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A COMPACTIFICATION WITH θ-CONTINUOUS LIFTING PROPERTY

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1. Let X be a topological space, and let X' be the set of all non-convergent ultrafilters on X. If $A \subseteq X$, let $A' = \{\mathscr{F} \in X' : A \in \mathscr{F}\}$, and $A^* = A \cup A'$. If $\mathscr{F}$ is a filter on X such that $\mathscr{F}' \neq \emptyset$ for all $F \in \mathscr{F}$, then let $\mathscr{F}'$ be the filter on X^* generated by $\{F' : F \in \mathscr{F}\}$; let $\mathscr{F}^*$ be the filter on X^* generated by $\{F^* : F \in \mathscr{F}\}$. If $\mathscr{F}'$ exists then $\mathscr{F}^* = \mathscr{F} \cap \mathscr{F}'$; otherwise, $\mathscr{F}^* = \mathscr{F}$.

A convergence is defined on X^* as follows: If $x \in X$, then a filter $A \to x$ in X^* if and only if $\mathscr{A} \geqq \mathscr{V}_X(x)^*$, where $V_X(x)$ is the X-neighborhood filter at x; if $\mathscr{G} \in X'$, then $\mathscr{A} \to \mathscr{G}$ in X^* if and only if $A \geqq \mathscr{G}^*$. The resulting space X^* is a pretopological space and the X^*-neighborhood filter of α is denoted by $\mathscr{V}_{X^*}(\alpha)$; if $\alpha = x \in X$, then $\mathscr{V}_{X^*}(\alpha) = \mathscr{V}_X(x)^*$, and if $\alpha = \mathscr{G} \in X'$, then $\mathscr{V}_{X^*}(\alpha) = \mathscr{G}^*$. The space X^* is not topological in many standard examples. It is shown in [**3**] that the space X^* is compact (meaning that each ultrafilter is convergent) and X is a subspace of X^*. Indeed, X^* is a convergence space compactification of X (see [**3**]).

In this paper, we obtain a toplogical compactification $X^\wedge$ of X by taking the "topological modification" of X^* (i.e., $X^\wedge$ and X^* have the same underlying set, and $X^\wedge$ has the finest topology coarser than X^*). The open sets for $X^\wedge$ are obtained as follows: $A \subseteq X^*$ is open if and only if $\alpha \in A$ implies $A \in \mathscr{V}_{X^*}(\alpha)$. We shall now show that $X^\wedge$ is a compactification of X, and give an explicit construction for an open base for $X^\wedge$ in terms of the open sets in X.

LEMMA 1.1. *If U is an open subset of X, then U^* is open in $X^\wedge$. If $x \in X$, then $\mathscr{V}_{X^*}(x) = \mathscr{V}_{X^\wedge}(x) = \mathscr{V}_X(x)^*$.*

Proof. Let $\alpha \in U^*$. If $\alpha = x \in U$, then $U \in \mathscr{V}_X(x)$ implies $U^* \in \mathscr{V}_X(x)^*$. If $\alpha = \mathscr{G} \in U'$, then $U \in G$ implies $U^* \in \mathscr{G}^*$. Thus U^* is an X^*-neighborhood of each of its elements, and hence open in $X^\wedge$, and the first assertion is proved.

Since $X^\wedge$ is coarser than X^*,

$$\mathscr{V}_{X^\wedge}(x) \leq \mathscr{V}_{X^*}(x) = \mathscr{V}_X(x)^*.$$

But the first assertion of the lemma implies $\mathscr{V}_{X^\wedge}(x) \geqq \mathscr{V}_X(x)^*$, and hence the second assertion is established.

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THEOREM 1.2. *$X^\wedge$ is a compactification of X.*

Proof. Since X^* is known to be a compactification of X, $X^\wedge$ is compact and X is dense in X'. The fact that X is a subspace of $X^\wedge$ is an immediate consequence of Lemma 1.

Sets of the form U^* for U open in X will not, in general, form a base for $X^\wedge$. We next describe another class of open sets in $X^\wedge$ which enable us to form a base for this topology.

