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THE MAXIMAL IDEAL SPACE OF SUBALGEBRAS OF THE DISK ALGEBRA

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1. **Introduction.** Let X be a compact Hausdorff space and $C(X)$ the complex-valued continuous functions on X. We say A is a *function algebra on* X if A is a point separating, uniformly closed subalgebra of $C(X)$ containing the constant functions. Equipped with the *sup-norm* $\|f\|=\sup\{|f(x)|:x\in X\}$ for $f\in A$, A is a Banach algebra. Let M_A denote the *maximal ideal space*.

Let D be the closed unit disk in $\mathbf{C}$ and let U be the open unit disk. We call $A(D)=\{f\in C(D):f$ is analytic on $U\}$ the *disk algebra*. Let T be the unit circle and set $C^1(T)=\{f\in C(T):f'(t)\in C(T)\}$.

In this paper we discuss conditions on a function algebra A on D contained in $A(D)$ which imply that $M_A=D$. Our main result is the following.

THEOREM 1. *Let A be a function algebra on D such that $A\subset A(D)$. Suppose there is $f\in A$ such that $f(t)\in C^1(T)$ and $Q_f=\{t\in T:f'(t)=0\}$ is countable. Then $M_A=D$.*

The following closely related result is due to Bjork ([2], Theorem 2.1).

THEOREM (Bjork). *Let A be a function algebra on D such that $A\subset A(D)$. Suppose there is a set $A_0\subset A$ such that $A_0|_T\subset C^1(T)$ and A_0 is uniformly dense in A. Then $M_A=D$.*

The hypothesis in Bjork's result that A_0 is uniformly dense in A can be replaced by the hypothesis that A_0 separates points on D. To see this, let $[A_0]$ be the smallest function algebra on D containing A_0 where we suppose now that A_0 separates points on D. By a result of Bjork ([2], Lemma 2.3) $[A_0]$ has a regular peak point $\alpha\in T$. (We say that $\alpha\in T$ is a *regular peak point* for $[A_0]$ if there is $f\in[A_0]$ with $f\in C^1(T)$ such that $f'(\alpha)\neq 0$, $\{\alpha\}=\{t\in T:f(t)=f(\alpha)\}$, and $f(\alpha)$ belongs to the boundary of the unbounded component of $\mathbf{C}\backslash f(T)$. But then α is also a regular peak point for A. This is precisely Bjork's condition for showing that $M_A=D$. (See [2]; p. 47.)

Hence, we may state the following more general result which is useful in applications. (See example 1.)

THEOREM 2. *Let A be a function algebra on D such that $A\subset A(D)$. Suppose there is a set $A_0\subset A$ such that A_0 separates points on D and $A_0|_T\subset C^1(T)$. Then $M_A=D$.*

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In §3 we give an example which shows that Theorem 1 does not contain Theorem 2. Also, we give an example which shows that the countability of Q_f in Theorem 1 can not be replaced by the condition that Q_f have measure zero in T. In §4 we give an application of Theorem 1.

In proving Theorem 1 we apply results of Gamelin [3] in the theory of function algebras on an arc. See Stout [9] for an exposition of this theory. We also use results of Bjork [2] on the structure of the maximal ideal space of a function algebra.

2. **Main result.** If A is a function algebra, let S_A be the *Shilov boundary*. Let $\hat{f}$ stand for the *Gelfand transform* of $f \in A$ and give M_A the Gelfand (weak-star) topology. If $f \in A$ and $z \in \mathbf{C}$, let $\pi_f^{-1}(z)=\{\Phi \in M_A : \hat{f}(\Phi)=z\}$ and let $\#\pi_f^{-1}(z)$ denote the cardinality of $\pi_f^{-1}(z)$.

