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A Determinantal Inequality Involving Partial Traces

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Abstract. Let $\mathbf{A}$ be a density matrix in $\mathbb{M}_m \otimes \mathbb{M}_n$. Audenaert [J. Math. Phys. 48(2007) 083507] proved an inequality for Schatten p-norms:

$$1 + \|\mathbf{A}\|_p \geq \|\operatorname{Tr}_1 \mathbf{A}\|_p + \|\operatorname{Tr}_2 \mathbf{A}\|_p,$$

where Tr_1 and Tr_2 stand for the first and second partial trace, respectively. As an analogue of his result, we prove a determinantal inequality

$$1 + \det \mathbf{A} \geq \det(\operatorname{Tr}_1 \mathbf{A})^m + \det(\operatorname{Tr}_2 \mathbf{A})^n.$$

1 Introduction

We denote by $\mathbb{M}_n$ the set of $n \times n$ complex matrices. The tensor product $\mathbb{M}_m \otimes \mathbb{M}_n$ is identified with the space $\mathbb{M}_m(\mathbb{M}_n)$, the set of $m \times m$ block matrices with each block in $\mathbb{M}_n$. Each element of $\mathbb{M}_m(\mathbb{M}_n)$ is also regarded as an $mn \times mn$ matrix with numerical entries. By convention, the $n \times n$ identity matrix is denoted by I_n; we use J_n to denote the $n \times n$ matrix with all entries equal to one.

In the sequel, a positive (semidefinite) matrix A is denoted by $A \geq 0$. For two Hermitian matrices A, B of the same size, $A \geq B$ means $A - B \geq 0$.

For any $\mathbf{A} \in \mathbb{M}_m(\mathbb{M}_n)$, we can write $\mathbf{A} = \sum_{i=1}^q X_i \otimes Y_i$ for some positive integer $q \leq m^2$ and some $X_i \in \mathbb{M}_m$, $Y_i \in \mathbb{M}_n$, $i = 1, \dots, q$. We can define two partial traces Tr_1 and Tr_2:

$$\operatorname{Tr}_1 \mathbf{A} = \sum_{i=1}^q (\operatorname{Tr} X_i) Y_i, \qquad \operatorname{Tr}_2 \mathbf{A} = \sum_{i=1}^q (\operatorname{Tr} Y_i) X_i,$$

where Tr stands for the usual trace. In other words, the first partial trace Tr_1 "traces out" the first factor and similarly for the second partial trace Tr_2. Clearly,

$$\operatorname{Tr}(\operatorname{Tr}_1 \mathbf{A})B = \operatorname{Tr}(I_m \otimes B)\mathbf{A}, \quad \text{for any } B \in \mathbb{M}_n;$$
$$\operatorname{Tr}(\operatorname{Tr}_2 \mathbf{A})C = \operatorname{Tr}(C \otimes I_n)\mathbf{A}, \quad \text{for any } C \in \mathbb{M}_m.$$

The actual forms of the partial traces are as follows (see [8, p. 12]):

$$\operatorname{Tr}_1 \mathbf{A} = \sum_{i=1}^m A_{i,i}, \qquad \operatorname{Tr}_2 \mathbf{A} = [\operatorname{Tr} A_{i,j}]_{i,j=1}^m.$$

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A density matrix on a bipartite system (see [8, pp. 4, 53]) is a positive semidefinite matrix in $\mathbb{M}_m \otimes \mathbb{M}_n$ with trace equal to one. Audenaert [1] recently proved an interesting norm inequality.

Theorem 1.1 ([1, Theorem 1]) *Let* $\mathbf{A} \in \mathbb{M}_m(\mathbb{M}_n)$ *be a density matrix. Then*

$$1 + \|\mathbf{A}\|_p \geq \|\operatorname{Tr}_1 \mathbf{A}\|_p + \|\operatorname{Tr}_2 \mathbf{A}\|_p, \tag{1.1}$$

where $\|\cdot\|_p$ *denotes the Schatten* p*-norm.*

Inequality (1.1) was called out to prove the subadditivity of the so-called Tsallis entropies; see [1] for more details. In this paper, as an analogue of (1.1), we prove the following determinantal inequality.

