THE INHOMOGENEOUS MINIMA OF INDEFINITE QUADRATIC FORMS

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If $f(\boldsymbol{x})$ is a real indefinite quadratic form in n variables with determinant $d \neq 0$, we set for any real $\boldsymbol{\alpha}$

$$\begin{aligned} m(f; \boldsymbol{\alpha}) &= \inf |f(\boldsymbol{x})| \qquad \text{over } \boldsymbol{x} \equiv \boldsymbol{\alpha} \pmod 1, \\ m(f) &= \sup_{\boldsymbol{\alpha}} m(f; \boldsymbol{\alpha}). \end{aligned}$$

A current problem in the geometry of numbers is the determination of bounds for the ratio $m(f)/|d|^{1/n}$.

Setting

$$c_n = \inf_f \frac{m(f)}{|d|^{1/n}},$$

Davenport [3] showed that $c_2 > 0$ (see Cassels [2] for a discussion of the literature on c_2). Cassels [2, p. 306] remarks that it is not known whether $c_3 > 0$.

It is in fact quite simple to show that

$$c_n = 0 \quad \text{for } n \geqq 3. \tag{1}$$

I establish here the sharper result:

Theorem. For any $n \geqq 3$ *and any signature* (r, s) $(r \geqq 1, s \geqq 1, r + s = n)$, *there exists a quadratic form* f *of signature* (r, s) *and determinant* ± 1 *with* $m(f) = 0$.

The result in fact holds for any f which represents arbitrarily small non-zero values for integral values of the variables.

We need the following lemma:

Suppose that f *has signature* (r, s) with $r \geqq 2$, $s \geqq 1$, $r + s = n$ *and determinant* $d \neq 0$, *and that* f *properly represents a value* $a > 0$; *then, for some* $\gamma > 0$ *depending only on* n,

$$m(f) \leqq (\gamma a^{n-2}|d|)^{1/(2(n-1))} + \tfrac{1}{4}a. \tag{2}$$

Proof. We may suppose, after a suitable equivalence transformation, that $f(1, 0, \cdots, 0) = a$; we may then write

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$$f(x) = a(x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n)^2 - g(x_2, \cdots, x_n)$$

where g is indefinite with determinant $-d/a$.

For any α, we may choose $x_j \equiv \alpha_j \pmod 1$ $(j = 2, \cdots, n)$ so that

$$g(x_2, \cdots, x_n) = b$$

with

$$0 \leqq b \leqq \left(\gamma \frac{|d|}{a}\right)^{1/(n-1)}; \tag{3}$$

this is a particular case of a result of Blaney [1] (Theorem 2). With this choice of $x_2, \cdots, x_n$, we have

$$f(x) = a(x_1 + \lambda)^2 - b.$$

We now choose $x_1 \equiv \alpha_1 \pmod 1$ so that

$$\left(\frac{b}{a}\right)^{\frac{1}{2}} - \tfrac{1}{2} \leqq x_1 + \lambda < \left(\frac{b}{a}\right)^{\frac{1}{2}} + \tfrac{1}{2};$$

an elementary calculation now gives

$$-(ab)^{\frac{1}{2}} + \tfrac{1}{4}a \leqq f(x) < (ab)^{\frac{1}{2}} + \tfrac{1}{4}a.$$

Thus, using (3), we have

$$m(f; \alpha) \leqq (ab)^{\frac{1}{2}} + \tfrac{1}{4}a \leqq (\gamma a^{n-2}|d|)^{1/(2(n-1))} + \tfrac{1}{4}a;$$

since this holds for all α, (1) follows immediately.

Proof of the Theorem. Take $n \geqq 3$ and f any indefinite form of signature (r, s), with determinant ± 1, which represents arbitrarily small non-zero values of each sign. (An example of such a form is

$$\theta x_1^2 - \frac{1}{\theta} x_2^2 + x_3^2 + \cdots + x_{r+1}^2 - x_{r+2}^2 - \cdots - x_n^2,$$

where θ has a continued fraction with unbounded partial quotients.) Since $m(-f) = m(f)$, we may suppose that $r \geqq 2$; then, by the lemma,

$$m(f) \leqq (\gamma a^{n-2})^{1/(2(n-1))} + \tfrac{1}{4}a$$

for any $a > 0$ represented by f. Since $n \geqq 3$ and a may be chosen as small as we please, it follows that $m(f) = 0$.

This establishes the theorem, and hence the assertion (1).

References

[1] Blaney, H. Indefinite quadratic forms in n variables, J. London Math. Soc. 23 (1948), 153–160.

[2] Cassels, J. W. S. An introduction to the geometry of numbers (Springer, 1959).

[3] Davenport, H. Indefinite quadratic forms and Euclid's algorithm in real quadratic fields, Proc. London Math. Soc. (2) 53 (1951), 65–82.

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