If A is a non-empty subset of X, we define a *neighborhood function for A* to be a function s such that, for each $x \in A$, $s(x)$ is an open neighborhood of x in X. Let $N(A)$ be the set of all neighborhood functions for A. If $s \in N(A)$, define

$$W_s(A) = A^* \cup (\quad \cup \{s(x)^* : x \in A\}).$$

PROPOSITION 1.3. *Under the assumption of the preceding paragraph $W_s(A)$ is open in $X^\wedge$.*

Proof. Let $\alpha \in W_s(A)$. If $\alpha = x \in W_s(A) \cap X$, then clearly $x \in s(y)^*$ for some $y \in A$, which is an $X^\wedge$-open subset of $W_s(A)$ by Lemma 1. If $\alpha = \mathcal{G} \in W_s(A) \cap X'$, and $\alpha \in s(y)^*$ for some $y \in A$, then again $W_s(A)$ is an $X^\wedge$-neighborhood of α. Otherwise, $\mathcal{G} \in A^*$, which implies $A \in \mathcal{G}$ and $A^* \in \mathcal{G}^* = \mathcal{V}_{X^*}(\alpha)$; thus $W_s(A) \in \mathcal{V}_{X^*}(\alpha)$ and the proof is complete.

PROPOSITION 1.4. *The collection*

$$\mathcal{W} = \{W_s(A) : A \subseteq X, A \neq \emptyset, s \in N(A)\}$$

is a base for the topology of $X^\wedge$.

Proof. Let $B \subseteq X^\wedge$ be open and $\alpha \in B$. If $\alpha \in B \cap X$, then by Lemma 1 there is an X-open set U such that $\alpha \in U^* \subseteq B$. If $\alpha = \mathcal{G} \in B \cap X'$, then let $A = B \cap X$. Since B is open in $X^\wedge$, there is for each $x \in A$ an open neighborhood $s(x)$ of x such that $s(x)^* \subseteq B$. Also, since $\mathcal{G} \in B$, there is $G \in \mathcal{G}$ such that $\mathcal{G}^* \subseteq B$. Since $G \subseteq A$, then if s' denotes the restriction of the neighborhood function s to G, it follows that $\alpha \in W_{s'}(G) \subseteq B$.

COROLLARY 1.5. *If $\alpha = \mathcal{G} \in X'$, then sets of the form*

$$\{W_s(G) : G \in \mathcal{G}, s \in N(G)\}$$

form an open base for $\mathcal{V}_{X^\wedge}(\alpha)$.

THEOREM 1.6. *If X is T_1, then $X^\wedge$ is T_1.*

Proof. Let $\alpha \in X^\wedge$ and $B = X^\wedge - \{\alpha\}$. If $\alpha = x \in X$, then $U = X - \{x\}$ is X-open, and so $B = U^*$ is $X^\wedge$-open. Suppose $\alpha = \mathcal{G} \in X'$. If $x \in X$, then there is an X-open neighborhood U of x such that $U \notin \mathcal{G}$, and $x \in U^* \subseteq B$. If $\beta = \mathscr{F} \in B \cap X'$, then choose $F \in \mathscr{F}$ such that $F \notin \mathscr{G}$. If $x \in F$, then as before there is an X-open neighborhood $s(x)$ of x such that $s(x)^* \subseteq B$. Also, $F \notin G$ implies $F^* \subseteq B$. Thus $\beta \in W_s(F) \subseteq B$, and B is X^*-open.

2. We next consider $f : X \to Y$, where X and Y are topological spaces and f is a continuous function. We first show that f can fail to have any continuous extension to the respective compactification spaces, but that a θ-continuous extension always exists.

A function $f:X \to Y$ is said to be θ-continuous (see [**1**]) at $x \in X$ if, for each closed neighborhood W of $f(x)$, there is a closed neighborhood V of x such that $f(V) \subseteq W$. Note that continuity always implies θ-continuity, and if Y is regular these concepts are equivalent.

Given $f:X \to Y$, let $A \subseteq X$ and $B \subseteq Y$. To minimize confusion, we shall use A^* to denote the "*-operation" relative to X, and B^{**} to denote the same operation relative to Y; a similar convention will apply to filters on X and Y, respectively.

Example 2.1. Let X be the set $\mathbf{R}$ of real numbers equipped with the discrete topology. Let Y be the set $\mathbf{R}$ with a topological base consisting of all open sets in the usual topology of $\mathbf{R}$ along with the set $\{\{x\}: x \text{ a rational number}\}$. Let $f:X \to Y$ be the identity map. We shall show that there is no continuous function $F:X^\wedge \to Y^\wedge$ which is an extension of f.

