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ON ANNIHILATOR IDEALS OF SKEW MONOID RINGS*

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Abstract. A ring R is called a left APP-ring if the left annihilator $l_R(Ra)$ is pure as a left ideal of R for every $a \in R$; R is called (left principally) quasi-Baer if the left annihilator of every (principal) left ideal of R is generated by an idempotent. Let R be a ring and M an ordered monoid. Assume that there is a monoid homomorphism $\phi : M \longrightarrow Aut(R)$. We give a necessary and sufficient condition for the skew monoid ring $R * M$ (induced by ϕ) to be left APP (left principally quasi-Baer, quasi-Baer, respectively).

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1. Introduction. Throughout this paper, R denotes a ring with unity. Recall that R is a right PP-ring if the right annihilator of an element of R is generated by an idempotent. Armendariz showed that polynomial rings over right PP-rings need not be right PP in the example in [**1**]. Also the concept of right PP-rings is not a Morita invariant property because $\mathbb{Z}[x]$ is PP but the 2×2 full matrix ring over $\mathbb{Z}[x]$ is not a right PP-ring [**1**]. In order to consider the natural question of how much of the right PP condition transfers to polynomial rings or matrix rings, a concept of left APP-rings was introduced and considered in [**15**]. By [**15**], Proposition 2.3, right PP-rings are left APP. It was shown in [**15**], Theorem 3.8 and Corollary 3.12, that the left APP condition is a Morita invariant property and transfers to a variety of polynomial extensions.

On the other hand, a ring R is (quasi-)Baer if the left annihilator of every non-empty subset (every left ideal) of R is generated by an idempotent of R. Clark defined quasi-Baer rings in [**7**] and used them to characterise when a finite dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semi-group algebra. As a generalisation of quasi-Baer rings, Birkenmeier, Kim and Park in [**5**] introduced the concept of left principally quasi-Baer rings. A ring R is called left principally quasi-Baer if the left annihilator of a principal left ideal of R is generated by an idempotent. Observe that biregular rings and quasi-Baer rings are left principally quasi-Baer. Clearly the concept of left APP-rings is a common generalisation of left principally quasi-Baer rings and right PP-rings. For more details and examples of quasi-Baer rings, left principally quasi-Baer rings and left APP-rings, see [**2**–**6**, **12**, **13** and **15**].

In this paper we consider the left APP property, left principal quasi-Baerness and quasi-Baerness of the skew monoid ring $R * M$. Some necessary and sufficient

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conditions for the skew monoid ring $R * M$ to be left APP (left principally quasi-Baer, quasi-Baer) are obtained.

Let R be a ring, and let M be an ordered monoid. Assume that there exists a monoid homomorphism $\phi : M \longrightarrow Aut(R)$. For any $g \in M$ and any $r \in R$, we denote by r^g the image of r under $\phi(g)$. Then we can form a skew monoid ring $R * M$ (induced by the monoid homomorphism ϕ) by taking its elements to be finite formal combinations $\sum_{g\in M} a_g g$, with multiplication induced by

$$(a_g g)(b_h h) = (a_g b_h^g)(gh).$$

If ϕ is weakly rigid (that is to say $ab = 0$ implies $a^g b = ab^g = 0$ for any $a, b \in R$ and any $g \in M$), then it has been proved in [**14**] that the skew monoid ring $R * M$ is quasi-Baer if and only if R is quasi-Baer. If R is ϕ-compatible, then it has been proved in [**8**] that $R * M$ is (left principally) quasi-Baer if and only if R is (left principally) quasi-Baer. It was shown in [**15**], Theorem 3.10, that if R is a left APP-ring, M is a u.p.-monoid, and the monoid homomorphism $\phi : M \longrightarrow Aut(R)$ satisfies the condition that for every $a \in R$, the left ideal $\sum_{g\in M} Ra^g$ is finitely generated, then the skew monoid ring $R * M$ (induced by the monoid homomorphism ϕ) is a left APP-ring. When M is a group or $Im(\phi)$ is a group, a necessary and sufficient condition for the skew monoid ring $R * M$ to be (left principally) quasi-Baer was given in [9]. In this paper we will show that for an ordered monoid M and a monoid homomorphism $\phi : M \longrightarrow Aut(R)$, the skew monoid ring $R * M$ is a left APP-ring (a left principally quasi-Baer ring, a quasi-Baer ring, respectively) if and only if the left annihilator $l_R(\sum_{g\in M} Ra^g)$ is right s-unital for every $a \in R$. (The left annihilator of $\sum_{g\in M} Ra^g$ is generated by an idempotent for every $a \in R$, and the left annihilator of left ideal $\sum_{b\in S} \sum_{g\in M} Rb^g$ of R is generated by an idempotent for every subset S of R, respectively.)

