotheses.

Theorem 2.4 adapts readily to partially ordered semigroups. Suppose that T is a partially ordered regular semigroup and let I, Λ be partially ordered sets. Further, suppose that P is an isotone $\Lambda \times I$ matrix over T; that is, P is isotone considered as a mapping from $\Lambda \times I$, under the cartesian ordering, into T. Then $\mathfrak{RM}(T; I, \Lambda; P)$ becomes a partially ordered semigroup under

$$(i, s, \lambda) \leqslant (j, t, \mu) \quad \text{if and only if } i \leqslant j, s \leqslant t, \lambda \leqslant \mu.$$

On the other hand, suppose that S is a partially ordered regular semigroup and that $S = SeS$ for some idempotent e. Let I, Λ, P be as in the proof of Theorem 2.3 and partially order I and Λ by

$$u \leqslant v \quad \text{for } u, v \in I \text{ if and only if } r_u \leqslant r_v,$$

$$u \leqslant v \quad \text{for } u, v \in \Lambda \text{ if and only if } r'_u \leqslant r'_v.$$

Then, since $p_{vu} = r'_v r_u$, P is an isotone matrix over eSe. Further, the map θ: $(u, s, v) \mapsto r_u s r'_v$ is an isotone mapping of $\mathfrak{RM}(eSe; I, \Lambda; P)$ onto S. Indeed, restricted to each local submonoid, θ is an order isomorphism. Hence the analog of Theorem 2.4 also holds in the category of partially ordered regular semigroups.

THEOREM 2.9. *Let S be a partially ordered regular semigroup and suppose that $S = SeS$ for some idempotent e. Then S is a (partially ordered semigroup) locally isomorphic image of a unital regular Rees matrix semigroup over eSe.*

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3. Naturally ordered regular semigroups with a greatest idempotent

Let S be a naturally ordered regular semigroup which has a greatest (under the imposed partial order $\leqslant$) idempotent u. Then, by Corollary 1.6, $S = SuS$ so we may apply the local structure theorem to obtain S as a locally isomorphic image of a regular Rees matrix semigroup over an inverse monoid. In order to produce the Rees matrix semigroup, from S, we need to choose sets I, Λ of idempotents of S, and construct the sandwich matrix P.

LEMMA 3.1. *Let S be a naturally ordered regular semigroup with greatest idempotent u. Let E denote the set of idempotents of S. Then Eu is a set of idempotent representatives of the $\mathcal{R}$-classes of S; uE is a set of idempotent representatives of the $\mathcal{L}$-classes of S.*

PROOF. Let $f \in E$; then $f = fuf$ so that $f \mathcal{R} fu$ and $fu \in E$. Hence Eu contains at least one idempotent from each $\mathcal{R}$-class of S. Suppose that $f, g \in Eu$ and $f \mathcal{R} g$. Then uf and ug are idempotents in $uEu \subseteq uSu$, which is inverse by Proposition 1.4. Further $uf \mathcal{R} ug$ since $f \mathcal{R} g$. Hence $uf = ug$. Thus $f = fuf = fug = fg$ since $f = fu$. But $f \mathcal{R} g$ implies $g = fg$ so we must have $f = g$. Hence Eu contains exactly one element of each $\mathcal{R}$-class of S; dually uE contains exactly one element of each $\mathcal{L}$-class of S.

We let $I = Eu$, $\Lambda = uE$. Then, for $f \in E$, $f \mathcal{R} fu = r_f$, $f \mathcal{L} uf = r'_f \in V(r_f)$ and $r_f r'_f = fuf = f$, $r'_f r_f = ufu\omega u$. Hence, if we let P be the $\Lambda \times I$ matrix with $p_{f,g} = r'_f r_g = ufgu = fg$, then θ defined by $(e, s, f)\theta = esf$ is an algebraic local isomorphism of $\mathcal{RM}(uSu; I, \Lambda; P)$ onto S.

LEMMA 3.2. *For each $(f, g) \in \Lambda \times I$, $p_{f,g}$ is idempotent.*

PROOF. Let B denote the subsemigroup generated by the idempotents of S. Then B has greatest element u and so, by Lemma 1.7, uBu is a semilattice. Now $f, g \in B$ and so $p_{f,g} = fg = ufgu \in uBu$. Hence $p_{f,g}$ is idempotent.