Let $\mathbf{N}$ be the set of natural numbers, and let $A = \{n\pi : n \in \mathbf{N}\}$. For each $n \in \mathbf{N}$, let $(x_{nm})_{m \in \mathbf{N}}$ be a sequence of rational numbers which converges in the usual topology on $\mathbf{R}$ to $n\pi$. Let $B_n = \{x_{nm} : m \in \mathbf{N}\}$, and let $\mathscr{F}$ be a free ultrafilter on $\mathbf{R}$ which contains the set $B = \bigcup \{B_n : n \in \mathbf{N}\}$ and has the property that each $F \in \mathscr{F}$ has an infinite intersection with infinitely many B_n's (e.g. let $\mathscr{F}$ be an ultrafilter containing $\{B - \bigcup_{n=1}^{\infty} A_n, B - B_n : n \in \mathbf{N}, A_n \text{ is a finite subset of } B_n\}$). Note that $\mathscr{F} \in Y'$, and since $\mathscr{F}$ has a filter base of Y-open sets, it follows from Corollary 1.5 that $\mathscr{V}_{Y^\wedge}(\mathscr{F}) = \mathscr{F}^{**}$.

Suppose $F:X^\wedge \to Y^\wedge$ is a continuous extension of f. From the fact that $\mathscr{V}_{Y^\wedge}(\mathscr{F}) = \mathscr{F}^{**}$, it follows necessarily that $F(\mathscr{F}) = \mathscr{F}$. If $U = R - A$, then U^* is $X^\wedge$-open, $U^* \in \mathscr{F}^*$, and $\mathscr{F}^* \to \mathscr{F}$ in $X^\wedge$. It is also true that U^{**} is $Y^\wedge$-open and $\mathscr{F} \in U^{**}$. But $U^{**} \notin F(\mathscr{F}^*)$. For by construction of $\mathscr{F}$, $(\mathrm{cl}_Y F) \cap A \neq \emptyset$ for all $F \in \mathscr{F}$, and so $F(D^*) \cap A^{**} \neq \emptyset$ for all $D \in \mathscr{F}$. It follows that $F(\mathscr{F}^*)$ does not $Y^\wedge$-converge to $F(\mathscr{F}) = \mathscr{F}$, and so F is not continuous.

THEOREM 2.2. *If $f:X \to Y$ is continuous, then there is a θ-continuous extension $F:X^\wedge \to Y^\wedge$.*

Proof. Let $F:X^\wedge \to Y^\wedge$ be any extension of f with the following properties:

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(a) If $\mathscr{F} \in X'$, and $f(\mathscr{F})$ converges in Y, then $F(\mathscr{F}) = y$, where y is any limit in Y of $f(\mathscr{F})$.

(b) If $\mathscr{F} \in X'$ and $f(\mathscr{F}) \in Y'$, then $F(\mathscr{F}) = f(\mathscr{F})$.

Next, observe that if A is a closed subset of X, then $A^* = X^* - (X - A)^*$ is closed in $X^\wedge$ by Lemma 1.1. This result, along with Proposition 1.4, enables us to deduce that sets of the form $(\text{cl}_X U)^*$, where U is X-open and $x \in U$, form a base of $X^\wedge$-closed neighborhoods at $x \in X$, and sets of the form $(\text{cl}_X U)^*$, where $U \in \mathscr{G}$ is X-open, form an $X^\wedge$-closed neighborhood base for $\mathscr{G} \in X'$.

Let $\alpha \in X^\wedge$ and $F(\alpha) = \beta$. If $\alpha = x \in X$, then $F(\alpha) = f(x) = y = \beta \in Y$; if V is any Y-open neighborhood of y, then $(\text{cl}_Y V)^{**}$ is a basic $Y^\wedge$-closed neighborhood of y in $Y^\wedge$ by our preceding discussion. By continuity of f, there is an X-open neighborhood U of x such that $f(U) \subseteq V$, and it is easy to verify that

$$F((\text{cl}_X U)^*) \subseteq (\text{cl}_Y V)^{**}.$$

Thus θ-continuity is established at all points α in X.

If $\alpha = \mathscr{G} \in X'$, then β may belong to Y or Y', depending on whether or not $f(\mathscr{G})$ converges in Y. If $\beta = y \in Y$, and $(\text{cl}_Y V)^{**}$ is a basic $Y^\wedge$-closed neighborhood of y in $Y^\wedge$ as described above, then $V \in f(\mathscr{G})$; if $U = f^{-1}(V)$, then $(\text{cl}_X U)^*$ is a closed $X^\wedge$-neighborhood of α in $X^\wedge$ and, as before,

$$F((\text{cl}_X)^*) \subseteq (\text{cl}_Y V)^{**}.$$

If $\beta = f(\mathscr{G}) \in Y'$, then V can be chosen to be any Y-open set in $f(\mathscr{G})$, and the same argument repeated.