LEMMA 1 ([1] p. 240). *Let A be a function algebra on X and let $f \in A$. Let Γ be a closed Jordan curve in $\mathbf{C}$ with interior V. Suppose Γ contains an open subarc J such that $\#\pi_f^{-1}(z) \leq n$ for all $z \in J$ and that $\pi_f^{-1}(V) \subset M_A \backslash S_A$. Then $\#\pi_f^{-1}(z) \leq n$ for all $z \in V$.*

Let A be a function algebra on X and suppose K is a compact subset of M_A. We set $\mathrm{Hull}_A(K)=\{\Phi \in M_A : |f(\Phi)| \leq \|f\|_K$ for all $f \in A\}$ and let $A \mid K$ denote the function algebra on K which is generated by the restriction to K of functions in A. Then $M_{A|K}=\mathrm{Hull}_A(K)$. If $V \in \mathbf{C}$, we let ∂V be the topological boundary of V.

Proof of Theorem 1. Let $F \subsetneqq T$ be compact. Let I be a proper closed subinterval of T containing F. Since there is $f \in A$ with $f \in C^1(T)$ and Q_f countable, it follows by [3], Theorem 5 that $A \mid I=C(I)$. Hence, $A \mid F=C(F)$. In particular, $S_A=T$.

Let $\Delta=M_A \backslash D$ and assume $\Delta \neq \varnothing$. We show this leads to a contradiction. Let $b\Delta$ be the topological boundary of Δ in M_A. By [2], Theorem 1.2 we have $\Delta \subset \mathrm{Hull}_A(b\Delta \cap T)$. If $b\Delta \cap T \neq T$, then $A \mid (b\Delta \cap T)=C(b\Delta \cap T)$. This implies that $\Delta \not\subset \mathrm{Hull}_A(b\Delta \cap T)=b\Delta \cap T$. Hence, $b\Delta \cap T=T$.

By [9], Lemma 30.29 there is a compact, totally disconnected set $J \subset f(T)$ such that the following conditions hold.

(i) At each point of $f(T)\backslash J$, $f(T)$ has the structure of an open arc.
(ii) If $K \subset f(T)\backslash J$ is compact, then f maps $f^{-1}(K)$ in a finite to one way onto K.

Let the bounded components of $\mathbf{C}\backslash f(T)$ be denoted by V_k for $k=0, 1, 2, \ldots$ and let V_∞ be the unbounded component. Then ∂V_∞ is not simply connected since $f(U)$ is contained in the polynomial hull of ∂V_∞. Consequently, ∂V_∞ is not totally disconnected ([7], Theorem 14.3, p. 123), and so $\partial V_\infty \not\subset J$.

Suppose $a_0 \in \partial V_\infty \backslash J$. By (ii) there are $t_1, \ldots, t_n \in T$ satisfying $f(t_i)=a_0$. Using (i) and (iii) we can find an open arc L' passing through a_0 which is contained in $f(T)$ and a subarc $L \subset L'$ with the following properties: L contains a_0 and the closure of L in $\mathbf{C}$ is contained in L', L is relatively open in $f(T)$ (that is, there is a connected open set Ω in $\mathbf{C}$ such that $\Omega \cap f(T)=L$), and there are pairwise disjoint open intervals I_i about t_i for $i=1, \ldots, n$ such that $f(I_i)=f(I_j)$ for all i and j and $\{t \in T: f(t) \in L\} = \bigcup_{i=1}^{n} I_i$.

Next we show that $L \subset \partial V_\infty$. Let Ω be a connected open set in $\mathbf{C}$ such that $\Omega \cap f(T)=L$. Then $(\Omega \cap \partial V_\infty) \subset L$. Since $a_0 \in L \cap \partial V_\infty$, it follows that $\Omega \backslash L$ meets both V_∞ and some bounded component V_0 of $\mathbf{C} \backslash f(T)$. From this we may conclude that $\Omega \backslash L$ is not connected. As a result, $\Omega \backslash L$ has exactly two open components which we will call E_1 and E_2 and L is contained in the boundaries of both E_1 and E_2 ([7], Theorem 11.7, p. 118 and Theorem 16.3, p. 127). Moreover, we have $E_1 \subset V_\infty$ and $E_2 \subset V_0$.