For a non-empty subset Y of R, $l_R(Y)$ and $r_R(Y)$ denote the left and right annihilators of Y in R, respectively. A monoid M is said to be ordered if the elements of M are linearly ordered with respect to the relation $<$ and if for all $x, y, z \in M$, $x < y$ implies $zx < zy$ and $xz < yz$. It is well known that any submonoid of a free group or a torsion-free nilpotent group is an ordered monoid. We denote by η the identity of a monoid M.

2. Left APP-rings. An ideal I of R is said to be right s-unital if for each $a \in I$ there exists an element $x \in I$ such that $ax = a$. Note that if I and J are right s-unital ideals, then so is $I \cap J$. (If $a \in I \cap J$, then $a \in aIJ \subseteq a(I \cap J)$.) It follows from [**17**], Theorem 1, that I is right s-unital if and only if for any finitely many elements $a_1, a_2, \ldots, a_n \in I$ there exists an element $x \in I$ such that $a_i = a_i x$, $i = 1, 2, \ldots, n$. A submodule N of a left R-module M is called a pure submodule if $L \otimes_R N \longrightarrow L \otimes_R M$ is a monomorphism for every right R-module L. By [**16**], Proposition 11.3.13, an ideal I is right s-unital if and only if R/I is flat as a left R-module if and only if I is pure as a left ideal of R.

By [**15**], a ring R is called a left APP-ring if the left annihilator $l_R(Ra)$ is right s-unital as an ideal of R for any element $a \in R$.

Right APP-rings may be defined analogously. Clearly every left principally quasi-Baer ring is a left APP-ring. (Thus the class of left APP-rings includes all biregular rings and all quasi-Baer rings.) From [**15**], it follows that right PP-rings are left APP and left APP-rings are quasi-Armendariz in the sense that whenever $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ satisfy $f(x)R[x]g(x) = 0$, we have $a_iRb_j = 0$ for each i and j (see for example [**10**]).

Let M be a monoid and $\phi : M \longrightarrow Aut(R)$ a monoid homomorphism. The ring R is called left M-APP if the left annihilator $l_R(\sum_{g\in M} Rb^g)$ is right s-unital for every $b \in R$. Clearly if $\phi(g) = 1$ for every $g \in M$, then R is left M-APP if and only if R is left APP. It is easy to see that if R is a left Noetherian and left APP-ring, then R is left M-APP for any monoid M (in fact, there exists a maximal element $\sum_{g\in N_0} Rb^g$ in the set $\{\sum_{g\in N} Rb^g | N \subseteq M, |N| < \infty\}$, which is unique, and so $l_R(\sum_{g\in M} Rb^g) = l_R(\sum_{g\in N_0} Rb^g) = \cap_{g\in N_0} l_R(Rb^g)$ is right s-unital).

Remark 1. (1) It follows from [**15**], Theorem 3.10, that if R is a left APP-ring and M an ordered monoid and if the monoid homomorphism $\phi : M \longrightarrow Aut(R)$ satisfies the condition that for every $a \in R$, the left ideal $\sum_{g\in M} Ra^g$ is finitely generated, then the skew monoid ring $R * M$ (induced by the monoid homomorphism ϕ) is a left APP-ring. Thus, by Theorem 2, R is a left M-APP-ring. Remark 3.11 of [**15**] gave some examples of left M-APP-rings.