Theorem 1.2 *Let* $\mathbf{A} \in \mathbb{M}_m(\mathbb{M}_n)$ *be a density matrix. Then*

$$1 + \det \mathbf{A} \geq \det(\operatorname{Tr}_1 \mathbf{A})^m + \det(\operatorname{Tr}_2 \mathbf{A})^n. \tag{1.2}$$

2 Auxiliary Results and Proofs

A linear map $\Phi: \mathbb{M}_n \to \mathbb{M}_k$ is positive if it maps positive matrices to positive matrices. A linear map $\Phi: \mathbb{M}_n \to \mathbb{M}_k$ is called m-positive if for $[A_{i,j}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$,

$$[A_{i,j}]_{i,j=1}^m \geq 0 \Longrightarrow \left[\Phi(A_{i,j})\right]_{i,j=1}^m \geq 0, \tag{2.1}$$

and Φ is completely positive if (2.1) is true for any positive integer m.

On the other hand, a linear map $\Phi: \mathbb{M}_n \to \mathbb{M}_k$ is m-copositive if

$$[A_{i,j}]_{i,j=1}^m \geq 0 \Longrightarrow \left[\Phi(A_{j,i})\right]_{i,j=1}^m \geq 0, \tag{2.2}$$

and Φ is completely copositive if (2.2) is true for any positive integer m.

We need the following result.

Proposition 2.1 *The map* $\Phi: \mathbb{M}_n \to \mathbb{M}_n$ *defined by* $\Phi(X) = (\operatorname{Tr} X)I_n - X$ *is completely copositive.*

Proof One may of course use the approach in [7] to prove this. Here we invoke a standard tool by Choi [4]. It suffices to prove that for any positive integer m ,

$$\left[\Phi(E_{j,i})\right]_{i,j=1}^m \geq 0,$$

where $E_{i,j} \in \mathbb{M}_n$ is the matrix with 1 in the (i,j)-entry and 0 elsewhere. But $[\Phi(E_{j,i})]_{i,j=1}^m$ is symmetric, row diagonally dominant with positive diagonal entries, implying

$$\left[\Phi(E_{j,i})\right]_{i,j=1}^m \geq 0. \qquad \blacksquare$$

The reader may easily observe that $\Phi(X) = (\operatorname{Tr} X)I_n - X$ is not 2-positive (see [3]).

In the proof of the next proposition, we only use the fact that $\Phi(X) = (\operatorname{Tr} X)I_n - X$ is 2-copositive. Proposition 2.2, first proved by Ando [2], plays a key role in our derivation of (1.2). We provide a proof here for the convenience of readers. Our proof is slightly more transparent than the original proof by Ando.

Proposition 2.2 *Let* $\mathbf{A} = [A_{i,j}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$ *be positive. Then*

$$(\operatorname{Tr}\mathbf{A})I_m \otimes I_n + \mathbf{A} \geq I_m \otimes (\operatorname{Tr}_1 \mathbf{A}) + (\operatorname{Tr}_2 \mathbf{A}) \otimes I_n.$$

Proof The proof is by induction on m. When $m = 1$, there is nothing to prove. We prove the base case $m = 2$ first. In this case, the required inequality is

$$\begin{pmatrix} (\operatorname{Tr}\mathbf{A})I_n & 0 \\ 0 & (\operatorname{Tr}\mathbf{A})I_n \end{pmatrix} + \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \geq \begin{pmatrix} A_{1,1} + A_{2,2} & 0 \\ 0 & A_{1,1} + A_{2,2} \end{pmatrix} + \begin{pmatrix} (\operatorname{Tr} A_{1,1})I_n & (\operatorname{Tr} A_{1,2})I_n \\ (\operatorname{Tr} A_{2,1})I_n & (\operatorname{Tr} A_{2,2})I_n \end{pmatrix},$$