If $\Omega \cap \partial V_\infty \neq L$, then there is $b \in L$ and an open disk B about b with $B \subset \Omega$ and $B \cap \partial V_\infty = \varnothing$. In this case we can find an arc from a point in V_∞ to a point in V_0 which does not pass through ∂V_∞ and this gives a contradiction.

We have just seen that $\pi_f^{-1}(w) \cap T$ contains n elements for each $w \in L$. Since L is also in the boundary of the unbounded component of $\mathbf{C} \backslash f(T)$, it follows that $\pi_f^{-1}(w) \subset T$. An elementary proof of this may be given, but the result also follows from a more general theorem of Björk ([1], theorem 1.7).

Let $\widehat{f(D)}$ be the polynomial hull of $f(D)$. The components of the interior of $\widehat{f(D)}$ are simply connected. Let G be the component which contains $f(U)$. Then $f(D) \subset \bar{G}$ and L is an open arc contained in ∂G. Moreover, $\partial G \subset f(T)$.

Furthermore, since L is open in $f(T)$, there is no $w_0 \in L$ with the property that a sequence $\{w_k\} \subset \partial G \backslash L$ converges to w_0. Let $\phi(z)$ be a conformal map of G onto U. From the previous remark it follows that $\phi(z)$ extends continuously to L and maps L homeomorphically into T ([5], p. 44). Consequently, $F(z)=\phi \circ f(z)$ maps I_i into T. By the Schwartz reflection principle $F(z)$ extends analytically across I_i for $i=1, \ldots, n$.

Let N be an open disk about $\phi(a_0)$ where N is chosen to be so small that $N \cap T \subset \phi(L)$ and $\phi^{-1}(N \cap U) \cap f(T) = \varnothing$. Since $\phi^{-1}(N \cap U)$ is connected, we must have $\phi^{-1}(N \cap U)$ contained in the single component V_0 of $\mathbf{C} \backslash f(T)$. Since $f(U)$ meets V_0, it follows that $\phi^{-1}(N \cap U) \subset V_0 \subset f(U)$. By reducing the radius of N, we can also find pairwise disjoint open sets W_i in $\mathbf{C}$ for $i=1, \ldots, n$ such that $t_i \in W_i$ and $N \subset F(W_i)$. It follows that $f(W_i \cap D) \supset \phi^{-1}(N \cap D)$ for each i.

The domain $\phi^{-1}(N \cap U)$ is bounded by the closed Jordan curve Γ where Γ is the image under ϕ^{-1} of $\partial(N \cap U)$. Also, a subarc of L lies in Γ. Lemma 1 implies that $\# \pi_f^{-1}(z) \leq n$ for $z \in \phi^{-1}(N \cap U)$. We have just noted that $\pi_f^{-1}(z) \geq n$ for $z \in \phi^{-1}(N \cap U)$, and so $\pi_f^{-1}(z)=n$ for $z \in \phi^{-1}(N \cap D)$.

Since $b\,\Delta \cap T = T$, there is a net $\{\Psi_\alpha\} \subset \Delta$ which converges to t_1. Then we have limit $\hat{f}(\Psi_\alpha) = a_0$ and consequently there is some a_0 so that $\hat{f}(\Psi_\alpha) \in \Omega$ for $\alpha \geq \alpha_0$. Since $f(\Psi_\alpha) \notin L \cup V_\infty$, we have $\hat{f}(\Psi_\alpha) \in V_0$ for $\alpha \geq \alpha_0$. Now $\phi(\hat{f}(\Psi_\alpha))$ converges to $\phi(a_0)$. Hence, there is some $\Psi_0 \in \{\Psi_\alpha\}$ so that $\phi(\hat{f}(\Psi_0)) \in N \cap U$. In this case $\hat{f}(\Psi_0) \in \phi^{-1}(N \cap U)$. This contradicts the equation $\# \pi_f^{-1}(\hat{f}(\Psi_0)) = n$ and we must conclude that $\Delta = \varnothing$.