(2) For a given left APP-ring T, let

$$R = \left\{(a_n)_{n\in\mathbb{Z}} \in \prod T | a_n \text{ is eventually constant}\right\},$$

which is a subring of the countably infinite direct product $\prod_{\mathbb{Z}} T$. Define an automorphism σ of R by $\sigma(a_n)_{n\in\mathbb{Z}} = (a_{n+1})_{n\in\mathbb{Z}}$. Let $M = \mathbb{N} \cup \{0\}$. Define $\phi : M \longrightarrow Aut(R)$ via $\phi(0) = 1$ and $\phi(n) = \sigma^n$ for every $n \in \mathbb{N}$. Suppose that $w = (\ldots, a, a, a_s, a_{s+1}, \ldots, a_t, a, a, \ldots) \in l_R(\sum_{g\in M} Rb^g)$, where $b = (b_n)_{n\in\mathbb{Z}} \in R$. Then $wRb^g = 0$ for each $g \in M$. Thus for any $s \le n \le t$, $a_nTb_n = 0$, $a_nTb_{n-1} = 0$, $a_nTb_{n-2} = 0, \ldots$. Since $(b_n)_{n\in\mathbb{Z}}$ is eventually constant, the left ideal $Tb_n + Tb_{n-1} + \cdots$ is finitely generated. By Proposition 2.6 of [**15**], $l_T(Tb_n + Tb_{n-1} + \cdots)$ is right s-unital. Thus $a_n = a_na'_n$ for some $a'_n \in l_T(Tb_n + Tb_{n-1} + \cdots)$. Similarly $a = aa'$ for some $a' \in l_T(\sum_{n\in\mathbb{Z}} Tb_n)$. Now it is easy to see that $w = ww'$, where $w' = (\ldots, a', a', a'_s, \ldots, a'_t, a', a', \ldots) \in l_R(\sum_{g\in M} Rb^g)$. Therefore R is left M-APP.

If we take $T = S[[x]]$, where

$$S = \left(\prod_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}\right) \Big/ \left(\bigoplus_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}\right),$$

then, by Example 2.5 of [**15**], T is an APP-ring, but T is neither PP nor principally quasi-Baer. Thus R is neither PP nor principally quasi-Baer.

(3) The following example (see [**9**]) shows that left M-APP rings need not be left APP. Let F be a field; let $A = F[s, t]$ be a commutative polynomial ring; and consider the ring $R = A/(st)$. Let $\bar{s} = s + (st)$ and $\bar{t} = t + (st)$ in R. Define an automorphism σ of R by $\sigma(\bar{s}) = \bar{t}$ and $\sigma(\bar{t}) = \bar{s}$. Then $l_R(R\bar{s}) = R\bar{t}$. Clearly this ideal is not right s-unital. Thus R is not a left APP-ring. By Example 2 of [**9**], any non-zero ideal I of R with $\sigma(I) = I$ is essential in R, and so $l_R(I) = 0$. (Note that R is a reduced ring.) Let $M = \mathbb{N} \cup \{0\}$. Define $\phi : M \longrightarrow Aut(R)$ via $\phi(0) = 1$ and $\phi(n) = \sigma^n$ for every $n \in \mathbb{N}$. Therefore R is M-APP.

The following is our main result which gives a necessary and sufficient condition for the skew monoid ring $R * M$ to be a left APP-ring.

THEOREM 2. *Let R be a ring, M an ordered monoid and $\phi : M \longrightarrow Aut(R)$ a monoid homomorphism. Then the following are equivalent:*

(1) *The skew monoid ring $R * M$ is a left APP-ring.*

(2) *R is a left M-APP-ring.*

Proof. (2)$\Longrightarrow$(1). Let $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n$, and let $\beta = b_1h_1 + b_2h_2 + \cdots + b_mh_m \in R * M$ satisfy $\alpha(R * M)\beta = 0$. Without loss of generality, we assume that $g_i < g_j$ and $h_i < h_j$ if $i < j$. Suppose that $c_1, c_2, \ldots, c_n \in R$ are such that $a_i = c_i^{g_i}$ for $i = 1, 2, \ldots, n$. We will show that $c_i \in l_R(\sum_{g\in M} Rb_j^g)$ for $i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$ by induction on n.

For any $c \in R$ and any $g \in M$, from $\alpha(R * M)\beta = 0$ it follows that

$$(a_1g_1 + a_2g_2 + \cdots + a_ng_n)(cg)(b_1h_1 + b_2h_2 + \cdots + b_mh_m) = 0.$$