The sets $I = Eu$, $\Lambda = uE$ being subsets of S inherit a partial order from S; each has greatest element u. Further the map P: $\Lambda \times I \to uSu$ is an isotone map and maps (u, u), the greatest element of $\Lambda \times I$ onto the identity of $T = uSu$. Further, when $R = \mathcal{RM}(T; I, \Lambda; P)$ is given the cartesian ordering, the mapping θ: $R \to S$, defined by $(f, s, g)\theta = fsg$, is a local isomorphism of partially ordered semigroups. Hence we have the converse half of the following theorem.

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THEOREM 3.3. *Let T be a naturally ordered inverse monoid and let Λ, I be partially ordered sets each with a greatest element* (1 *say*); *let P be an isotone map of $\Lambda \times I$ into the idempotents of T such that $p_{1,1} = 1$, the identity of T. Then, under the cartesian ordering, $\mathfrak{R}\mathfrak{M}(T; I, \Lambda; P)$ is a naturally ordered regular semigroup with a greatest idempotent.*

A partially ordered regular semigroup is naturally ordered with a greatest idempotent if and only if it is a locally isomorphic image of a semigroup constructed in this way.

PROOF. It is straightforward to show that $\mathfrak{R}\mathfrak{M}(T; I, \Lambda; P)$ is a naturally ordered regular semigroup with greatest idempotent (1, 1, 1). Suppose that S is any isotone image of $\mathfrak{R}\mathfrak{M}(T; I, \Lambda; P)$, say by a homomorphism θ. Suppose that $\bar{e}, \bar{f}$, are idempotents in S such that $\bar{e}\omega\bar{f}$. Then, by Lallement's Lemma ([6], p. 52), twice, there are idempotents $e, f \in \mathfrak{R}\mathfrak{M}(T; I, \Lambda; P)$ such that $e\theta = \bar{e}$, $f\theta = \bar{f}$ and $e\omega f$. It follows that $e \leqslant f$, since $\mathfrak{R}\mathfrak{M}(T; I, \Lambda; P)$ is naturally ordered, and thus $\bar{e} = e\theta \leqslant f\theta = \bar{f}$. Hence S is naturally ordered. It clearly has a greatest idempotent.

COROLLARY 3.4 (to the proof). *If S is a naturally ordered regular semigroup, so is any isotone homomorphic image of S.*

COROLLARY 3.5. *Let T be a semilattice with greatest element* 1 *and let Λ, I be partially ordered sets each with greatest element* 1, *say*; *let P be an isotone map of $\Lambda \times I$ into T such that $p_{1,1} = 1$. Then, under the cartesian ordering, $\mathfrak{R}\mathfrak{M}(T; I, \Lambda; P)$ is a naturally ordered regular semigroup with a greatest element.*

A partially ordered regular semigroup is naturally ordered with a greatest element if and only if it is a locally isomorphic image of a semigroup constructed as above.

PROOF. It is straightforward to show that $\mathfrak{R}\mathfrak{M}(T; I, \Lambda; P)$ is naturally ordered and, clearly, it has greatest element (1, 1, 1). Further any isotone image of a partially ordered semigroup with a greatest element also has a greatest element. And, by Corollary 3.4, any isotone image of a naturally ordered regular semigroup is naturally ordered.

On the other hand, if S is naturally ordered with greatest elements u then, by Theorem 3.3, it is a locally isomorphic image of $\mathfrak{R}\mathfrak{M}(uSu; I, \Lambda; P)$, where $I = Eu$, $\Lambda = uE$ and E is the set of idempotents of S. Further, by Lemma 1.7, uSu is a semilattice.

COROLLARY 3.6. *A simple regular semigroup can be naturally ordered with a greatest element if and only if it is a rectangular band.*

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PROOF. Suppose S is a naturally ordered simple regular semigroup with greatest element u. Then S is a locally isomorphic image of a regular Rees matrix semigroup over uSu, where uSu is a semilattice. But S simple implies uSu simple and thus, since it is a semilattice, uSu is trivial. But then any Rees matrix semigroup over uSu is a rectangular band. Hence, so is S.

Conversely it is clear that any rectangular band can be partially ordered in the required way.

An analogous result to Corollary 3.5 holds for Dubreil–Jacotin regular semigroups which are naturally ordered. The proof is, however, somewhat more intricate so we state this result as a separate theorem.