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3. **Examples.** Example 1 shows that Theorem 1 does not contain Theorem 2.

EXAMPLE 1. There is a function algebra A on D with $A \subset A(D)$ with the following properties:

(i) If $f \in A$ satisfies $f|_T \in C^1(T)$, then Q_f is uncountable.

(ii) There is $A_0 \subset A$ such that A_0 separates points on D and $A_0|_T \subset C^1(T)$.

Proof. Let $\{z_k\}$ be a Blaschke sequence in U which accumulates to a closed uncountable set K of T of measure zero. Define $A = \{f \in A(D) : f'(z_k) = 0 \text{ for all } k\}$. Let $B(z)$ be a Blaschke product with zeros at the z_k and let $g(z) \in A(D)$ be equal to zero precisely on K. If we set $A_0 = \{F(z) : F(z) = \int_0^z f(\zeta)g(\zeta)B(\zeta)\,d\zeta \text{ for } f \in A(D)\}$, then $A_0 \subset A$ and $A_0|_T \subset C^1(T)$. We show that A_0 separates points on D.

Given a and b in D with $a \neq b$, consider $f(z) = (z-a)(z-b)g(z)B(z)$. Define $F_n(z) \in A_0$ by $F_n(z) = \int_0^z f(\zeta)\exp(2\pi n(\zeta-a)/(b-a))\,d\zeta$ for $n = 0, \pm 1, \pm 2, \ldots$. Since $f(a) = f(b)$, we can regard f as a continuous periodic function on the interval from a to b. If $0 = F_n(b) - F_n(a) = \int_a^b f(\zeta)\exp(2\pi n(\zeta-a)/(b-a))\,d\zeta$ for all n, then all the Fourier coefficients of f are zero. This implies that f is zero on a line segment in D which is a contradiction.

Finally, if $f \in A$ satisfies $f|_T \in C^1(T)$, then $f'(z) \in A(D)$ and hence $f'(z)$ is equal to zero on K. q.e.d.

EXAMPLE 2. We use an example of Glicksberg [4] to show that the countability of Q_f in Theorem 1 cannot be replaced by the condition that Q_f have measure zero in T.

Proof. Let $E \subset T$ be a Cantor set of measure zero with the following property. If $T \backslash E = \bigcup_{n=1}^{\infty} I_n$ where the I_n's are disjoint open intervals and ε_n = the length of I_n, then $-\infty < \sum_{n=1}^{\infty} \varepsilon_n \log \varepsilon_n$. Let K be a Cantor set in $\mathbf{C}$ having positive planar measure and let ϕ be a homeomorphism of E onto K. Let S^2 be the Riemann sphere. If $A_K = \{f \in C(K) : f \in C(S^2) \text{ and } f \text{ is analytic on } S^2 \backslash K\}$, then $A = \{f \in A(D) : f \circ \phi^{-1} \in A_K\}$ is a function algebra on D with maximal ideal space properly containing D ([4]). However, there are functions $f(z) \in A$ such that $f \in C^1(T)$ and $f(t) = f'(t) = 0$ precisely on E ([8], p. 85).

4. **Application.** Let A be a function algebra on D with $A \subset A(D)$. In [6] it is shown that if A contains an ideal J of $A(D)$ such that $\{z \in D : f(z) = 0 \text{ for all } f \in J\}$ is a countable set, then $M_A = D$. The converse is not true. That is, there is a function algebra A on D with $A \subset A(D)$ and $M_A = D$ but such that A contains no nonzero ideal of $A(D)$. To see this let $f_1(z) = (z-1)\exp((z+1)/(z-1))$ and $f_2(z) = (z-1)^2\exp((z+1)/(z-1))$. Then f_1 and f_2 generate a function algebra A on D and $A \subset A(D)$. By applying Theorem 1 (or the proof of Theorem 2), we see $M_A = D$. It is straightforward but lengthy calculation to show that A contains no nonzero deal of $A(D)$.

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