Considering the coefficient of the largest element g_ngh_m in above, we obtain $a_nc^{g_n}b_m^{g_ng} = 0$. This implies that $(c_ncb_m^g)^{g_n} = 0$. Thus $c_ncb_m^g = 0$, since $(-)^{g_n}$ is an automorphism of R. So $c_nRb_m^g = 0$ for all $g \in M$, which implies that $c_n(\sum_{g\in M} Rb_m^g) = 0$. That is to say $c_n \in l_R(\sum_{g\in M} Rb_m^g)$. By (2), $l_R(\sum_{g\in M} Rb_m^g)$ is right s-unital. Thus there exists $e \in l_R(\sum_{g\in M} Rb_m^g)$ such that $c_n = c_ne$. Now for every $c \in R$ and every $g \in M$, we have

$$\begin{aligned} 0 &= (a_1g_1 + a_2g_2 + \cdots + a_ng_n)(ecg)(b_1h_1 + b_2h_2 + \cdots + b_mh_m) \\ &= a_1e^{g_1}c^{g_1}b_1^{g_1g}g_1gh_1 + \cdots + a_{n-1}e^{g_{n-1}}c^{g_{n-1}}b_m^{g_{n-1}g}g_{n-1}gh_m \\ &\quad + a_ne^{g_n}c^{g_n}b_{m-1}^{g_ng}g_ngh_{m-1} + a_ne^{g_n}c^{g_n}b_m^{g_ng}g_ngh_m. \end{aligned}$$

Since $a_ne^{g_n}c^{g_n}b_m^{g_ng} = a_n(ecb_m^g)^{g_n} = 0$, $a_{n-1}e^{g_{n-1}}c^{g_{n-1}}b_m^{g_{n-1}g} = a_{n-1}(ecb_m^g)^{g_{n-1}} = 0$, considering the coefficient of the largest element g_ngh_{m-1} in above, we have $a_ne^{g_n}c^{g_n}b_{m-1}^{g_ng} = 0$. Thus

$$\left(c_ncb_{m-1}^g\right)^{g_n} = \left(c_necb_{m-1}^g\right)^{g_n} = c_n^{g_n}e^{g_n}c^{g_n}b_{m-1}^{g_ng} = a_ne^{g_n}c^{g_n}b_{m-1}^{g_ng} = 0,$$

which implies that $c_nRb_{m-1}^g = 0$ for every $g \in M$. Hence $c_n \in l_R(\sum_{g\in M} Rb_m^g) \cap l_R(\sum_{g\in M} Rb_{m-1}^g)$. Noting that $l_R(\sum_{g\in M} Rb_m^g)$ and $l_R(\sum_{g\in M} Rb_{m-1}^g)$ are right s-unital ideals of R, so is $l_R(\sum_{g\in M} Rb_m^g) \cap l_R(\sum_{g\in M} Rb_{m-1}^g)$. Thus there exists $f \in l_R(\sum_{g\in M} Rb_m^g) \cap l_R(\sum_{g\in M} Rb_{m-1}^g)$ such that $c_n = c_nf$. Now for any $g \in M$ and any $c \in R$ we have $(a_1g_1 + a_2g_2 + \cdots + a_ng_n)(fcg)(b_1h_1 + b_2h_2 + \cdots + b_mh_m) = 0$. Continuing this process, we obtain

$$c_n \in \cap_{j=1}^m l_R\left(\sum_{g\in M} Rb_j^g\right).$$

Thus for any $c \in R$ and any $g \in M$, $(a_ng_n)(cg)(b_1h_1 + b_2h_2 + \cdots + b_mh_m) = a_nc^{g_n}b_1^{g_ng}g_ngh_1 + \cdots + a_nc^{g_n}b_m^{g_ng}g_ngh_m = (c_ncb_1^g)^{g_n}g_ngh_1 + \cdots + (c_ncb_m^g)^{g_n}g_ngh_m = 0$. So we have

$$(a_1g_1 + a_2g_2 + \cdots + a_{n-1}g_{n-1})(R * M)(b_1h_1 + b_2h_2 + \cdots + b_mh_m) = 0.$$

Now using induction on n, we obtain that $c_i \in \cap_{j=1}^m l_R(\sum_{g\in M} Rb_j^g)$ for all i. Since $l_R(\sum_{g\in M} Rb_1^g), \ldots, l_R(\sum_{g\in M} Rb_m^g)$ are right s-unital ideals, it is clear that