or equivalently,

$$H := \begin{pmatrix} (\operatorname{Tr} A_{2,2})I_n - A_{2,2} & A_{1,2} - (\operatorname{Tr} A_{1,2})I_n \\ A_{2,1} - \operatorname{Tr} A_{2,1})I_n & (\operatorname{Tr} A_{1,1})I_n - A_{1,1} \end{pmatrix} \geq 0. \tag{2.3}$$

By Proposition 2.1,

$$\begin{pmatrix} (\operatorname{Tr} A_{1,1})I_n - A_{1,1} & (\operatorname{Tr} A_{2,1})I_n - A_{2,1} \\ (\operatorname{Tr} A_{1,2})I_n - A_{1,2} & (\operatorname{Tr} A_{2,2})I_n - A_{2,2} \end{pmatrix} \geq 0,$$

and so

$$H = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} (\operatorname{Tr} A_{1,1})I_n - A_{1,1} & (\operatorname{Tr} A_{2,1})I_n - A_{2,1} \\ (\operatorname{Tr} A_{1,2})I_n - A_{1,2} & (\operatorname{Tr} A_{2,2})I_n - A_{2,2} \end{pmatrix} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \geq 0,$$

confirming (2.3).

Suppose the result is true for $m = k - 1 > 1$. When $m = k$,

$$\begin{aligned}
\Gamma &:= (\operatorname{Tr}\mathbf{A})I_k \otimes I_n + \mathbf{A} - I_k \otimes (\operatorname{Tr}_1 \mathbf{A}) + (\operatorname{Tr}_2 \mathbf{A}) \otimes I_n. \\
&= \Big(\operatorname{Tr}\sum_{i=1}^k A_{i,i}\Big) I_k \otimes I_n + \mathbf{A} - I_k \otimes \Big(\sum_{j=1}^k A_{j,j}\Big) - \big([\operatorname{Tr} A_{i,j}]_{i,j=1}^k\big) \otimes I_n \\
&= \begin{bmatrix} \sum_{i=1}^{k-1}(\operatorname{Tr} A_{i,i})I_n & & & \\ & \ddots & & \\ & & \sum_{i=1}^{k-1}(\operatorname{Tr} A_{i,i})I_n & \\ & & & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} (\operatorname{Tr} A_{k,k})I_n & & & \\ & \ddots & & \\ & & (\operatorname{Tr} A_{k,k})I_n & \\ & & & \sum_{i=1}^{k}(\operatorname{Tr} A_{i,i})I_n \end{bmatrix} \\
&\quad + \begin{bmatrix} A_{1,1} & \cdots & A_{1,k-1} & 0 \\ \vdots & & \vdots & \vdots \\ A_{k-1,1} & \cdots & A_{k-1,k-1} & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 & A_{1,k} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & A_{k-1,k} \\ A_{k,1} & \cdots & A_{k,k-1} & A_{k,k} \end{bmatrix}
\end{aligned}$$

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$$
-\begin{bmatrix} \sum_{i=1}^{k-1} A_{i,i} & & & \\ & \ddots & & \\ & & \sum_{i=1}^{k-1} A_{i,i} & \\ & & & 0 \end{bmatrix} - \begin{bmatrix} A_{k,k} & & & \\ & \ddots & & \\ & & A_{k,k} & \\ & & & \sum_{i=1}^{k} A_{i,i} \end{bmatrix}
$$

$$
-\begin{bmatrix} (\operatorname{Tr} A_{1,1})I_n & \cdots & (\operatorname{Tr} A_{1,k-1})I_n & 0 \\ \vdots & & \vdots & \vdots \\ (\operatorname{Tr} A_{k-1,1})I_n & \cdots & (\operatorname{Tr} A_{k-1,k-1})I_n & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix}
$$

$$
-\begin{bmatrix} 0 & \cdots & 0 & (\operatorname{Tr} A_{1,k}))I_n \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & (\operatorname{Tr} A_{k-1,k})I_n \\ (\operatorname{Tr} A_{k,1})I_n & \cdots & (\operatorname{Tr} A_{k,k-1})I_n & (\operatorname{Tr} A_{k,k})I_n \end{bmatrix}.
$$