THEOREM 3.7. *Let T be a Dubreil–Jacotin inverse monoid, with greatest idempotent the identity* 1. *Let I, Λ be partially ordered sets each with a greatest element,* 1 *say, and let P be an isotone map of $\Lambda \times I$ into the semilattice of idempotents of T such that $p_{1,1} = 1$. Then $\mathfrak{R}\mathfrak{M}(T; I, \Lambda; P)$ is a naturally ordered Dubreil–Jacotin regular semigroup, under the cartesian ordering.*

A partially ordered regular semigroup is a naturally ordered Dubreil–Jacotin semigroup if and only if it is a locally isomorphic image of a semigroup constructed as above.

PROOF. Let T, I, Λ, P be as above and let θ denote the natural homomorphism of T onto its maximum group homomorphic image G. Then G is partially ordered by $a\theta \leq b\theta$ if and only if $ea \leq eb$ for some idempotent e and θ is an isotone homomorphism of T onto G such that $a\theta \leq 1\theta$ if and only if $a \leq 1$.

Define a mapping φ: $\mathfrak{R}\mathfrak{M}(T; I, \Lambda; P) \to G$ by setting $(i, s, \lambda)\varphi = s\theta$. Then, since each $p_{\lambda i}$ is idempotent, it is immediate that φ is an isotone homomorphism of $\mathfrak{R}\mathfrak{M}(T; I, \Lambda; P)$ onto the partially ordered group G. Further, $(i, s, \lambda)\varphi \leq 1\theta$, the identity of G, implies $s\theta \leq 1\theta$ and so, since T is a Dubreil–Jacotin semigroup, $s \leq 1$. Hence $(i, s, \lambda) \leq (1, 1, 1)$ so that $\mathfrak{R}\mathfrak{M}(T; I, \Lambda; P)$ is a Dubreil–Jacotin semigroup; we have already seen that it is naturally ordered.

On the other hand, suppose that S is a naturally ordered Dubreil–Jacotin regular semigroup and let u be the greatest idempotent of S. Then uSu is a Dubreil–Jacotin inverse monoid and S is a locally isomorphic image of $\mathfrak{R}\mathfrak{M}(uSu; I, \Lambda; P)$, for some I, Λ, P.

To complete the proof of the theorem, we show that any locally isomorphic image of a naturally ordered Dubreil–Jacotin regular semigroup is itself a Dubreil–Jacotin semigroup. To this end, let T be a naturally ordered Dubreil–Jacotin regular semigroup and let θ be a local isomorphism (of partially ordered semigroups) of T onto a partially ordered semigroup S. Denote by φ the, unique, isotone homomorphism of T onto a group G such that $\{x \in T: x\varphi \leqslant 1$, the identity of $G\}$ has a greatest element v and let $u = v\theta$; then u is the greatest idempotent of S, since θ is isotone.

Suppose that $a\theta = b\theta$ then, for each $b' \in V(b)$, $ab'\theta = (bb')\theta$ so that, by Lemma 2.3, ab' is idempotent, because θ is a local isomorphism. Hence $(ab')\varphi = 1$; that is $a\varphi = b\varphi$. It follows therefore that there is a unique homomorphism ψ: $S \to G$ such that $\varphi = \theta\psi$.

Suppose that $a\theta \leqslant b\theta$. Then $(bb'ab')\theta \leqslant bb'\theta$, for $b' \in V(b)$ where $bb'ab'$ and bb' both belong to $bb'Tbb'$. Hence, since θ is a local isomorphism, $bb'ab' \leqslant bb'$ and thus $(bb'ab')\varphi \leqslant (bb')\varphi$. This implies $(bb'ab'b)\theta\psi \leqslant b\theta\psi$ and so, since G is a group, $a\theta\psi \leqslant b\theta\psi$. Hence ψ is isotone and further

$$\begin{aligned} a\theta\psi \leqslant 1 \quad & \text{if and only if } a\varphi \leqslant 1, \\ & \text{if and only if } a \leqslant v, \\ & \text{if and only if } a\theta \leqslant u \end{aligned}$$

since ψ is isotone and u is idempotent. Thus S is a Dubreil–Jacotin regular semigroup and, since T is naturally ordered, so is S.

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