$\cap_{j=1}^{m} l_R(\sum_{g\in M} Rb_j^g)$ is right s-unital. Thus there exists $e \in \cap_{j=1}^{m} l_R(\sum_{g\in M} Rb_j^g)$ such that $c_i = c_i e$, $i = 1, 2, \ldots, n$. Now we have

$$\begin{aligned}\alpha(e\eta) &= (a_1g_1 + a_2g_2 + \cdots + a_ng_n)(e\eta)\\ &= \sum_{i=1}^{n} a_i e^{g_i} g_i = \sum_{i=1}^{n} (c_i e)^{g_i} g_i = \sum_{i=1}^{n} c_i^{g_i} g_i = \sum_{i=1}^{n} a_i g_i\\ &= \alpha.\end{aligned}$$

For every $r \in R$ and every $g \in M$, $(e\eta)(rg)\beta = \sum_{j=1}^{m} erb_j^g(gh_j) = 0$. Thus $e\eta \in l_{R*M}((R * M)\beta)$. This shows that $R * M$ is a left APP-ring.

(1)$\Longrightarrow$(2). Suppose that $R * M$ is a left APP-ring. Let $b \in R$ and $a \in l_R(\sum_{g\in M} Rb^g)$. Then $(a\eta)(R * M)(b\eta) = 0$. Thus there exists $c_0\eta + c_1g_1 + \cdots + c_ng_n \in R * M$ such that $a\eta = (a\eta)(c_0\eta + c_1g_1 + \cdots + c_ng_n)$ and $(c_0\eta + c_1g_1 + \cdots + c_ng_n)(R * M)(b\eta) = 0$, where $c_i \in R$ and $\eta, g_1, \ldots, g_n$ are distinct elements of M. It is easy to see that $a = ac_0$. Note that M is cancellative. For any $r \in R$ and any $g \in M$, from $(c_0\eta + c_1g_1 + \cdots + c_ng_n)(rg)(b\eta) = 0$ it follows that $c_0rb^g g = 0$, which implies that $c_0rb^g = 0$. Thus $c_0(\sum_{g\in M} Rb^g) = 0$. This shows that R is a left M-APP-ring. □

COROLLARY 3. *Let R be a ring and M an ordered monoid. Then the monoid ring $R[M]$ is left APP if and only if R is left APP.*

COROLLARY 4. *Let R be a ring and σ a ring automorphism of R. Then the ring $R[x;\sigma]$ (respectively $R[x, x^{-1};\sigma]$) is left APP if and only if the left annihilator of $\sum_{i=0}^{\infty} R\sigma^i(b)$ (respectively $\sum_{i=-\infty}^{\infty} R\sigma^i(b)$) is right s-unital for every $b \in R$.*

Proof. Define a homomorphism $\phi : \mathbb{N} \cup \{0\} \longrightarrow Aut(R)$ ($\phi : \mathbb{Z} \longrightarrow Aut(R)$) of monoids via $\phi(i) = \sigma^i$. Then the result follows from Theorem 2. □

3. Left principally quasi-Baer rings and quasi-Baer rings. Let R be a ring, M a monoid and $\phi : M \longrightarrow Aut(R)$ a monoid homomorphism; R is called a left M-principally quasi-Baer ring if for any $a \in R$, the left annihilator of $\sum_{g\in M} Ra^g$ is generated by an idempotent. For the condition that M is a group, left M-principally quasi-Baer rings were considered by Y. Hirano in [**9**]. Note that by Remark 1(3), left M-principally quasi-Baer rings need not be left principally quasi-Baer.

There are a lot of results concerning quasi-Baerness and left principal quasi-Baerness of polynomial extensions of a ring. G. F. Birkenmeier, J. Y. Kim and J. K. Park showed in [**4**], Theorem 1.8, that R is quasi-Baer if and only if $R[X]$ is quasi-Baer if and only if $R[[X]]$ is quasi-Baer if and only if $R[x, x^{-1}]$ is quasi-Baer if and only if $R[[x, x^{-1}]]$ is quasi-Baer, where X is an arbitrary non-empty set of not necessarily commuting indeterminates. Furthermore, it was shown in [**4**], Theorem 1.2, that if R is quasi-Baer, then so are $R[x;\sigma]$, $R[[x;\sigma]]$, $R[x, x^{-1};\sigma]$ and $R[[x, x^{-1};\sigma]]$. It was proved in [**3**], Theorem 2.1, that a ring R is left principally quasi-Baer if and only if $R[x]$ is left principally quasi-Baer. C. Y. Hong, N. K. Kim and T. K. Kwak showed in [**11**], Corollaries 12, 15 and 22, that if σ is a rigid endomorphism of R, then R is a quasi-Baer (respectively left principally quasi-Baer) ring if and only if $R[x;\sigma,\delta]$ is a quasi-Baer (respectively left principally quasi-Baer) ring if and only if $R[[x;\sigma]]$ is a quasi-Baer ring. If R is a ring and $(S, \leq)$ a strictly totally ordered monoid which satisfies the condition that $0 \leq s$ for every $s \in S$, then it is shown in [**13**] that R is a quasi-Baer ring if and only if the ring $[[R^{S,\leq}]]$ of generalised power series over R is a quasi-Baer ring. If M