After some rearrangement, we have $\Gamma = \mathbf{P} + \mathbf{Q}$, where

$$
\mathbf{P} := \begin{bmatrix} \sum_{i=1}^{k-1} (\operatorname{Tr} A_{i,i})I_n & & & \\ & \ddots & & \\ & & \sum_{i=1}^{k-1} (\operatorname{Tr} A_{i,i})I_n & \\ & & & 0 \end{bmatrix} + \begin{bmatrix} A_{1,1} & \cdots & A_{1,k-1} & 0 \\ \vdots & & \vdots & \vdots \\ A_{k-1,1} & \cdots & A_{k-1,k-1} & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix}
$$

$$
-\begin{bmatrix} \sum_{i=1}^{k-1} A_{i,i} & & & \\ & \ddots & & \\ & & \sum_{i=1}^{k-1} A_{i,i} & \\ & & & 0 \end{bmatrix} - \begin{bmatrix} (\operatorname{Tr} A_{1,1})I_n & \cdots & (\operatorname{Tr} A_{1,k-1})I_n & 0 \\ \vdots & & \vdots & \vdots \\ (\operatorname{Tr} A_{k-1,1})I_n & \cdots & (\operatorname{Tr} A_{k-1,k-1})I_n & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix}
$$

and

$$
\mathbf{Q} := \begin{bmatrix} (\operatorname{Tr} A_{k,k})I_n & & & \\ & \ddots & & \\ & & (\operatorname{Tr} A_{k,k})I_n & \\ & & & \sum_{i=1}^{k} (\operatorname{Tr} A_{i,i})I_n \end{bmatrix}
$$

$$
+\begin{bmatrix} 0 & \cdots & 0 & A_{1,k} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & A_{k-1,k} \\ A_{k,1} & \cdots & A_{k,k-1} & A_{k,k} \end{bmatrix} - \begin{bmatrix} A_{k,k} & & & \\ & \ddots & & \\ & & A_{k,k} & \\ & & & \sum_{i=1}^{k} A_{i,i} \end{bmatrix}
$$

$$
-\begin{bmatrix} 0 & \cdots & 0 & (\operatorname{Tr} A_{1,k}))I_n \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & (\operatorname{Tr} A_{k-1,k})I_n \\ (\operatorname{Tr} A_{k,1})I_n & \cdots & (\operatorname{Tr} A_{k,k-1})I_n & (\operatorname{Tr} A_{k,k})I_n \end{bmatrix}
$$

$$
= \begin{bmatrix} (\operatorname{Tr} A_{k,k})I_n - A_{k,k} & & & A_{1,k} - (\operatorname{Tr} A_{1,k}))I_n \\ & \ddots & & \vdots \\ & & (\operatorname{Tr} A_{k,k})I_n - A_{k,k} & A_{k-1,k} - (\operatorname{Tr} A_{k-1,k})I_n \\ (\operatorname{Tr} A_{k,1})I_n & \cdots & (\operatorname{Tr} A_{k,k-1})I_n & \sum_{i=1}^{k-1} \Big((\operatorname{Tr} A_{i,i})I_n - A_{i,i} \Big) \end{bmatrix}.
$$

Now by induction hypothesis, $\mathbf{P} \geq 0$. It remains to show that $\mathbf{Q} \geq 0$.