It follows that F is θ-continuous for all $\alpha \in X^\wedge$, and the proof is complete.

COROLLARY 2.3. *If $f:X \to Y$ is continuous, and $Y^\wedge$ is regular, then there is a continuous extension $F:X^\wedge \to Y^\wedge$.*

COROLLARY 2.4. *If $X^\wedge$ is T_2, then $X^\wedge = \beta X$.*

Proof. If Y is a compact, T_2 space, and $f:X \to Y$ is continuous, then $Y = Y^\wedge$ and, by Corollary 2.3, there is a continuous extension $F: X^\wedge \to Y$. This extension is unique because Y is T_2, and so $X^\wedge$ is the largest T_2 compactification of X; i.e., $X^\wedge = \beta X$.

A T_3 topological space X is defined to be a *G-space* if each non-convergent ultrafilter has a filter base of closed sets. This condition (but not the terminology) is due to Gazik [**2**], who showed that the pretopological compactification X^* of a T_3-topological space X is equivalent to βX if and only if X is a G-space. When X^* is a topological space, $X^* = X^\wedge$. Thus if X is a G-space, it follows from [**2**] that $X^\wedge$ is equivalent to βX.

THEOREM 2.5. *For a T_3 topological space X, the following statements are equivalent:*

(1) *X is a G-space.*

(2) *If $\mathscr{F}$ and $\mathscr{G}$ are distinct non-convergent ultrafilters on X, then there are disjoint open sets U, V such that $U \in \mathscr{F}$, $\mathscr{V} \in \mathscr{G}$.*

(3) *$X^\wedge$ is T_2.*

(4) *$X^\wedge$ is equivalent to βX.*

Proof. (3) $\Leftrightarrow$ (4). This was established in Corollary 2.4.

(1) $\Rightarrow$ (4). This was established in the paragraph preceding the theorem.

(3) $\Rightarrow$ (1). If X is not a G-space, then there is $\mathscr{F} \in X'$ such that $\text{cl}_X \mathscr{F} \neq \mathscr{F}$, and so there is an ultrafilter $\mathscr{G} \geqq \text{cl}_X \mathscr{F}$ such that $\mathscr{G} \neq \mathscr{F}$. If $\tilde{\mathscr{F}}$ is the filter on $X^\wedge$ generated by $\mathscr{F}$, then $\tilde{\mathscr{F}} \to \mathscr{F}$ in $X^\wedge$. If $\tilde{\mathscr{G}}$ is the filter on $X^\wedge$ generated by $\mathscr{G}$, then, because $X^\wedge$ is regular, $\tilde{\mathscr{G}} \to \mathscr{F}$ in $X^\wedge$. But either $\mathscr{G} \in X'$, in which case $\tilde{\mathscr{G}} \to \mathscr{G} \neq \mathscr{F}$, or else there is $x \in X$ such that $\mathscr{G} \to x$ in X, in which case $\tilde{\mathscr{G}} \to x$ in $X^\wedge$; either way there is a contradiction, since $X^\wedge$ is assumed to be T_2.

(3) $\Rightarrow$ (2). Let $\mathscr{F}, \mathscr{G} \in X'$. Since $X^\wedge$ is T_2, it follows by Corollary 1.5 that there are disjoint sets of the form $W_s(F)$ and $W_{s'}(G)$, where $F \in \mathscr{F}$, $G \in \mathscr{G}$. If $U = W_s(F) \cap X$ and $V = W_{s'}(G) \cap X$, then U and V satisfy the conditions of (2).

(2) $\Rightarrow$ (3). Let α, β be distinct elements of $X^\wedge$. If $\alpha, \beta \in X$, then there are disjoint X-open neighborhoods U and V of α and β, respectively, and U^*, V^* are disjoint $X^\wedge$-open neighborhoods of these elements. If $\alpha \in X$, $\beta \in X'$, then because X is T_3 there are disjoint X-open sets U and V such that $\alpha \in U$ and $V \in \mathscr{G} = \beta$, and again it follows that U^* and V^* are disjoint $X^\wedge$-open neighborhoods of α and β. Finally, if $\alpha = \mathscr{F}$ and $\beta = \mathscr{G}$ and both in X', then the sets U, V given in (2) yield disjoint $X^\wedge$-open neighborhoods of α and β. Thus $X^\wedge$ is T_2.

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