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is an ordered monoid, then it is proved in [**9**], Theorem 1, that $R[M]$ is quasi-Baer if and only if R is quasi-Baer. This result has been generalised by G. F. Birkenmeier and J. K. Park in [**6**], Theorem 1.2, by showing that if M is a u.p.-monoid, then $R[M]$ is quasi-Baer (respectively left principally quasi-Baer) if and only if R is quasi-Baer (respectively left principally quasi-Baer). For skew monoid rings it was proved in [**9**], Theorem 2, that if R is a ring and M an ordered group acting on R, then $R * M$ is a left principally quasi-Baer ring if and only if R is a left M-principally quasi-Baer ring. It was also noted in [**9**], Remark, that if M is an ordered monoid and if there exists a monoid homomorphism $\phi : M \longrightarrow Aut(R)$ such that $Im(\phi)$ is a group, then the skew monoid ring $R * M$ is a left principally quasi-Baer ring if and only if R is a left $Im(\phi)$-principally quasi-Baer ring. Here we have the following result.

THEOREM 5. *Let R be a ring, M an ordered monoid and $\phi : M \longrightarrow Aut(R)$ a monoid homomorphism. Then the following are equivalent:*

(1) *The skew monoid ring $R * M$ is a left principally quasi-Baer ring.*

(2) *R is a left M-principally quasi-Baer ring.*

Proof. (2)$\Longrightarrow$(1). Suppose that $b_1h_1 + b_2h_2 + \cdots + b_mh_m$ belongs to $R * M$, and consider the principal left ideal $I = (R * M)(b_1h_1 + b_2h_2 + \cdots + b_mh_m)$ of $R * M$. Without loss of generality, we assume that $h_1 < h_2 < \cdots < h_m$. Let J denote the set of all coefficients of elements of I. Then it is easy to see that

$$J = \sum_{g\in M} Rb_1^g + \sum_{g\in M} Rb_2^g + \cdots + \sum_{g\in M} Rb_m^g.$$

By point (2), there exists an idempotent $e_j \in R$ such that $l_R(\sum_{g\in M} Rb_j^g) = Re_j$, $j = 1, 2, \ldots, m$. Let $e = e_1e_2 \ldots e_m$. Then $l_R(\sum_{g\in M} Rb_1^g + \sum_{g\in M} Rb_2^g + \cdots + \sum_{g\in M} Rb_m^g) = \cap_{j=1}^m l_R(\sum_{g\in M} Rb_j^g) = Re$. Clearly $e\eta \in l_{R*M}(I)$. Suppose $a_1g_1 + a_2g_2 + \cdots + a_ng_n \in l_{R*M}(I)$. Then $(a_1g_1 + a_2g_2 + \cdots + a_ng_n)(R * M)(b_1h_1 + b_2h_2 + \cdots + b_mh_m) = 0$. Without loss of generality, we assume that $g_1 < g_2 < \cdots < g_n$. Suppose that $c_1, c_2, \ldots, c_n \in R$ are such that $a_i = c_i^{g_i}$ for $i = 1, 2, \ldots, n$. Then, by analogy with the proof of Theorem 2, we have $c_i \in l_R(\sum_{g\in M} Rb_j^g)$ for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. Thus $c_i \in l_R(J)$, and so $c_i = c_ie$, $i = 1, 2, \ldots, n$. Now

$$\left(\sum_{i=1}^n a_ig_i\right)(e\eta) = \sum_{i=1}^n a_ie^{g_i}g_i = \sum_{i=1}^n (c_ie)^{g_i}g_i = \sum_{i=1}^n c_i^{g_i}g_i = \sum_{i=1}^n a_ig_i,$$

which implies $\sum_{i=1}^n a_ig_i \in (R * M)(e\eta)$. Thus $l_{R*M}(I) \leq (R * M)(e\eta)$. Therefore $l_{R*M}(I) = (R * M)(e\eta)$, and so $R * M$ is a principally quasi-Baer ring.