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It is easy to see that $\mathbf{Q}$ can be written as a sum of $k-1$ matrices with each summand $*$-congruent to

$$H_i := \begin{bmatrix} (\operatorname{Tr} A_{k,k})I_n - A_{k,k} & A_{i,k} - (\operatorname{Tr} A_{i,k}))I_n \\ A_{k,i} - (\operatorname{Tr} A_{k,i})I_n & (\operatorname{Tr} A_{i,i})I_n - A_{i,i} \end{bmatrix}, \quad i = 1, \dots, k-1.$$

Just as in the proof of the base case, we infer that $H_i \geq 0$ for all $i = 1, \dots, k-1$. Therefore, $\mathbf{Q} \geq 0$, thus the proof of induction step is complete. ∎

The next corollary is known as a Cauchy–Khinchin matrix inequality in the literature (see [9, Theorem 1]). Here we present a simple proof using Proposition 2.2.

Corollary 2.3 *Let $X = (x_{ij})$ be a real $m \times n$ matrix. Then*

$$\Big(\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}\Big)^2 + mn\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}^2 \geq m\sum_{i=1}^{m}\Big(\sum_{j=1}^{n} x_{ij}\Big)^2 + n\sum_{j=1}^{n}\Big(\sum_{i=1}^{m} x_{ij}\Big)^2.$$

Proof Let $\operatorname{vec} X = [x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, \dots, x_{m1}, \dots, x_{mn}]^T$ be a vectorization of X. Then a simple calculation gives

$$(\operatorname{vec} X)^T (J_m \otimes J_n) \operatorname{vec} X = (\operatorname{vec} X)^T J_{mn} \operatorname{vec} X = \Big(\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}\Big)^2,$$

$$(\operatorname{vec} X)^T (I_m \otimes I_n) \operatorname{vec} X = (\operatorname{vec} X)^T \operatorname{vec} X = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}^2,$$

$$(\operatorname{vec} X)^T (I_m \otimes J_n) \operatorname{vec} X = \sum_{i=1}^{m}\Big(\sum_{j=1}^{n} x_{ij}\Big)^2,$$

$$(\operatorname{vec} X)^T (J_m \otimes I_n) \operatorname{vec} X = \sum_{j=1}^{n}\Big(\sum_{i=1}^{m} x_{ij}\Big)^2.$$

Thus the desired inequality is equivalent to

$$(\operatorname{vec} X)^T (J_m \otimes J_n + mnI_m \otimes I_n - mI_m \otimes J_n - nJ_m \otimes I_n) \operatorname{vec} X \geq 0. \tag{2.4}$$

Setting $\mathbf{A} = J_m \otimes J_n$ in Proposition 2.2 yields

$$J_m \otimes J_n + mnI_m \otimes I_n - mI_m \otimes J_n - nJ_m \otimes I_n \geq 0,$$

and so (2.4) follows. ∎

We require one more result for our purpose.

Proposition 2.4 *Let $X, Y, W, Z \in \mathbb{M}_\ell$ be positive. If $X + Y \geq W + Z$, $X \geq W$, and $X \geq Z$, then*

$$\det X + \det Y \geq \det W + \det Z. \tag{2.5}$$

Proof Without loss of generality, assume that $X = I_\ell$ (for we can assume first X is invertible by a standard continuity argument, then pre-post multiply all matrices by $X^{-1/2}$). After this, by a unitary similarity, we can further assume that $Y = D$, a

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diagonal matrix. Thus, we need to show that if $I_\ell + D \geq W + Z$ and $I_\ell \geq W, Z \geq 0$, then

$$1 + \det D \geq \det W + \det Z. \tag{2.6}$$

By the Hadamard inequality (see [5, Theorem 7.8.1]), (2.6) would follow from

$$1 + \det D \geq \det(\operatorname{diag}(W)) + \det(\operatorname{diag}(Z)), \tag{2.7}$$

where $\operatorname{diag}(\cdot)$ means the diagonal part of a matrix.