(1)$\Longrightarrow$(2). Suppose that the skew monoid ring $R * M$ is a left principally quasi-Baer ring. Let $b \in R$. We consider the left annihilator $l_R(\sum_{g\in M} Rb^g)$. By Hypothesis (1), there exists an idempotent $\alpha \in R * M$ such that $l_{R*M}((R * M)(b\eta)) = (R * M)\alpha$. We may write $\alpha = e_0\eta + e_1g_1 + \cdots + e_ng_n \in R * M$, where $e_i \in R$ and $\eta, g_1, \ldots, g_n$ are distinct elements of M. Note that the monoid M is cancenllative. For any $r \in R$ and any $g \in M$, from $0 = (e_0\eta + e_1g_1 + \cdots + e_ng_n)(rg)(b\eta) = e_0rb^gg + e_1r^{g_1}b^{g_1g}g_1g + \cdots + e_nr^{g_n}b^{g_ng}g_ng$ it follows $e_0rb^gg = 0$, and so $e_0rb^g = 0$. Since g is an arbitrary element of M, we have $e_0(\sum_{g\in M} Rb^g) = 0$. Thus $Re_0 \subseteq l_R(\sum_{g\in M} Rb^g)$. To prove the converse inclusion, let $a \in l_R(\sum_{g\in M} Rb^g)$. Then for any $r \in R$ and any $g \in M$, $(a\eta)(rg)(b\eta) = arb^gg = 0$. Thus $(a\eta)(R * M)(b\eta) = 0$, and so $(a\eta) = (a\eta)\alpha = (a\eta)(e_0\eta + e_1g_1 + \cdots + e_ng_n) = ae_0\eta + ae_1g_1 + \cdots + ae_ng_n$. Considering the coefficient of η we obtain $a = ae_0$.

Hence $l_R(\sum_{g\in M} Rb^g) \subseteq Re_0$. In particular e_0 is an idempotent of R. Hence R is a left M-principally quasi-Baer ring. □

Let M be an ordered monoid and $\phi : M \longrightarrow Aut(R)$ a monoid homomorphism. R is called a left M-quasi-Baer ring if for any subset S of R, the left annihilator of left ideal $\sum_{b\in S}\sum_{g\in M} Rb^g$ of R is generated by an idempotent of R. For the condition that G is a group, left G-quasi-Baer rings was considered by Y. Hirano in [**9**]. Note that by Remark 1(3), left M-quasi-Baer rings need not be left quasi-Baer. By analogy with the proof of Theorem 5 we have the following result.

Theorem 6. *Let R be a ring, M an ordered monoid and $\phi : M \longrightarrow Aut(R)$ a monoid homomorphism. Then the following are equivalent:*

(1) *The skew monoid ring $R * M$ is a quasi-Baer ring.*
(2) *R is a left M-quasi-Baer ring.*

Proof. Let I be a left ideal of $R * M$. Denote by I_0 the set of all coefficients of elements of I. Let

$$J = \sum_{b\in I_0}\sum_{g\in M} Rb^g.$$

If R is left M-quasi-Baer, then there exists an idempotent $e \in R$ such that $l_R(J) = Re$. Now by analogy with the proof of Theorem 5 we can complete the proof. □

Corollary 7. *Let R be a ring and σ a ring automorphism of R. Then*

(i) *the ring $R[x;\sigma]$ (respectively $R[x, x^{-1};\sigma]$) is left principally quasi-Baer if and only if the left annihilator of $\sum_{i=0}^{\infty} R\sigma^i(b)$ (respectively $\sum_{i=-\infty}^{\infty} R\sigma^i(b)$) is generated by an idempotent for every $b \in R$;*
(ii) *the ring $R[x;\sigma]$ (respectively $R[x, x^{-1};\sigma]$) is quasi-Baer if and only if the left annihilator of $\sum_{b\in S}\sum_{i=0}^{\infty} R\sigma^i(b)$ (respectively $\sum_{b\in S}\sum_{i=-\infty}^{\infty} R\sigma^i(b)$) is generated by an idempotent for any subset S of R.*

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