Let d_i, w_i, z_i, $i = 1, \ldots, \ell$, be the diagonal entries of D, W, and Z, respectively. Then $d_i \geq 0$, $0 \leq w_i, z_i \leq 1$ for $i = 1, \ldots, \ell$. We will prove (2.7) by induction. The base case is clear. Assume that (2.7) is true for $\ell = k - 1 \geq 1$. When $\ell = k$, there are two cases.

Case I: If $1 \geq \prod_{j=1}^{k-1} w_j + \prod_{j=1}^{k-1} z_j$, then

$$1 + \prod_{j=1}^{k} d_j \geq 1 \geq \prod_{j=1}^{k-1} w_j + \prod_{j=1}^{k-1} z_j \geq \prod_{j=1}^{k} w_j + \prod_{j=1}^{k} z_j.$$

Case II: If $\prod_{j=1}^{k-1} w_j + \prod_{j=1}^{k-1} z_j > 1$, then

$$\begin{aligned}
1 + \prod_{j=1}^{k} d_j &\geq 1 + \Big(\prod_{j=1}^{k-1} w_j + \prod_{j=1}^{k-1} z_j - 1\Big) d_k \geq 1 + \Big(\prod_{j=1}^{k-1} w_j + \prod_{j=1}^{k-1} z_j - 1\Big)(w_k + z_k - 1) \\
&= 1 + \prod_{j=1}^{k} w_j + \prod_{j=1}^{k} z_j + (w_k - 1)\prod_{j=1}^{k-1} z_j + (z_k - 1)\prod_{j=1}^{k-1} w_j - (w_k + z_k - 1) \\
&\geq \prod_{j=1}^{k} w_j + \prod_{j=1}^{k} z_j + \Big(1 - \prod_{j=1}^{k-1} z_j\Big)(1 - w_k) + \Big(1 - \prod_{j=1}^{k-1} w_j\Big)(1 - z_k) \\
&\geq \prod_{j=1}^{k} w_j + \prod_{j=1}^{k} z_j.
\end{aligned}$$

Thus, (2.7) holds for $\ell = k$, so the proof of the induction step is complete. ■

We are now in a position to present the proof of Theorem 1.2.

Proof of Theorem 1.2 Let $X = (\operatorname{Tr} \mathbf{A}) I_m \otimes I_n$, $Y = \mathbf{A}$, $W = I_m \otimes (\operatorname{Tr}_1 \mathbf{A})$, $Z = (\operatorname{Tr}_2 \mathbf{A}) \otimes I_n$, respectively. Clearly,

$$(\operatorname{Tr} \mathbf{A}) I_n \geq \operatorname{Tr}_1 \mathbf{A} \geq 0 \quad \text{and} \quad (\operatorname{Tr} \mathbf{A}) I_m \geq \operatorname{Tr}_2 \mathbf{A} \geq 0$$

imply that $X \geq W \geq 0$ and $X \geq Z \geq 0$. Moreover, by Proposition 2.2, $X + Y \geq W + Z$. That is, the conditions in Proposition 2.4 are met. Therefore,

$$\begin{aligned}
(\operatorname{Tr} \mathbf{A})^{mn} + \det \mathbf{A} &\geq \det\big(I_m \otimes (\operatorname{Tr}_1 \mathbf{A})\big) + \det\big((\operatorname{Tr}_2 \mathbf{A}) \otimes I_n\big) \\
&= \det(\operatorname{Tr}_1 \mathbf{A})^m + \det(\operatorname{Tr}_2 \mathbf{A})^n.
\end{aligned}$$

Taking into account that $\mathbf{A}$ is a density matrix, the desired result (1.2) follows. ■

Remark 2.5 In [6, Lemma 2.5], the author proved (2.5) under a stronger assumption: $X + Y \geq W + Z$, $X \geq W \geq Y \geq 0$ and $X \geq Z \geq Y \geq 0$. However, from the present proof of Theorem 1.2, we see that [6, Lemma 2.5] could not be directly applied here.

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